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Two Polynomial Division Inequalities in L^p

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This paper is a first attempt to give numerical values for constants C_p and C'_p in classical estimates $\|P\| \leq C_p n\|xP\|$ and $\|P\| \leq C'_p n^2\|(1-x)P\|$ where P is an algebraic polynomial of degree at most n $(n > 0)$ and $\|\cdot\|$ denotes the p-metric on $[-1, 1]$. The basic tools are Markov and Bernstein inequalities.

Keywords: Polynomial inequalities; Schur inequality; Explicit constants

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1. NOTATION AND BASIC INEQUALITIES

Let I be an interval, $p \geq 1$ and f a measurable function. We set

$$\|f\|_{p,I} = \left[\int_I |f(x)|^p \,\mathrm{d}x\right]^{1/p} \quad (p < \infty),$$
$$\|f\|_{\infty,I} = \operatorname*{ess\,sup}_{x\in I} |f(x)|.$$

If $I = [-1, 1]$ the subscript I is omitted:

$$\|f\|_p = \|f\|_{p,[-1,1]} \quad (1 \leq p \leq \infty).$$

For any measurable 2π-periodic function g, we define

$$\|g\|_p^* = \|g\|_{p,[0,2\pi]} \quad (1 \leq p \leq \infty).$$

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We denote by $\mathbb{P}_n$ ($\mathbb{T}_n$ resp.) the set of algebraic (2π-periodic trigonometric resp.) polynomial of degree (order resp.) at most n. For $P \in \mathbb{P}_n$, $\deg(P)$ stands for degree of P.

We recall the following three classical inequalities:

Markov inequality [1, p. 141]: for any $P \in \mathbb{P}_n$ we have

$$\|P'\|_\infty \le n^2 \|P\|_\infty. \tag{1.1}$$

Bernstein inequality [1, p. 90]: for any $T \in \mathbb{T}_n$ we have

$$\|T'\|^*_\infty \le n \|T\|^*_\infty. \tag{1.2}$$

Tchebicheff inequality [1, p. 51]: let $h > 1$ and (T_n) be the sequence of Tchebicheff polynomials. For any $P \in \mathbb{P}_n$ we have

$$\|P\|_{\infty,[-h,h]} \le T_n(h) \|P\|_\infty. \tag{1.3}$$

2. INTRODUCTION AND RESULTS

In 1919 Schur [2] gave estimates that we can rewrite in the following form: for any $P \in \mathbb{P}_n$,

$$\|P\|_\infty \le (n+1) \|xP\|_\infty$$

and

$$\|P\|_\infty \le \frac{n+1}{2} \cot\left(\frac{\pi}{4n+4}\right) \|(1-x)P\|_\infty.$$

More generally, it is well known (see for example [3–6]) that for any $p \ge 1$ there exist absolute constants C_p and C'_p such that for any $P \in \mathbb{P}_n$ $(n > 0)$

$$\|P\|_p \le C_p n \|xP\|_p, \tag{2.1}$$

$$\|P\|_p \le C'_p n^2 \|(1-x)P\|_p. \tag{2.2}$$

Furthermore exponents 1 and 2 of n in estimates (2.1) and (2.2) are optimal.

These inequalities are extensively used in approximation theory. Unfortunately, numerical values for C_p and C'_p have never been provided. So our aim is to prove the following two results:

THEOREM 2.1 *Let $n>0$. For any $p\geq 1$ and any $P\in\mathbb{P}_n$, $\|P\|_p\leq 6n\|xP\|_p$.*

THEOREM 2.2 *Let $n>0$. For any $p\geq 1$ and any $P\in\mathbb{P}_n$, $\|P\|_p\leq 6.3n^2\|(1-x)P\|_p$.*

3. PROOF OF THEOREM 2.1

LEMMA 3.1 *Let $n\in\mathbb{N}^*$, $T\in\mathbb{T}_n$, $\theta_0\in[0,2\pi]$ be such that $|T(\theta_0)|=\|T\|^*_\infty$ and $J=[\theta_0-(\sqrt{2}/n),\theta_0+(\sqrt{2}/n)]$.*

For any $\theta\in J$ we have

$$|T(\theta)|\geq\left(1-\tfrac{1}{2}(\theta-\theta_0)^2n^2\right)\|T\|^*_\infty.$$

Proof From Taylor's formula we get

$$T(\theta)=T(\theta_0)+(\theta-\theta_0)T'(\theta_0)+\tfrac{1}{2}(\theta-\theta_0)^2T''(\theta_1)$$

for some θ_1 between θ_0 and θ.

Then, since $T'(\theta_0)=0$ and $|T(\theta_0)|=\|T\|^*_\infty$ we have

$$T(\theta)-T(\theta_0)=\tfrac{1}{2}(\theta-\theta_0)^2T''(\theta_1)$$

and

$$\|T\|^*_\infty-|T(\theta)|=|T(\theta)-T(\theta_0)|=\tfrac{1}{2}(\theta-\theta_0)^2|T''(\theta_1)|.$$

Thus, applying twice Bernstein inequality (1.2) yields

$$\|T\|^*_\infty-|T(\theta)|\leq\tfrac{1}{2}(\theta-\theta_0)^2n^2\|T\|^*_\infty$$

whence the lemma follows immediately.

LEMMA 3.2 *Let $n\in\mathbb{N}^*$. The absolute minimum of*

$$I(t)=\int_{t-\sqrt{2}/n}^{t+\sqrt{2}/n}\left(1-\tfrac{1}{2}(\theta-t)^2n^2\right)^p|\cos\theta|^p|\sin\theta|\,\mathrm{d}\theta$$

is $I(\pi/2)$.

Proof Let $g(\theta)=|\cos\theta|^p|\sin\theta|$ and g' be the derivative of g for $\theta\neq 0$ (mod $\pi/2$). It is easily computed that

$$\frac{\partial I}{\partial t}=\int_{t-\sqrt{2}/n}^{t+\sqrt{2}/n}\left(1-\tfrac{1}{2}(\theta-t)^2n^2\right)^p g'(\theta)\,\mathrm{d}\theta.$$

A short study of $I(t)$ shows that

- $I(-t)=I(t)$,
- $I(\pi-t)=I(t)$,
- $\partial I(0)/\partial t=\partial I(\alpha)/\partial t=\partial I(\pi/2)/\partial t=0$ for some $\alpha\in(0,\pi/2)$,
- I is an increasing function for $t\in[0,\alpha]$,
- I is a decreasing function for $t\in[\alpha,\pi/2]$,
- $I(\pi/2)\leq I(0)$ (since $p\geq 1$).

Then, the absolute minimum of $I(t)$ is attained when $t=\pi/2$.

LEMMA 3.3 *For any* $n\in\mathbb{N}^*$ *we have*

$$\int_{\pi/2-\sqrt{2}/n}^{\pi/2+\sqrt{2}/n}\left(1-\tfrac{1}{2}(\theta-(\pi/2))^2n^2\right)^p|\cos\theta|^p|\sin\theta|\,\mathrm{d}\theta$$
$$\geq 2^{1-p}(\sin(1/n))^{p+1}/(p+1).$$

Proof

$$\int_{\pi/2-\sqrt{2}/n}^{\pi/2+\sqrt{2}/n}\left(1-\tfrac{1}{2}(\theta-(\pi/2))^2n^2\right)^p|\cos\theta|^p|\sin\theta|\,\mathrm{d}\theta$$
$$=\int_{-\sqrt{2}/n}^{\sqrt{2}/n}\left(1-\tfrac{1}{2}n^2t^2\right)^p|\sin t|^p|\cos t|\,\mathrm{d}t$$
$$=2\int_0^{\sqrt{2}/n}\left(1-\tfrac{1}{2}n^2t^2\right)^p(\sin t)^p\cos t\,\mathrm{d}t$$
$$>2\int_0^{1/n}(\sin t)^p\cos t\,\mathrm{d}t\,\left(1-\tfrac{1}{2}n^2(1/n)^2\right)^p$$
$$=2^{1-p}(\sin(1/n))^{p+1}/(p+1).$$

COROLLARY 3.4 *Let* $P\in\mathbb{P}_n$ $(n>0)$ *and* $T(\theta)=P(\cos\theta)$. *We have*

$$\|xP\|_p^p\geq\|T\|_\infty^{*p}\,2^{-p}(\sin(1/n))^{p+1}/(p+1).$$

Proof We have

$$\begin{aligned}\|xP\|_p^p &= \int_{-1}^{1} |x|^p |P(x)|^p \,\mathrm{d}x \\ &= \int_0^{\pi} |\cos\theta|^p |P(\cos\theta)|^p \sin\theta \,\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi} |\cos\theta|^p |T(\theta)|^p |\sin\theta| \,\mathrm{d}\theta.\end{aligned}$$

Let $\theta_0 \in [0, \pi]$ be such that $|T(\theta_0)| = \|T\|_\infty^*$ and $J = [\theta_0 - \sqrt{2}/n, \theta_0 + \sqrt{2}/n]$. Then,

$$\frac{1}{2}\int_0^{2\pi} |\cos\theta|^p |T(\theta)|^p |\sin\theta| \,\mathrm{d}\theta \geq \frac{1}{2}\int_J |\cos\theta|^p |T(\theta)|^p |\sin\theta| \,\mathrm{d}\theta$$

and using successively Lemmas 3.1–3.3 we get

$$\begin{aligned}\|xP(x)\|_p^p &\geq \frac{1}{2}\int_J \left(1 - \tfrac{1}{2}(\theta - \theta_0)^2 n^2\right)^p |\cos\theta|^p |\sin\theta| \,\mathrm{d}\theta\, \|T\|_\infty^{*p} \\ &\geq \frac{1}{2}\int_{\pi/2-\sqrt{2}/n}^{\pi/2+\sqrt{2}/n} \left(1 - \tfrac{1}{2}(\theta - (\pi/2))^2 n^2\right)^p |\cos\theta|^p |\sin\theta| \,\mathrm{d}\theta \|T\|_\infty^{*p} \\ &\geq \|T\|_\infty^{*p}\, 2^{-p} (\sin(1/n))^{p+1}/(p+1).\end{aligned}$$

LEMMA 3.5 *Let $P \in \mathbb{P}_n$ $(n > 0)$ and $T(\theta) = P(\cos\theta)$. If for some B we have $\|T\|_\infty^{*p} \leq B\,\|xP\|_p^p$, then,*

$$\|P\|_p \leq (2B/p)^{1/(p+1)} (p+1)^{1/p} \|xP\|_p.$$

Proof Let $a \in [0, 1]$ and

$$\begin{aligned}K = {} & [\pi/2 - \arcsin a, \pi/2 + \arcsin a] \\ & \cup [3\pi/2 - \arcsin a, 3\pi/2 + \arcsin a].\end{aligned}$$

Clearly,

$$\int_K |T(\theta)|^p |\sin\theta| \,\mathrm{d}\theta \leq \|T\|_\infty^{*p} \int_K |\sin\theta| \,\mathrm{d}\theta = 4a \|T\|_\infty^{*p},$$

then,

$$\int_K |T(\theta)|^p |\sin\theta| \,d\theta \le 4aB\,\|xP\|_p^p. \tag{3.1}$$

Furthermore

$$\begin{aligned}\int_{[0,2\pi]\setminus K} |T(\theta)|^p |\sin\theta| \,d\theta &\le a^{-p} \int_{[0,2\pi]\setminus K} |T(\theta)|^p |\sin\theta| |\cos\theta|^p \,d\theta \\ &\le a^{-p} \int_{[0,2\pi]} |T(\theta)|^p |\sin\theta| |\cos\theta|^p \,d\theta,\end{aligned}$$

then

$$\int_{[0,2\pi]\setminus K} |T(\theta)|^p |\sin\theta| \,d\theta \le a^{-p} 2\|xP\|_p^p. \tag{3.2}$$

Inequalities (3.1) and (3.2) together give: for every $a \in [0, 1]$, $\|P\|_p^p \le (a^{-p} + 2aB)\|xP\|_p^p$. In order to minimize the coefficient $(a^{-p} + 2aB)$ we choose $a = (p/(2B))^{1/(p+1)}$ and we get

$$\|P\|_p \le (2B/p)^{1/(p+1)}(p+1)^{1/p}\|xP\|_p.$$

LEMMA 3.6 *If* $P \in \mathbb{P}_1$ *then* $\|P\|_p \le 6\|xP\|_p$.

Proof If P is a constant polynomial we have

$$\|P\|_p \le (p+1)^{1/p}\|xP\|_p \le 6\|xP\|_p.$$

If $\deg(P) = 1$, we can assume that $P(x) = x + b$, $(b \ge 0)$. For $x \in [0, 1]$ we have $x + b \ge (1 + b)x$ then

$$\int_{-1}^1 |xP(x)|^p \,dx \ge \int_0^1 |xP(x)|^p \,dx \ge (1+b)^p \int_0^1 x^{2p} \,dx = \frac{(1+b)^p}{2p+1}.$$

On the other hand, $\int_{-1}^1 |P(x)|^p \,dx \le 2(1+b)^p$; thus

$$\|P\|_p \le (4p+2)^{1/p}\|xP\|_p \le 6\|xP\|_p.$$

Proof of Theorem 2.1 Lemma 3.6 shows that in the following we can assume $\deg(P) \geq 2$. Using the result of Corollary 3.4 and applying Lemma 3.5 with $B = 2^p(p+1)(\sin(1/n))^{-p-1}$ yields

$$\|P\|_p \leq (\sin(1/n))^{-1} 2(1 + (1/p))^{1/(p+1)}(p+1)^{1/p}\|xP\|_p.$$

For $n \geq 2$,

$$\frac{\sin(1/n)}{1/n} \geq \frac{\sin(1/2)}{1/2},$$

then

$$\frac{1}{\sin(1/n)} \leq \frac{n}{2\sin(1/2)}. \tag{3.3}$$

Furthermore, for $p \geq 1$, $(1+(1/p))^{1/(p+1)}(p+1)^{1/p}$ is a decreasing function of p, whence

$$(1 + (1/p))^{1/(p+1)}(p+1)^{1/p} \leq 2\sqrt{2}. \tag{3.4}$$

Taking account of inequalities (3.3) and (3.5) we get

$$\|P\|_p \leq 2\sqrt{2}(\sin(1/2))^{-1} n \|xP\|_p$$

which completes the proof of Theorem 2.1 since $2\sqrt{2}/(\sin(1/2)) = 5.8996\ldots$

4. PROOF OF THEOREM 2.2

4.1. Preliminary Lemmas

LEMMA 4.1 *Let* $I_n = [-1, 1 - (1/n^2)]$ $(n > 0)$. *For any* $P \in \mathbb{P}_n$, *we have*

$$\|P\|_\infty \leq T_n\left(\frac{2n^2+1}{2n^2-1}\right)\|P\|_{\infty, I_n}.$$

Proof This is an immediate corollary of Tchebicheff inequality (1.3).

LEMMA 4.2 *For* $n \geq 2$,

$$T_n\left(\frac{2n^2+1}{2n^2-1}\right) \leq 113/49$$

holds true.

Proof For $h \geq 1$,

$$\begin{aligned} T_n(h) &= \frac{1}{2}\left[\left(h+\sqrt{h^2-1}\right)^n + \left(h-\sqrt{h^2-1}\right)^n\right] \\ &= \cosh\left[n \ln\left(h+\sqrt{h^2-1}\right)\right]. \end{aligned}$$

Therefore

$$T_n\left(\frac{2n^2+1}{2n^2-1}\right) = \cosh\left(n \ln\frac{n\sqrt{2}+1}{n\sqrt{2}-1}\right)$$

and the right-hand member of this last equality is easily proved to be a decreasing function of n. Then, for $n \geq 2$

$$T_n\left(\frac{2n^2+1}{2n^2-1}\right) \leq T_2(9/7) = 113/49.$$

LEMMA 4.3 *If* $P \in \mathbb{P}_1$ *then* $\|P\|_p \leq 6\|(1-x)P\|_p$.

Proof If P is a constant polynomial we have

$$\|P\|_p = \frac{1}{2}(p+1)^{1/p}\|(1-x)P\|_p \leq \|(1-x)P\|_p.$$

If $\deg(P) = 1$, we can assume that $P(x) = x + b$. We consider four cases:

(1) *Case* $b \leq 0$. In this case we have $\int_{-1}^{1} |x+b|^p \, dx \leq 2(1-b)^p$ and since for $x \in [-1, 1]$, $1 - x \geq 1$ and $|x+b| \geq (1-b)|x|$ we have

$$\int_{-1}^{1} |x+b|^p (1-x)^p \, dx \geq (1-b)^p \int_{-1}^{0} |x|^p \, dx = \frac{1}{p+1}(1-b)^p;$$

then

$$\|P\|_p \leq (2p+2)^{1/p}\|(1-x)P\|_p \leq 4\|(1-x)P\|_p.$$

(2) *Case* $b \in [0,1]$. We have

$$\int_{-1}^{1} |x+b|^p \,\mathrm{d}x \leq 2\int_{-b}^{1} (x+b)^p \,\mathrm{d}x = 2\frac{(1+b)^{p+1}}{p+1}$$

and

$$\begin{aligned}
&\int_{-1}^{1} |x+b|^p (1-x)^p \,\mathrm{d}x \\
&= \int_{-1}^{b} |x+b|^p (1-x)^p \,\mathrm{d}x + \int_{-b}^{1} |x+b|^p (1-x)^p \,\mathrm{d}x \\
&= \int_{0}^{1-b} |t-1+b|^p (2-t)^p \,\mathrm{d}t + \int_{0}^{1+b} t^p (1+b-t)^p \,\mathrm{d}t \\
&\geq (1+b)^p \int_{0}^{1-b} (1-b-t)^p \,\mathrm{d}t + 2\left(\frac{1+b}{2}\right)^p \int_{0}^{(1+b)/2} t^p \,\mathrm{d}t \\
&= \frac{(1+b)^p (1-b)^{p+1}}{p+1} + \frac{2}{p+1}\left(\frac{1+b}{2}\right)^{2p+1}.
\end{aligned}$$

Therefore we have $\|P\|_p \leq C\|(1-x)P\|_p$ with

$$C = \left(\frac{2}{((1+b)/4)^p + (1-b)^{p+1}/(1+b)}\right)^{1/p}.$$

Then if $b \geq 1/2$ we have

$$C \leq \left(\frac{2}{([1+(1/2)]/4)^p}\right)^{1/p} = \frac{8}{3}2^{1/p} \leq 6.$$

If $b < 1/2$ then $(1-b)/(1+b) \geq 1/3$; thus

$$C \leq \left(\frac{2}{1/4^p + 1/(3\times 2^p)}\right)^{1/p} \leq \left(\frac{2}{1/4^p + 1/(3\times 4^p)}\right)^{1/p} = 4\left(\frac{3}{2}\right)^{1/p} \leq 6.$$

Finally, in this case, $\|P\|_p \leq 6\|(1-x)P\|_p$.

(3) *Case* $b \in [1, 2]$. We have

$$\int_{-1}^{1} (x+b)^p \, dx \leq \frac{(1+b)^{p+1}}{p+1}$$

and

$$\int_{-1}^{1} (x+b)^p (1-x)^p \, dx \geq \int_{0}^{1} (x+b)^p (1-x)^p \, dx$$
$$\geq \left(\frac{1+b}{2}\right)^p \int_0^1 (1-x)^p \, dx = \frac{1}{p+1}\left(\frac{1+b}{2}\right)^p,$$

then $\|P\|_p \leq (2+2b)^{1/p} \|(1-x)P\|_p \leq 6\|(1-x)P\|_p$.
(4) *Case* $b \geq 2$. In this case we have

$$\int_{-1}^{1} (x+b)^p \, dx \leq 2\int_{0}^{1} (x+b)^p \, dx$$
$$= 2\int_{-1}^{0} (x+b+1)^p \, dx$$
$$= 2\int_{-1}^{0} (x+b)^p \left(\frac{x+b+1}{x+b}\right)^p dx$$
$$\leq 2^{p+1} \int_{-1}^{0} (x+b)^p \, dx$$
$$\leq 2^{p+1} \int_{-1}^{0} (x+b)^p (1-x)^p \, dx$$
$$\leq 2^{p+1} \int_{-1}^{1} (x+b)^p (1-x)^p \, dx.$$

Then $\|P\|_p \leq 2^{1+(1+p)} \|(1-x)P\|_p \leq 4\|(1-x)P\|_p$.
In the four cases we have $\|P\|_p \leq 6\|(1-x)P\|_p$.

4.2. Proof of Theorem 2.2

Let $P \in \mathbb{P}_n$. Using Lemma 4.3 we can assume that $2 \leq \deg(P) \leq n$. Let $x_n \in I_n = [-1, 1-(1/n^2)]$ be such that $|P(x_n)| = \|P\|_{\infty, I_n}$.

Lemmas 4.1 and 4.2 give

$$\int_{1-(1/n^2)}^{1} |P(x)|^p \,\mathrm{d}x \le n^{-2}\|P\|_{\infty}^{p} \le n^{-2}(113/49)^p\|P\|_{\infty,I_n}^{p}. \qquad (4.1)$$

For $x \in I_n$ Markov inequality (1.1) yields

$$|P(x) - P(x_n)| \le |x - x_n|\frac{n^2}{1-(1/(2n^2))}\|P\|_{\infty,I_n},$$

then for $|x - x_n| \le n^{-2}[1 - 1/(2n^2)]$ we have

$$|P(x)| \ge \left(1 - \frac{n^2}{1-(1/(2n^2))}|x - x_n|\right)\|P\|_{\infty,I_n}. \qquad (4.2)$$

There exists an interval J_n with an end at x_n satisfying $J_n \subset I_n$, length$(J_n) = n^{-2}[1 - 1/(2n^2)]$.

Either $J_n = [x_n, x_n + n^{-2}(1 - 1/(2n^2))]$ or $J_n = [x_n - n^{-2}(1 - 1/(2n^2)), x_n])$. For any $x \in J_n$ inequality (4.2) is satisfied. Then we can write

$$\begin{aligned}\int_{I_n} |P(x)|^p \,\mathrm{d}x &\ge \int_{J_n} |P(x)|^p \,\mathrm{d}x \\ &\ge \|P\|_{\infty,I_n}^{p}\int_{J_n}\left(1 - \frac{n^2}{1-(1/(2n^2))}|x - x_n|\right)^p \mathrm{d}x \\ &= \|P\|_{\infty,I_n}^{p}\int_0^{n^{-2}[1-1/(2n^2)]}\left(1 - \frac{n^2}{1-(1/(2n^2))}t\right)^p \mathrm{d}t \\ &= \frac{1 - 1/(2n^2)}{(p+1)n^2}\|P\|_{\infty,I_n}^{p}.\end{aligned}$$

This yields

$$\|P\|_{\infty,I_n}^{p} \le \frac{(p+1)n^2}{1-(1/(2n^2))}\|P\|_{p,I_n}^{p}$$

and due to inequality (4.1)

$$\int_{1-(1/n^2)}^{1} |P(x)|^p \,\mathrm{d}x \le \frac{(113/49)^p(p+1)}{1-(1/(2n^2))}\|P\|_{p,I_n}^{p}.$$

Then, clearly,

$$\|P\|_p^p \le \left(1 + \frac{(113/49)^p(p+1)}{1-(1/(2n^2))}\right)\|P\|_{p,I_n}^p$$

and since for $x \in I_n$, $1 - x \ge n^{-2}$ we have

$$\int_{-1}^{1-(1/n^2)} |P(x)|^p \, dx \le n^{2p} \int_{-1}^{1-(1/n^2)} (1-x)^p |P(x)|^p \, dx,$$

then

$$\|P\|_p \le n^2 \left(1 + \frac{(113/49)^p(p+1)}{1-(1/(2n^2))}\right)^{1/p} \|(1-x)P\|_p.$$

To complete the proof, we note that

$$\begin{aligned}\left(1 + \frac{(113/49)^p(p+1)}{1-(1/(2n^2))}\right)^{1/p} &\le 1 + \frac{(113/49)(p+1)^{1/p}}{(1-(1/(2n^2)))^{1/p}} \\ &\le 1 + (113/49)[8(p+1)/7]^{1/p} \\ &\le 1 + (113/49)(16/7) = 6.27\ldots\end{aligned}$$

since $[8(p+1)/7]^{1/p}$ is a decreasing function of p.

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