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An Inequality on Solutions of Heat Equation*

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Let $v(x, t)$ be the solution of the initial value problem for the n dimensional heat equation. Then, for any a and for any $t_0 > 0$, an inequality about $v(a, t)$ and $v(x, t_0)$ is obtained.

Keywords: Heat equation; Integral transform; Positive matrix; Reproducing kernel

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1. INTRODUCTION

For a positive integer n, we consider the n dimensional heat equation

$$\begin{cases} \Delta v(x,t) = \partial_t v(x,t), & x \in \mathbb{R}^n \text{ and } t > 0; \\ v(x,0) = F(x), & x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where Δ is the n dimensional Laplacian and F is a member in the space $L^2(\mathbb{R}^n)$ for the Lebesgue measure on $\mathbb{R}^n$. Then, the solution is represented by

$$v(x,t) = \left(\frac{1}{2\sqrt{\pi t}}\right)^n \int_{\mathbb{R}^n} F(\xi) \exp\left\{-\frac{|x-\xi|^2}{4t}\right\} \mathrm{d}\xi. \tag{1.2}$$

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Furthermore, from the expression (1.2) we know that the solution $v(x, t)$ can be holomorphically extended on the n dimensional complex space $\mathbb{C}^n$ with respect to the space variable x. For the time variable t also, $v(x, t)$ can be holomorphically extended on the right half plane $D = \{z \mid \Re z > 0\}$ of the complex plane $\mathbb{C}$. These facts are found in [4,6].

In this paper, for any $a \in \mathbb{R}^n$ and for a fixed time t_0, we derive an inequality which expresses the relation of $v(a, t)$ and $v(x, t_0)$. Our inequality is the generalization of an inequality in [6] for the n dimension.

THEOREM *For an initial values F in $L^2(\mathbb{R}^n)$ let $v(x, t)$ be the solution of the n dimensional heat equation* (1.1). *Then, for any $a \in \mathbb{R}^n$ and for any $t_0 > 0$, the following inequality is valid:*

$$\iint_D |\partial_t v(a, z)|^2 x^{n/2} \, \mathrm{d}x \, \mathrm{d}y \leq C(n, t_0) \iint_{\mathbb{C}^n} |v(w, t_0)|^2 \exp\left(-\frac{|\tau|^2}{2t_0}\right) \mathrm{d}\lambda \, \mathrm{d}\tau, \tag{1.3}$$

where $z = x + \mathrm{i}y$ $(x, y \in \mathbb{R})$, $w = \lambda + \mathrm{i}\tau$ $(\lambda, \tau \in \mathbb{R}^n)$. Moreover, the equality holds if and only if F is a member in $M(n, a)$. Here, $C(n, t_0) = n\Gamma(n/2)/(2^{2n+1}\pi^{n-1}t_0^{n/2})$ and $M(n, a)$ is the closure of the space spanned linearly by

$$\left\{f(\xi) = \mathrm{e}^{-\alpha|\xi - a|^2} \text{ on } \mathbb{R}^n \,\middle|\, \alpha \in D\right\}$$

in $L^2(\mathbb{R})$.

2. SOME HOLOMORPHIC FUNCTION SPACES

We let $K(z, u)$ be the Bergman kernel on the domain D with respect to the measure $\mathrm{d}x \, \mathrm{d}y/\pi$. It is explicitly represented by $K(z, u) = 1/(z + \bar{u})^2$. For any $q \geq 1$, we consider the Hilbert space

$$H_q = \left\{ f\colon \text{holomorphic in } D \,\middle|\, \|f\|^2_{H_q} = \frac{1}{\pi\Gamma(2q - 1)} \iint_D |f(z)|^2 K(z, z)^{1-q} \, \mathrm{d}x \, \mathrm{d}y < \infty, \; z = x + \mathrm{i}y \right\}.$$

Then, the kernel function

$$K_q(z,u) = \Gamma(2q)K(z,u)^q, \quad (z,u) \in D \times D,$$

is the reproducing kernel of H_q in the following sense: for any $z \in D$, $K(\cdot, z)$ is the member in H_q and every member f in H_q is represented by

$$f(z) = \langle f, K_q(\cdot, z)\rangle_{H_q}, \quad z \in D,$$

where $\langle\cdot,\cdot\rangle_{H_q}$ is the inner product in the Hilbert space H_q (refer to [2,3]). Meanwhile, the kernel function K_q can be represented by

$$K_q(z,u) = \int_0^\infty \mathrm{e}^{-\xi z}\mathrm{e}^{-\xi \bar{u}}\xi^{2q-1}\,\mathrm{d}\xi, \quad z, u \in D, \tag{2.1}$$

and the right hand side of (2.1) converges for all $q > 0$. Hence, for any q with $0 < q < 1$, the function K_q also determines the H_q that admits the reproducing kernel $K_q(z,u)$ (see [1,7]). For any $q > 0$, we denote

$$K_q(z,u) = \Gamma(2q)K(z,u)^q, \quad z, u \in D,$$

and we also consider the Hilbert space

$$A_q = \left\{ g\text{: holomorphic in } D \mid \|g\|^2_{A_q} = \frac{1}{\pi\Gamma(2q+1)}\iint_D |g'(z)|^2 K(z,z)^{-q}\,\mathrm{d}x\,\mathrm{d}y < \infty, \ \lim_{x\to\infty} g(x) = 0 \right\}.$$

Since the mapping $f \longmapsto f'$ is the isometry from H_q onto H_{q+1}, $H_q = A_q$, and $K_q(z,u)$ is the reproducing kernel of A_q (see [3]).

3. PROOF OF THEOREM

Following the theory of generalized integral transforms [5], we prove our theorem. First, for $a = 0$, we consider the integral transform

$$\mathcal{H}F(z) = \left(\frac{1}{2\sqrt{\pi z}}\right)^n \int_{\mathbb{R}^n} F(\xi)\exp\left(-\frac{|\xi|^2}{4z}\right)\mathrm{d}\xi = v(0,z), \quad z \in D.$$

For any $t_0 > 0$, we calculate the kernel form

$$\begin{aligned} T_n(z,u) &= \left(\frac{1}{4\pi\sqrt{z\bar{u}}}\right)^n \int_{\mathbb{R}^n} \exp\left(-\frac{\xi^2}{4z}\right)\overline{\exp\left(-\frac{\xi^2}{4u}\right)} \mathrm{d}\xi \\ &= \left(\frac{1}{2\sqrt{\pi}}\right)^n K(z,u)^{n/4}. \end{aligned}$$

Since the function $T_n(z,u)$ is positive matrix on D, it determines the reproducing kernel Hilbert space S_n (see [1,7]). On the other hand, the space S_n is characterized by

$$S_n = \Big\{f\text{: holomorphic in } D \,|\, \|f\|^2_{S_n} = \frac{2^{3n/2+1}\pi^{n/2-1}}{n\Gamma(\frac{n}{2})} \iint_D |f'(z)|^2 x^{n/2}\,\mathrm{d}x\,\mathrm{d}y < \infty\Big\}.$$

Hence we have the norm inequality

$$\|v(0,z)\|^2_{S_n} \le \int_{\mathbb{R}^n} |F(\xi)|^2\,\mathrm{d}\xi. \tag{3.1}$$

For the orthogonal complement $N^\perp$ of the null space

$$N = \bigcap_{z\in D}\{F \in L^2(\mathbb{R}^n) \,|\, \mathcal{H}F(z) = 0\},$$

the equality in (3.1) holds if and only if F is a member in $N^\perp$. In fact, $N^\perp$ is the closure of the space in $L^2(\mathbb{R}^n)$ which is linearly spanned by members of the family

$$\{G(\xi) = \exp(-\alpha|\xi|^2) \,|\, \alpha \in D\},$$

and so $N^\perp = M(n,0)$. From [4], the norm equality

$$\left(\frac{1}{2\pi t_0}\right)^{n/2} \iint_{\mathbb{C}^n} |v(w,t_0)|^2 \exp\left(-\frac{|\tau|^2}{2t_0}\right) \mathrm{d}\lambda\,\mathrm{d}\tau = \int_{\mathbb{R}^n} |F(\xi)|^2\,\mathrm{d}\xi. \tag{3.2}$$

holds, and from (3.1) and (3.2) our inequality is obtained for $a = 0$.

For any $a \in \mathbb{R}^n$, since

$$\begin{aligned} v(a,t) &= \left(\frac{1}{2\sqrt{\pi t}}\right)^n \int_{\mathbb{R}^n} F(\xi)\exp\left(-\frac{|a-\xi|^2}{4t}\right) \mathrm{d}\xi \\ &= \left(\frac{1}{2\sqrt{\pi t}}\right)^n \int_{\mathbb{R}^n} F(\xi+a)\exp\left(-\frac{|\xi|^2}{4t}\right) \mathrm{d}\xi, \end{aligned}$$

we have

$$\|v(a,t)\|^2_{S_n} \leq \int_{\mathbb{R}^n} |F(\xi+a)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^n} |F(\xi)|^2 \mathrm{d}\xi. \tag{3.3}$$

From (3.2) and (3.3), the inequality (1.3) is valid. Meanwhile, the equality in (3.3) holds if and only if $F(\xi + \mathrm{a}) \in M(n, 0)$. Therefore the proof has been completed.

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