

On Some Inequalities and Stability Results Related to the Exponential Function

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Some inequalities related to the exponential function are solved and the stability of the functional equations $f'(x) = f(x)$ and $(f(y) - f(x))/(y - x) = f((x + y)/2)$ is studied.

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One of the most classical characterizations of the real exponential function $f(x) = e^x$ is the fact that the exponential function is the only (modulo a multiplicative constant) nontrivial solution of the differential equation $f' = f$. Our aim in this note is to study the Hyers–Ulam stability of this equation, i.e. to solve for a given $\varepsilon > 0$ the inequality

$$|f'(x) - f(x)| \leq \varepsilon, \quad (1)$$

and to study also the related inequality (for all $x \neq y$)

$$\left| f\left(\frac{x+y}{2}\right) - \frac{f(y) - f(x)}{y-x} \right| \leq \varepsilon. \quad (2)$$

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In dealing with (1) and (2) we will solve several inequalities which have their own interest. In what follows I will stand for any real interval and \mathbb{R}^+ for the set of all nonnegative real numbers. A function f will be termed Jensen concave if f satisfies the inequality $f((x+y)/2) \geq (f(x)+f(y))/2$ and f will be said to be k -lipschitz whenever $|f(x)-f(y)| \leq k|x-y|$ for all x, y in the (convex) domain of f .

We begin the study of (1) with the following

LEMMA 1 *Let $g: I \rightarrow \mathbb{R}$ be a differentiable function. Then:*

- (i) *the inequality $g(x) \leq g'(x)$ holds for all x in I if and only if g can be represented in the form*

$$g(x) = i(x) \cdot e^x, \quad x \in I, \quad (3)$$

where $i: I \rightarrow \mathbb{R}$ is an arbitrary nondecreasing differentiable function;

- (ii) *the inequality $g'(x) \leq g(x)$ holds for all x in I if and only if g admits the representation*

$$g(x) = d(x)e^x, \quad x \in I, \quad (4)$$

where $d: I \rightarrow \mathbb{R}$ is an arbitrary nonincreasing differentiable function.

Proof If $g(x) \leq g'(x)$ $x \in I$, then the function $i: I \rightarrow \mathbb{R}$ defined by the formula $i(x) = g(x)e^{-x}$, $x \in I$, is differentiable and satisfies

$$i'(x) = g'(x)e^{-x} - g(x)e^{-x} = (g'(x) - g(x))e^{-x} \geq 0,$$

for all $x \in I$.

Therefore i is nondecreasing and (4) follows. The converse is immediate. Part (ii) follows from (i) by replacing g in (i) by $-g$.

Now we can solve (1) completely.

THEOREM 1 *Given an $\varepsilon > 0$ let $f: I \rightarrow \mathbb{R}$ be a differentiable function. Then*

$$|f'(x) - f(x)| \leq \varepsilon, \quad x \in I,$$

holds for all x in I if and only if f can be represented in the form

$$f(x) = \varepsilon + e^x \ell(e^{-x}), \quad x \in I, \quad (5)$$

where ℓ is an arbitrary differentiable function defined on the interval $J = \{e^{-x} \mid x \in I\}$, nonincreasing and 2ε -lipschitz.

Proof If (1) holds then

$$f(x) - \varepsilon \leq f'(x) \leq \varepsilon + f(x),$$

for all $x \in I$.

On one hand $g(x) = f(x) - \varepsilon$ will satisfy $g(x) \leq g'(x)$ so by part (i) of Lemma 1 we obtain the representation

$$f(x) - \varepsilon = i(x)e^x, \quad x \in I, \tag{6}$$

with i differentiable and nondecreasing. On the other hand $h(x) = f(x) + \varepsilon$ will satisfy $h'(x) \leq h(x)$ and by part (ii) of Lemma 1 we obtain the representation

$$f(x) + \varepsilon = d(x)e^x, \quad x \in I, \tag{7}$$

with d differentiable and nonincreasing. By (6) and (7) necessarily

$$i(x)e^x + \varepsilon = d(x)e^x - \varepsilon, \quad x \in I, \tag{8}$$

and by differentiation

$$i'(x)e^x + i(x)e^x = d'(x)e^x + e^x d(x) = d'(x)e^x + i(x)e^x + 2\varepsilon, \quad x \in I.$$

This together with the nonpositivity of d' yields

$$d'(x) = \frac{i'(x)e^x - 2\varepsilon}{e^x} = i'(x) - 2\varepsilon e^{-x} \leq 0, \quad x \in I,$$

i.e.

$$0 \leq i'(x) \leq 2\varepsilon e^{-x}, \quad x \in I.$$

If we define $J = \{e^{-x} \mid x \in I\}$ and $\ell: J \rightarrow \mathbb{R}$ by $\ell(z) = i(-\ln z)$, $z \in J$, then ℓ is differentiable and

$$\ell'(z) = -i'(-\ln z)/z \leq 0, \quad z \in J.$$

Therefore ℓ is nonincreasing and, moreover, ℓ is 2ε -lipschitz because for all z_1, z_2 in J , $z_1 \neq z_2$, by the mean value theorem there exists z_3 in $(\min(z_1, z_2), \max(z_1, z_2))$ such that

$$\begin{aligned} |\ell(z_1) - \ell(z_2)| &= |\ell'(z_3)| |z_1 - z_2| = \left| \frac{-i'(-\ln z_3)}{z_3} \right| |z_1 - z_2| \\ &= i'(-\ln z_1) \cdot e^{-\ln z_3} |z_1 - z_2| \leq 2\varepsilon |z_1 - z_2|. \end{aligned}$$

Thus we have the desired representation

$$f(x) = \varepsilon + i(x)e^x = \varepsilon + \ell(e^{-x}) \cdot e^x.$$

It is immediate to prove the converse implication.

Remark In the study of the Hyers–Ulam stability of a functional equation one hopes that if a function “ ε -satisfies” an equation (e.g. $|f'(x) - f(x)| \leq \varepsilon$) then there must exist a constant k such that the function must be $k\varepsilon$ -uniformly close to the general solution of the corresponding functional equation (resp., $|f(x) - e^x| \leq k\varepsilon$). It is quite interesting to note, using Theorem 1, that from $|f'(x) - f(x)| \leq \varepsilon$ one can deduce the existence of a solution $g(x) = ce^x, x \in I$, of the equation $g' = g$ such that

$$|f(x) - g(x)| \leq 3\varepsilon \quad \text{for all } x \in I. \quad (*)$$

Actually, since f has to be of the form (5) with a differentiable nonincreasing and 2ε -lipschitz function $\ell : J = \inf\{e^{-z} : z \in I\} \rightarrow \mathbb{R}$, putting $a := \inf J \in [0, \infty)$ and $c := \lim_{t \rightarrow a^+} \ell(t)$ we see that c must be finite. Now, for every $x \in I$ one has

$$\begin{aligned} |f(x) - ce^x| &\leq \varepsilon + e^x |\ell(e^{-x}) - c| \leq \varepsilon + e^x \cdot 2\varepsilon |e^{-x} - a| \\ &= \varepsilon(1 + 2|1 - ae^x|). \end{aligned}$$

Therefore, since $b = \infty$ implies $a = 0$ we get (*) in that case whereas the finiteness of b gives $a = e^{-b}$ and, consequently,

$$|f(x) - ce^x| \leq \varepsilon(1 + 2|1 - e^{x-b}|) = \varepsilon(3 - 2e^{x-b})$$

for all $x \in I$ which leads to (*) as well.

Now we turn our considerations to (2). First we observe that for the class of differentiable functions, (2) reduces to (1) so Theorem 1 describes the general differentiable solutions of (2). Since it is well-known that the only differentiable solution of the functional equation

$$\frac{f(y) - f(x)}{y - x} = f\left(\frac{x + y}{2}\right) \quad \text{for } x \neq y, \tag{9}$$

is the zero function, we can conclude also that if (9) were stable in the sense of Hyers and Ulam then each differentiable solution of (2) would have to be bounded jointly with its derivative.

Let us consider now some special classes of functions (see e.g. [2,4,5]) satisfying weaker conditions than the equality (9).

LEMMA 2 *For every real number $x \neq y$ the exponential function satisfies the inequalities*

$$e^{(x+y)/2} \leq \frac{e^y - e^x}{y - x} \leq \frac{e^x + e^y}{2}. \tag{10}$$

Proof Using the power series expansion for the exponential function and the fact that $2 \leq n + 1 \leq 2^n$, for all positive integers n , it is immediate to show that $e^{t/2} \leq (e^t - 1)/t \leq (e^t + 1)/2$, for all $t > 0$, and (10) follows.

LEMMA 3 *A function $f: I \rightarrow \mathbb{R}^+$ satisfies the inequality*

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(y) - f(x)}{y - x}, \tag{11}$$

for all $x \neq y$ in I , if and only if f can be represented in the form $f(x) = i(x)e^x$ where $i: I \rightarrow \mathbb{R}^+$ is an arbitrary nondecreasing function.

Proof If f satisfies (11) then for any x in I and $h > 0$ such that $x + h \in I$ we have

$$f(x + h) \geq f(x) + hf\left(x + \frac{h}{2}\right) \geq f(x) + hf(x) = (1 + h)f(x),$$

because, clearly, f has to be nondecreasing.

By an obvious induction we get

$$f(x + ih) \geq (1 + h)^i f(x)$$

whenever $x + ih \in I$ and $i \in \mathbb{N}$. Thus, for an arbitrarily fixed $n \in \mathbb{N}$, for every $x < y$ from I , one eventually obtains

$$f(y) = f\left(x + n\frac{y-x}{n}\right) \geq \left(1 + \frac{y-x}{n}\right)^n f(x),$$

and if we let n tend to infinity $f(y) \geq e^{y-x}f(x)$, i.e., the function $i: I \rightarrow \mathbb{R}^+$ defined by $i(x) = f(x)e^{-x}$ is nondecreasing. Conversely, if we have the representation $f(x) = i(x)e^x$, $x \in I$, with $i: I \rightarrow \mathbb{R}^+$ nondecreasing then, since for $x < y$ we have $i(x) \leq i(x+y/2) \leq i(y)$, we can deduce that

$$i(y)e^{y-x} - i\left(\frac{x+y}{2}\right)e^{y-x} \geq 0 \geq i(x) - i\left(\frac{x+y}{2}\right),$$

that is,

$$i(y)e^{y-x} - i(x) \geq i\left(\frac{x+y}{2}\right)(e^{y-x} - 1),$$

and multiplying both terms by $e^x/(y-x)$ with the aid of Lemma 2 we have

$$\begin{aligned} \frac{f(y) - f(x)}{y-x} &= \frac{i(y)e^y - i(x)e^x}{y-x} \geq i\left(\frac{x+y}{2}\right) \frac{e^y - e^x}{y-x} \\ &\geq i\left(\frac{x+y}{2}\right) e^{(x+y)/2} = f\left(\frac{x+y}{2}\right), \end{aligned}$$

i.e. (11) follows. Moreover, f is nonnegative because so is i .

THEOREM 2 *Given an $\varepsilon > 0$ let $f: I \rightarrow \mathbb{R}^+$ be a function such that $f(x) \geq \varepsilon$ for all x in I . Then f satisfies the inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(y) - f(x)}{y-x} + \varepsilon, \quad (12)$$

for all $x < y$ in I , if and only if f can be represented in the form $f(x) = \varepsilon + i(x)e^x$, $x \in I$, where $i: I \rightarrow \mathbb{R}^+$ is a nondecreasing function.

Proof By (12) we can apply the previous lemma to the function $f - \varepsilon$.

LEMMA 4 *If a function $f: I \rightarrow \mathbb{R}^+$ is nondecreasing and satisfies the inequality*

$$\frac{f(y) - f(x)}{y - x} \leq f\left(\frac{x + y}{2}\right) \tag{13}$$

for all $x < y$ in I , then f can be represented in the form $f(x) = d(x)e^x$, $x \in I$, with a nonincreasing function $d: I \rightarrow \mathbb{R}^+$.

Proof If f is a nondecreasing solution of (13) then for any x in I and $h \in (0, 1)$ such that $x + h$ is in I we have

$$0 \leq f(x + h) - f(x) \leq hf\left(x + \frac{h}{2}\right) \leq hf(x + h),$$

i.e., $f(x + h) \leq f(x)/(1 - h)$. Thus if $x < y$ in I , for sufficiently large $n \in \mathbb{N}$ we have $(y - x)/n \in (0, 1)$ and one obtains

$$f(y) = f\left(x + (y - x)\right) = f\left(x + n\frac{y - x}{n}\right) \leq f(x) / \left(1 - \frac{y - x}{n}\right)^n,$$

whence by letting n tend to infinity $f(y) \leq f(x)e^{y-x}$, i.e., $d(x) = f(x) \cdot e^{-x}$, $x \in I$, is nonincreasing.

LEMMA 5 *A nondecreasing Jensen concave function $f: I \rightarrow \mathbb{R}^+$ satisfies (13) if and only if there exists a nonincreasing function $d: I \rightarrow \mathbb{R}^+$ such that $I \ni x \mapsto d(x)e^x$ is concave and $f(x) = d(x)e^x$, $x \in I$.*

Proof Necessity follows from Lemma 4. To prove sufficiency assume that $f(x) = d(x)e^x$, $x \in I$, is Jensen concave and nondecreasing with d nonincreasing. If $x < y$ in I then $d(x) = f(x)e^{-x} \geq f(y)e^{-y} = d(y)$, i.e., $f(x) \geq f(y)e^{x-y}$ and therefore since all functions are positive and we can apply Lemma 2 obtaining

$$\begin{aligned} \frac{f(y) - f(x)}{y - x} &\leq \frac{f(y) - f(y)e^{y-x}}{y - x} = \frac{f(y)}{e^y} \cdot \frac{e^y - e^x}{y - x} \leq \frac{f(y)}{e^y} \frac{e^x + e^y}{2} \\ &= \frac{f(y)e^{x-y} + f(y)}{2} \leq \frac{f(x) + f(y)}{2} \leq f\left(\frac{x + y}{2}\right), \end{aligned}$$

which states that (13) holds.

Remark Note that a monotonic (and hence measurable) Jensen concave function $f: I \rightarrow \mathbb{R}$ has to be necessarily concave in the usual sense, i.e. to satisfy the inequality

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and all $\lambda \in [0, 1]$ (see e.g. [3]).

THEOREM 3 Given an $\varepsilon > 0$, a nondecreasing Jensen concave function $f: I \rightarrow \mathbb{R}$ satisfying $f(x) \geq -\varepsilon$ for all $x \in I$, is a solution of the inequality

$$\frac{f(y) - f(x)}{y - x} - \varepsilon \leq f\left(\frac{x + y}{2}\right) \quad (14)$$

if, and only if, $f(x) = d(x)e^x - \varepsilon$ where $d: I \rightarrow \mathbb{R}^+$ is nonincreasing and $I \ni x \mapsto d(x)e^x$ is Jensen concave.

Proof Apply Lemma 5 to the function $f + \varepsilon$.

Thus given $\varepsilon > 0$ for the class of functions $f: I \rightarrow \mathbb{R}$ such that $f(x) \geq \varepsilon$ for all x in I and f is nondecreasing and Jensen concave, by combining Theorems 2 and 3 it follows a representation for the solutions of the inequality (2). To find solutions of (2) in a wider class of functions is an open problem.

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