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On the Comparison of Trigonometric Convolution Operators with their Discrete Analogues for Riemann Integrable Functions

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This note is concerned with a comparison of the approximation-theoretical behaviour of trigonometric convolution processes and their discrete analogues. To be more specific, for continuous functions it is a well-known fact that under suitable conditions the relevant uniform errors are indeed equivalent, apart from constants. It is the purpose of this note to extend the matter to the frame of Riemann integrable functions. To establish the comparison for the corresponding Riemann errors, essential use is made of appropriate stability inequalities.

Keywords: Trigonometric convolution operators; Discrete analogues; Comparison; Stability inequalities

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1 INTRODUCTION AND RESULTS

Let $C_{2\pi}$ be the Banach space of 2π-periodic functions f, continuous on the real axis $\mathbb{R}$, endowed with the usual sup-norm $\|f\|_C := \sup\{|f(u)| : u \in \mathbb{R}\}$. In connection with bounded linear operators T from $C_{2\pi}$ into itself, i.e., $T \in [C_{2\pi}]$, we use the notation $\|T\|_{[C]} := \sup\{\|Tf\|_C : \|f\|_C \leq 1\}$. For $n \in \mathbb{N}$, the set of natural numbers, let Π_n be

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the set of all trigonometric polynomials of degree at most n. Given a positive, even, polynomial kernel $(\chi_n)_{n=1}^{\infty}$ of the form

$$\chi_n(x) := \sum_{k=-n}^{n} \rho_{k,n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \rho_{k,n} \cos kx, \tag{1.1}$$

thus $0 \leq \chi_n(x) \in \Pi_n$ with $\rho_{-k,n} = \rho_{k,n}$ and $\rho_{0,n} = 1$, consider the (positive trigonometric) convolution operator

$$T_n f(x) := (f * \chi_n)(x) := \frac{1}{2\pi} \int_0^{2\pi} f(u)\chi_n(x-u)\,\mathrm{d}u. \tag{1.2}$$

Obviously, $T_n \in [C_{2\pi}]$; in fact $\|T_n\|_{[C]} = \|\chi_n\|_I := (1/2\pi)\int_0^{2\pi} |\chi_n(u)|\,\mathrm{d}u$, and therefore $\|T_n\|_{[C]} = 1$ in view of the positivity of the kernel. With the aid of the equidistant knots $u_{j,n} := 2\pi j/(2n+1)$, a discrete analogue of T_n may be introduced via the finite sum

$$J_n f(x) := \frac{1}{2n+1} \sum_{j=0}^{2n} f(u_{j,n})\chi_n(x - u_{j,n}), \tag{1.3}$$

which may be interpreted either as a quadrature formula for the integral (1.2) (mid-point or trapezoidal rule) or as a linear mean of the Lagrange interpolation polynomial associated with $\{u_{j,n}\}$ (cf. [7, p. 413]). As an immediate consequence of these definitions we note that for each $n \in \mathbb{N}$ one has $T_n p = J_n p$ for all $p \in \Pi_n$ (cf. [10, p. 8]). Therefore $\|J_n\|_{[C]} = 1$ as well, and there hold true the Berman conditions ($n \in \mathbb{N}$)

$$T_n J_n = J_n J_n, \qquad J_n T_n = T_n T_n. \tag{1.4}$$

Moreover, using the first part of (1.4), for example, it follows that

$$\begin{aligned}\|T_n f - f\|_C &\leq \|T_n f - T_n J_n f\|_C + \|J_n J_n f - J_n f\|_C + \|J_n f - f\|_C \\ &\leq \|J_n f - f\|_C \Big[\|T_n\|_{[C]} + \|J_n\|_{[C]} + 1\Big].\end{aligned}$$

Consequently, for all $f \in C_{2\pi}$ and $n \in \mathbb{N}$ one has the comparison assertion (see [1])

$$\frac{1}{3}\|T_n f - f\|_C \leq \|J_n f - f\|_C \leq 3\|T_n f - f\|_C. \tag{1.5}$$

In other words, apart from a constant, one may immediately pass from one (uniform) error to the other.

Dealing with the discrete operators (1.3), it is not meaningful to look for extensions of (1.5) to Lebesgue spaces $L^p_{2\pi}$ since point evaluation functionals are not bounded in connection with L^p-metrics. On the other hand (cf. [10, p. 5]), a reasonable candidate for an extension of (1.5) is given by the linear space $R_{2\pi}$ of 2π-periodic functions, Riemann integrable over $[0, 2\pi]$. Indeed, inspired by the work of Pólya [8], a notion of a convergence for sequences of elements was introduced in $R_{2\pi}$ (see [4]) such that $R_{2\pi}$ is not only (sequentially) complete, but trigonometric polynomials are in fact dense in $R_{2\pi}$. As a consequence of these considerations it also turned out that it is appropriate to measure errors in $R_{2\pi}$ via the functionals

$$\left\| \sup_{k\geq n} |T_k f - f| \right\|_{\delta_n}, \qquad \left\| \sup_{k\geq n} |J_k f - f| \right\|_{\delta_n} \tag{1.6}$$

for some suitable positive nullsequence (δ_n). Here the δ-norms are given for $f \in B_{2\pi}$, $\delta > 0$ by (cf. [3,9])

$$\begin{aligned} \|f\|_\delta &:= \overline{\int_0^{2\pi}} M(f, x, \delta)\,\mathrm{d}x, \\ M(f, x, \delta) &:= \sup\{|f(u)| : u \in U_\delta(x) := [x - \delta, x + \delta]\}, \end{aligned} \tag{1.7}$$

where $\overline{\int} f := \overline{\int_0^{2\pi}} f(u)\,\mathrm{d}u$ denotes the upper Riemann integral of $f \in B_{2\pi}$, the Banach space of functions, 2π-periodic and bounded on $\mathbb{R}$. Clearly, $C_{2\pi} \subset R_{2\pi} \subset B_{2\pi}$ and $\overline{\int} |f| = 2\pi \|f\|_I$ if $f \in R_{2\pi}$. Moreover, note that $f \in R_{2\pi}$ in turn implies $M(f, x, \delta) \in R_{2\pi}$ (cf. [5]). On the other hand, dealing with upper Riemann integrals of bounded functions ensures that the error functionals (1.6) are indeed well-defined.

In these terms one has the following counterpart to (1.5) for Riemann integrable functions.

THEOREM 1.1 *Let the convolution process $(T_n)_{n=1}^\infty$ and its discrete analogue $(J_n)_{n=1}^\infty$ be given via ((1.1)–(1.3)) and suppose that the kernel (χ_n) additionally satisfies conditions ((2.1)–(2.2)) for some nullsequence (δ_n) with $1/n = \mathcal{O}(\delta_n)$. Then for $f \in R_{2\pi}$ there holds true the comparison assertion*

$$c_1 \left\| \sup_{k\geq n} |T_k f - f| \right\|_{\delta_n} \leq \left\| \sup_{k\geq n} |J_k f - f| \right\|_{\delta_n} \leq c_2 \left\| \sup_{k\geq n} |T_k f - f| \right\|_{\delta_n},$$

the constants $0 < c_1$, $c_2 < \infty$ being independent of $f \in R_{2\pi}$ and $n \in \mathbb{N}$.

Indeed, since $J_k p = T_k p$ for every $p \in \Pi_k$ and since $J_n f, T_n f \in \Pi_n$ for all $f \in R_{2\pi}$, one has, for example, that for $k, n \in \mathbb{N}$ with $k \geq n$ (cf. (1.4))

$$J_k f - f = J_k(f - T_n f) + T_k(T_n f - f) + T_k f - f,$$

and therefore the proof of Theorem 1.1 is an immediate consequence of the stability inequalities, established in the next section.

2 STABILITY INEQUALITIES

For the convolution process (T_n) there are several possibilities to prove stability in connection with the Riemann errors (1.6). Let us just mention the following result, actually valid for functions $f \in B_{2\pi}$ which are additionally Lebesgue measurable, i.e., for $f \in BM_{2\pi}$ (note that $R_{2\pi} \subset BM_{2\pi}$).

THEOREM 2.1 *Let a positive, even, polynomial kernel* $(\chi_n)_{n=1}^{\infty}$ *be given via* (1.1). *Suppose there exist some* $\beta > 0$ *and a monotonely decreasing sequence* $(c_n)_{n=1}^{\infty}$ *of positive constants such that*

$$\chi_n(u) \leq c_n u^{-\beta} \quad \textit{for } n \in \mathbb{N} \textit{ and } u \in [\delta_n, \pi], \tag{2.1}$$

where $(\delta_n)_{n=1}^{\infty}$ *with* $0 < \delta_n \leq \pi$ *is a further monotonely decreasing null-sequence satisfying*

$$c_n \int_{\delta_n}^{\pi} u^{-\beta}\, \mathrm{d}u \leq M \quad \textit{for } n \in \mathbb{N}. \tag{2.2}$$

Then for the convolution process (1.2) *and* $f \in BM_{2\pi}$ *there holds true the stability inequality*

$$\left\| \sup_{k \geq n} |T_k f| \right\|_{\delta_n} \leq c\|f\|_{\delta_n}, \tag{2.3}$$

the constant $c < \infty$ *being independent of* $f \in BM_{2\pi}$ *and* $n \in \mathbb{N}$.

Proof First let us recall the elementary fact (cf. [5]) that for $f \in B_{2\pi}$, $\delta > 0$ and $m \in \mathbb{N}$ or $\lambda > 0$

$$\|f\|_{m\delta} \leq m\|f\|_{\delta}, \qquad \|f\|_{\lambda\delta} \leq (1 + \lambda)\|f\|_{\delta}, \tag{2.4}$$

respectively. Moreover (cf. [6]), for $f \in BM_{2\pi}$, $g \in R_{2\pi}$ and $\delta > 0$

$$\|f * g\|_{\delta} \leq \|f\|_{\delta}\|g\|_{I}. \tag{2.5}$$

Since the kernel is even, one has ($k \geq n$)

$$T_k f(x) = \frac{1}{2\pi}\left(\int_0^{\delta_n} + \int_{\delta_n}^{\pi}\right)[f(x+u) + f(x-u)]\chi_k(u)\,\mathrm{d}u$$

and therefore in view of (2.1)

$$|T_k f(x)| \leq M(f, x, \delta_n) + \frac{c_k}{2\pi}\int_{\delta_n}^{\pi}[|f(x+u)| + |f(x-u)|]u^{-\beta}\,\mathrm{d}u.$$

Because of $c_k \leq c_n$ for $k \geq n$ one obtains by (2.5) that

$$\left\| \sup_{k\geq n} |T_k f| \right\|_{\delta_n} \leq \|M(f, x, \delta_n)\|_{\delta_n}$$
$$+ c_n 2\|f\|_{\delta_n}\int_{\delta_n}^{\pi} u^{-\beta}\,\mathrm{d}u \leq 2\|f\|_{\delta_n} + 2M\|f\|_{\delta_n},$$

where we have used (2.2) and (2.4).

As a first application, actually needed for the following, let us consider the Fejér means (cf. (1.2))

$$\begin{aligned}\sigma_n f(x) &:= (f * F_n)(x) := \frac{1}{2\pi}\int_0^{2\pi} f(u)F_n(x-u)\,\mathrm{d}u,\\ F_n(u) &:= \frac{1}{n+1}\left[\frac{\sin(n+1)u/2}{\sin u/2}\right]^2 = 1 + 2\sum_{j=1}^{n}\left(1 - \frac{j}{n+1}\right)\cos ju.\end{aligned} \tag{2.6}$$

Since (cf. [2, p. 51])

$$F_n(u) \leq \frac{\pi^2}{n+1}u^{-2} \quad \text{for } n \in \mathbb{N} \text{ and } u \in \left[\frac{1}{n}, \pi\right],$$

the Fejér kernel satisfies all the assumptions of Theorem 2.1 with $\beta = 2$, $c_n = \pi^2/(n+1)$ and $\delta_n = 1/n$, and hence (cf. [6])

COROLLARY 2.1 *For the Fejér means of $f \in BM_{2\pi}$ there holds true the stability inequality*

$$\left\| \sup_{k\geq n} |\sigma_k f| \right\|_{1/n} \leq c\|f\|_{1/n},$$

the constant $c < \infty$ being independent of $f \in BM_{2\pi}$ and $n \in \mathbb{N}$.

Turning to the discrete operators (1.3) it will in fact be shown that the discrete process (J_n) will always be stable, provided the convolution process (T_n) shares this property.

THEOREM 2.2 *For processes* $(T_n)_{n=1}^{\infty}, (J_n)_{n=1}^{\infty}$ *given via* ((1.1)–(1.3)) *there exists a constant* $c<\infty$, *independent of* $f\in R_{2\pi}$, $n\in\mathbb{N}$, $\delta>0$, *such that*

$$\left\|\sup_{k\geq n}|J_k f|\right\|_{\delta}\leq c\left\|\sup_{k\geq n}T_k\left[M\left(f,\cdot,\frac{1}{k}\right)\right]\right\|_{\delta+1/n}.$$

Proof Without loss of generality one may assume $k\geq n\geq 2$. Since $f\in R_{2\pi}$, $\chi_k\in\Pi_k\subset R_{2\pi}$, one has $f(u)\chi_k(x-u)\in R_{2\pi}$ as a function of u and thus $M(f(\cdot)\chi_k(x-\cdot),t,\delta)\in R_{2\pi}$ as a function of t for each $x\in\mathbb{R}$, $\delta>0$. In view of the definitions (1.3) and (1.7) it follows that for each $x\in\mathbb{R}$

$$\begin{aligned}
|J_k f(x)| &\leq \sum_{j=0}^{2k}\frac{1}{2\pi}\int_{u_{j,k}}^{u_{j+1},k}|f(u_{j,k})|\chi_k(x-u_{j,k})\,\mathrm{d}t\\
&\leq \frac{1}{2\pi}\int_0^{2\pi}M\left(f(\cdot)\chi_k(x-\cdot),t,\frac{2\pi}{2k+1}\right)\mathrm{d}t\\
&\leq \frac{7}{2\pi}\int_0^{2\pi}M\left(f(\cdot)\chi_k(x-\cdot),t,\frac{1}{2k}\right)\mathrm{d}t,
\end{aligned}$$

the latter because of $2\pi/(2k+1)\leq 7(1/2k)$ and (2.4). Using the fact that for $\delta>0$

$$\sup_{u\in U_\delta(t)}|f(u)\chi_k(x-u)|\leq\sup_{u\in U_\delta(t)}|f(u)|\cdot\sup_{z\in U_\delta(x-t)}\chi_k(z), \tag{2.7}$$

the previous estimate may be continued to

$$\begin{aligned}
|J_k f(x)| &\leq \frac{7}{2\pi}\int_0^{2\pi}M\left(f,t,\frac{1}{2k}\right)M\left(\chi_k,x-t,\frac{1}{2k}\right)\mathrm{d}t\\
&= \frac{7}{2\pi}\int_0^{2\pi}M\left(f,x-v,\frac{1}{2k}\right)M\left(\chi_k,v,\frac{1}{2k}\right)\mathrm{d}v,
\end{aligned}$$

employing the 2π-periodicity of the expressions involved. Obviously, since $0\leq\chi_k\in\Pi_k$, one obtains (cf. (1.7))

$$\begin{aligned}
M\left(\chi_k,v,\frac{1}{2k}\right) &\leq \chi_k(v)+\sup_{s,t\in U_{1/2k}(v)}|\chi_k(s)-\chi_k(t)|\\
&\leq \chi_k(v)+\int_{-1/2k}^{1/2k}|\chi_k'(v+r)|\,\mathrm{d}r,
\end{aligned}$$

and therefore (cf. (1.2))

$$
\begin{aligned}
|J_k f(x)| &\le \frac{7}{2\pi}\int_0^{2\pi} M\left(f, x-v, \frac{1}{k}\right)\chi_k(v)\,\mathrm{d}v \\
&\quad + \frac{7}{2\pi}\int_0^{2\pi} M\left(f, x-v, \frac{1}{2k}\right)\left[\int_{-1/2k}^{1/2k} |\chi_k'(v+r)|\,\mathrm{d}r\right]\mathrm{d}v \\
&=: 7\left(T_k\left[M\left(f, \cdot, \frac{1}{k}\right)\right](x) + A_k f(x)\right),
\end{aligned} \tag{2.8}
$$

say. Concerning $A_k f(x)$, to apply a Fubini-type theorem for Riemann integrable functions, let us just mention that $g \in R_{2\pi}$ indeed implies $g(y-t)$ to be Riemann integrable with regard to (y, t) over $[0, 2\pi]^2$ (this may be shown via the Lebesgue characterization of the set of points of discontinuity of a Riemann integrable function). Hence the interchange of the order of integration is justified, delivering

$$
\begin{aligned}
A_k f(x) &= \frac{1}{2\pi}\int_{-1/2k}^{1/2k}\left[\int_0^{2\pi} M\left(f, x-v, \frac{1}{2k}\right)|\chi_k'(v+r)|\,\mathrm{d}v\right]\mathrm{d}r \\
&= \frac{1}{2\pi}\int_{-1/2k}^{1/2k}\left[\int_0^{2\pi} M\left(f, x-u+r, \frac{1}{2k}\right)|\chi_k'(u)|\,\mathrm{d}u\right]\mathrm{d}r,
\end{aligned}
$$

again using the 2π-periodicity of the expressions involved. But for $|r| \le 1/2k$

$$
M\left(f, x-u+r, \frac{1}{2k}\right) \le M\left(f, x-u, \frac{1}{2k}+|r|\right) \le M\left(f, x-u, \frac{1}{k}\right)
$$

and thus

$$
A_k f(x) \le \frac{1}{k}\frac{1}{2\pi}\int_0^{2\pi} M\left(f, x-u, \frac{1}{k}\right)|\chi_k'(u)|\,\mathrm{d}u.
$$

Now let us employ the representation (cf. [2, p. 99])

$$
\chi_k'(u) = \frac{1}{2\pi}\int_0^{2\pi} \chi_k(u+v)F_{k-1}(v)2k\sin kv\,\mathrm{d}v \tag{2.9}
$$

of the derivative of the polynomial χ_k. It may be mentioned that (2.9) was used by F. Riesz (1914) for his elegant proof of Bernstein's inequality. Since the polynomial χ_k is positive, one has (cf. (1.2), (2.6))

$$\begin{aligned} A_k f(x) &\leq \frac{1}{\pi}\int_0^{2\pi} M\left(f, x-u, \frac{1}{k}\right)\left[\frac{1}{2\pi}\int_0^{2\pi} \chi_k(u-v)F_{k-1}(v)\,\mathrm{d}v\right]\mathrm{d}u \\ &= 2\left(M\left(f, \cdot, \frac{1}{k}\right) * \chi_k * F_{k-1}\right)(x). \end{aligned}$$

Again by Fubini's theorem (within the frame of Riemann integration) it is well-known that these convolutions are commutative. Moreover, since T_k and σ_k are positive linear operators, they are monotone. Hence, setting $S_k\ f(x) := T_k[M(f, \cdot, 1/k)](x)$, it follows that $\|S_k f\|_C \leq \sup\{|f(x)|\}$ and (cf. (1.2), (2.6))

$$A_k f(x) \leq 2\sigma_{k-1}[S_k f](x) \leq 2\sigma_{k-1}\left[\sup_{j\geq n} S_j f\right](x).$$

In view of Corollary 2.1 this implies that for $k\geq n\geq 2$ and every $\delta > 0$ (cf. (1.7), (2.7))

$$\begin{aligned} \left\|\sup_{k\geq n} A_k f\right\|_\delta &\leq 2\overline{\int_0^{2\pi}} M\left(\sup_{k\geq n-1}\sigma_k\left[\sup_{j\geq n} S_j f\right], x, \delta\right)\mathrm{d}x \\ &\leq 2\overline{\int_0^{2\pi}}\left(\sup_{k\geq n-1}\sigma_k\left[M\left(\sup_{j\geq n} S_j f, \cdot, \delta\right)\right](x)\right)\mathrm{d}x \\ &\leq 2\left\|\sup_{k\geq n-1}\sigma_k\left[M\left(\sup_{j\geq n} S_j f, \cdot, \delta\right)\right]\right\|_{1/(n-1)} \\ &\leq 2c\left\|M\left(\sup_{j\geq n} S_j f, x, \delta\right)\right\|_{1/(n-1)} = 2c\left\|\sup_{j\geq n} S_j f\right\|_{\delta+1/(n-1)}, \end{aligned}$$

where we have used the fact that $\sup_{j\geq n} S_j f \in BM_{2\pi}$ and therefore $M(\sup_{j\geq n} S_j f, x, \delta) \in BM_{2\pi}$ for each $f \in R_{2\pi}$ (cf. [6]). Thus in view of (2.8) we have obtained

$$\left\|\sup_{k\geq n}|J_k f|\right\|_\delta \leq 7\left\|\sup_{k\geq n} S_k f\right\|_\delta + 14c\left\|\sup_{j\geq n} S_j f\right\|_{\delta+1/(n-1)},$$

and a further application of (2.4) establishes the assertion.

THEOREM 2.3 *Let the processes* $(T_n)_{n=1}^{\infty}, (J_n)_{n=1}^{\infty}$ *be given via* ((1.1)–(1.3)) *and suppose the convolution process* (T_n) *to be stable in the sense* (*cf.* (2.3)) *that for* $f \in R_{2\pi}$, $n \in \mathbb{N}$ *and a monotonely decreasing nullsequence* (δ_n) *with* $1/n = \mathcal{O}(\delta_n)$

$$\left\| \sup_{k \geq n} |T_k f| \right\|_{\delta_n} \leq c\|f\|_{\delta_n}. \tag{2.10}$$

Then the discrete analogue (1.3) *also satisfies the stability inequality*

$$\left\| \sup_{k \geq n} |J_k f| \right\|_{\delta_n} \leq c\|f\|_{\delta_n},$$

the constant $c < \infty$ *being independent of* $f \in R_{2\pi}$, $n \in \mathbb{N}$.

Proof By assumption there exists a constant $M < \infty$ such that $1/n \leq M\delta_n$. Therefore by Theorem 2.2 and by (2.4), (2.10)

$$\begin{aligned}
\left\| \sup_{k \geq n} |J_k f| \right\|_{\delta_n} &\leq c \left\| \sup_{k \geq n} T_k \left[M\left(f, \cdot, \frac{1}{k}\right) \right] \right\|_{\delta_n + M\delta_n} \\
&\leq c(2 + M) \left\| \sup_{k \geq n} T_k \left[M\left(f, \cdot, \frac{1}{n}\right) \right] \right\|_{\delta_n} \\
&\leq c^* \left\| M\left(f, \cdot, \frac{1}{n}\right) \right\|_{\delta_n} = c^* \|f\|_{\delta_n + \frac{1}{n}} \leq c\|f\|_{\delta_n}.
\end{aligned}$$

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