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Some Remarks about Poincaré Type Inequalities and Representation Formulas in Metric Spaces of Homogeneous Type*

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We derive an integral representation formula for a function in terms of its vector field gradient, assuming a less restrictive growth condition on the volumes of balls than was previously known. We give the explicit form of the constants involved in the formula. We also show that the required growth condition is satisfied by a large class of Carnot–Carathéodory vector fields.

Keywords: Poincaré inequality; Representation formulas; Carnot–Carathéodory metrics; Riemannian manifolds

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In a previous joint paper with Lu [FLW], the authors proved the equivalence between suitable forms of the L^1-Poincaré inequality in metric spaces of homogeneous type and the representation formula for a function with zero average in a ball in terms of a singular integral (of potential type) of its 'gradient'. In this note we shall show that the assumptions of [FLW] can be considerably relaxed, by dropping a restrictive growth condition on the volume of metric balls. In this way,

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the statement is sharp for the fundamental example of the Carnot–Carathéodory metric associated with a family of Lipschitz continuous vector fields. In addition, in the present note we shall give explicit dependence of the constants appearing in our inequalities, so that the result is more convenient for Riemannian manifolds.

From now on, $(\mathcal{S}, \rho, m)$ will denote a quasimetric space with quasimetric ρ, endowed with a measure m, and we will denote by K the quasimetric constant of $(\mathcal{S}, \rho)$, i.e., for all $x, y, z \in \mathcal{S}$,

$$\rho(x,y) \leq K[\rho(x,z) + \rho(z,y)]. \tag{1}$$

Moreover, we shall assume that the measure m is *locally doubling*, i.e., that for all $R \in (0, R_0)$ ($R_0 \leq \infty$) there exists $A(R) \geq 1$ such that

$$m(B(x,2r)) \leq A(R)m(B(x,r)) \tag{2}$$

for $x \in \mathcal{S}, r \in (0, R]$, where, by definition, $B(x,r) = \{y \in \mathcal{S}$ such that $\rho(x,y) < r\}$, and $m(B(x,r))$ denotes the m-measure of $B(x,r)$. As usual, we refer to $B(x,r)$ as the ball with center x and radius r, and if B is a ball, we write $r(B)$ for its radius and cB for the ball of radius $cr(B)$ having the same center as B. We shall call *doubling constant of m* (*at the radius* $R < R_0$) the real number

$$\sup\left\{\frac{m(B(x,2r))}{m(B(x,r))}, r \leq R, x \in \mathcal{S}\right\}.$$

For the sake of simplicity, we still denote this constant by $A(R)$.

We shall say that $(\mathcal{S}, \rho)$ has *the segment property* if for each pair of points $x, y \in \mathcal{S}$, there exists a continuous curve γ connecting x and y such that $\rho(\gamma(t), \gamma(s)) = |t-s|$.

THEOREM 1 *Let* $(\mathcal{S}, \rho, m)$ *be a complete quasimetric space satisfying* (1) *and* (2) *such that* $(\mathcal{S}, \rho)$ *has the segment property. Let* μ, ν *be locally doubling measures on* $(\mathcal{S}, \rho, m)$ *with doubling constants* $A_\mu(R)$ *and* $A_\nu(R)$, *respectively. Let* $B_0 = B(x_{B_0}, r_0)$ *be a ball, let* $\tau > 1$ *be a fixed constant and let* $f, g \in L^1(\tau K B_0)$ *be given functions. Assume there exists* $P(r_0) > 0$ *such that, for all balls* $B \subseteq \tau K B_0$,

$$\frac{1}{\nu(B)} \int_B |f - f_{B,\nu}| \, \mathrm{d}\nu \leq P(r_0) \frac{r(B)}{\mu(B)} \int_B |g| \, \mathrm{d}\mu, \tag{3}$$

where $f_{B,\nu} = 1/\nu(B)\int_B f\,\mathrm{d}\nu = ⨍_B f\,\mathrm{d}\nu$. *If there is a constant* $\theta(r_0) > 0$ *such that for all balls* $B, \tilde{B}$ *with* $\tilde{B} \subseteq B \subseteq \tau K B_0$,

$$\frac{\mu(B)}{\mu(\tilde{B})} \geq \theta(r_0)\frac{r(B)}{r(\tilde{B})}, \tag{4}$$

then for $(\mathrm{d}\nu)$*-a.e.* $x \in B_0$,

$$|f(x) - f_{B_0,\nu}| \leq C\int_{\tau K B_0} |g(y)| \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,\mathrm{d}\mu(y) \tag{5}$$

with

$$C = C(K,\tau)\frac{P(r_0)}{\theta(r_0)}\left[A_\nu((\tau K^2 + K)r_0)A_\mu((\tau K^2 + K)r_0)\right]^{\eta(\tau,K)}.$$

Aside from the explicit form of the constants, the difference between this result and the corresponding result in [FLW] is that condition (4) above is replaced in [FLW] by the stronger reverse doubling condition

$$\frac{\mu(B)}{\mu(\tilde{B})} \geq c\left(\frac{r(B)}{r(\tilde{B})}\right)^{1+\epsilon} \quad \text{if } \tilde{B} \subseteq B \subseteq \tau K B_0$$

for some $\epsilon > 0$, but the segment property is not assumed in [FLW]. The segment property, however, automatically holds in many spaces, in particular in those for which the metric is induced by a collection of Carnot–Carathéodory vector fields: see Remark 3 later in this paper.

Remark 1 (i) As we shall see in the proof of Theorem 1, hypothesis (4) is needed only in case $(\tau - 1)Kr_0 \leq r(B) \leq \tau K r_0$.

(ii) In addition, the proof shows that if we replace hypothesis (3) by

$$\frac{1}{\nu(B)}\int_B |f - f_{B,\nu}|\,\mathrm{d}\nu \leq C\phi(B)\sigma(B), \tag{3'}$$

for $B \subseteq \tau K B_0$, where σ is any measure and ϕ is a nonnegative function of balls B, and if we also replace hypothesis (4) by the assumptions

$$\phi(B) \leq C\phi(\tilde{B}) \tag{4$'$a}$$

if $\tilde{B} \subseteq cB$ and $r(B)$, $r(B_0)$ are comparable, and

$$\phi(B) \approx \phi(\tilde{B}) \tag{4$'$b}$$

if B, $\tilde{B}$ have comparable radii and the metric distance between them is at most a multiple of their radii, where c depends only on τ and K, then we obtain the conclusion that for $(\mathrm{d}\nu)$-a.e. $x \in B_0$ and an appropriate constant C,

$$|f(x) - f_{B_0,\nu}| \leq C \int_{\tau K B_0} \phi(B(x, \rho(x,y)))\,\mathrm{d}\sigma(y). \tag{5$'$}$$

Note that assumptions (4$'$a) and (4$'$b) both follow from assuming that

$$\phi(B) \leq C\phi(\tilde{B})$$

if $\tilde{B} \subseteq cB$.

Proof of Theorem 1 By hypothesis, for a fixed $\tau > 1$ and all balls $B \subset \tau K B_0$,

$$\frac{1}{\nu(B)} \int_B |f - f_{B,\nu}|\,\mathrm{d}\nu \leq P(r_0)\frac{r(B)}{\mu(B)} \int_B |g|\,\mathrm{d}\mu.$$

Let $x \in B_0$ be given. There is a constant $\eta > 0$ independent of x and B_0 such that $B(x, \eta r(B_0)) \subset \tau K B_0$. In fact, it is enough to choose $\eta = \tau - 1$, since if $y \in B(x, \eta r_0)$ then

$$\begin{aligned}\rho(x_{B_0}, y) &\leq K(\rho(x_{B_0}, x) + \rho(x,y)) \\ &< K(r_0 + \eta r_0) = \tau K r_0.\end{aligned}$$

Denote $B(x, \eta r_0) = B_1$ and let $r_1 = r(B_1) = \eta r_0$. Now

$$|f(x) - f_{B_0,\nu}| \leq |f(x) - f_{B_1,\nu}| + |f_{B_1,\nu} - f_{B_0,\nu}|. \tag{6}$$

For the second term on the right in (6), we first note that

$$\tau K B_0 \subseteq B(x, (\tau K^2 + K) r_0) = \frac{\tau K^2 + K}{\eta} B_1,$$

since if $y \in \tau K B_0$, then

$$\rho(x, y) \leq K[\rho(y, x_{B_0}) + \rho(x_{B_0}, x)] \leq K[\tau K + 1] r_0.$$

Hence

$$\nu(\tau K B_0) \leq \nu\left(\frac{\tau K^2 + K}{\eta} B_1\right).$$

Now let $m \in \mathbb{Z}$, $m \geq 1$, be such that

$$2^{m-1} < \frac{\tau K^2 + K}{\eta} \leq 2^m,$$

so that

$$m \leq \log_2 \frac{2(\tau K^2 + K)}{\eta} := \eta_1 > 1,$$

since $\tau + 1 > \tau - 1 = \eta$. Then

$$r\left(\frac{\tau K^2 + K}{\eta} B_1\right) \leq 2^m r(B_1),$$

and we can apply formula (2) m times to ν and $r = 2^k r(B_1)$, with k varying from $m-1$ to 0 and

$$\mathfrak{R} = 2^{m-1} r(B_1) < \frac{\tau K^2 + K}{\eta} r(B_1) = (\tau K^2 + K) r_0.$$

Hence, keeping in mind that $A_\nu \geq 1$,

$$\nu(\tau K B_0) \leq A_\nu^{\eta_1}((\tau K^2 + K) r_0) \nu(B_1),$$

so that

$$\frac{1}{\nu(B_1)} \leq A_\nu^{\eta_1}((\tau K^2 + K)r_0)\frac{1}{\nu(\tau KB_0)}.$$

On the other hand, arguing as above and taking into account that $\tau K \leq \tau K^2 + K$ and that we can assume $\eta < 1$, we obtain

$$\nu(\tau KB_0) \leq A_\nu^{\eta_1}((\tau K^2 + K)r_0)\nu(B_0),$$

so that

$$\frac{1}{\nu(B_1)} + \frac{1}{\nu(B_0)} \leq 2A_\nu^{\eta_1}((\tau K^2 + K)r_0)\frac{1}{\nu(\tau KB_0)}.$$

Thus

$$\begin{aligned}
&|f_{B_1,\nu} - f_{B_0,\nu}| \\
&\quad \leq |f_{B_1,\nu} - f_{\tau KB_0,\nu}| + |f_{B_0,\nu} - f_{\tau KB_0,\nu}| \\
&\quad \leq \frac{1}{\nu(B_1)}\int_{B_1}|f(y) - f_{\tau KB_0,\nu}|\,\mathrm{d}\nu(y) + \frac{1}{\nu(B_0)}\int_{B_0}|f(y) - f_{\tau KB_0,\nu}|\,\mathrm{d}\nu(y) \\
&\quad \leq \left(\frac{1}{\nu(B_1)} + \frac{1}{\nu(B_0)}\right)\int_{\tau KB_0}|f(y) - f_{\tau KB_0,\nu}|\,\mathrm{d}\nu(y) \ \ (\text{since} B_1, B_0 \subset \tau KB_0) \\
&\quad \leq 2\frac{A_\nu^{\eta_1}((\tau K^2 + K)r_0)}{\nu(\tau KB_0)}\int_{\tau KB_0}|f - f_{\tau KB_0,\nu}|\,\mathrm{d}\nu \\
&\quad \leq 2A_\nu^{\eta_1}((\tau K^2 + K)r_0)P(r_0)\frac{r(\tau KB_0)}{\mu(\tau KB_0)}\int_{\tau KB_0}|g|\,\mathrm{d}\mu,
\end{aligned}$$

by the Poincaré inequality (3).

We shall now prove that if $y \in \tau KB_0$, then the kernel

$$\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}$$

which appears in (5) can be estimated from below (up to a suitable multiplicative constant) by the term $r(\tau KB_0)(\mu(\tau KB_0))^{-1}$. This estimate will basically follow by applying assumption (4); however, the estimate first requires some manipulations since in general $B(x,\rho(x,y))$ is not contained in τKB_0. Indeed, if $x\in B_0$, $y\in\tau KB_0$ and $z\in B(x,\rho(x,y)/M)$ with $M=(\tau K^2+K)/\eta$, we have

$$\begin{aligned}\rho(z,x_{B_0}) &\le K[\rho(z,x)+\rho(x,x_{B_0})]\\ &\le K\left[\frac{\rho(x,y)}{M}+r_0\right]\le K\left[\frac{K}{M}(\rho(x,x_{B_0})+\rho(y,x_{B_0}))+r_0\right]\\ &\le K\left[\frac{K+\tau K^2}{M}+1\right]r_0=\tau Kr_0,\end{aligned}$$

so that $B(x,\rho(x,y)/M)\subseteq\tau KB_0$. Thus, arguing as before and then applying (4) with $\tilde{B}=B(x,\rho(x,y)/M)$ and $B=\tau KB_0$, we get

$$\begin{aligned}\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))} &=\frac{\tau K^2+K}{\eta}\frac{\rho(x,y)/M}{\mu(B(x,M\rho(x,y)/M))}\\ &\ge\frac{\tau K^2+K}{\eta}A_\mu^{-\eta_1}((\tau K^2+K)r_0)\frac{\rho(x,y)/M}{\mu(B(x,\rho(x,y)/M))}\\ &\ge\frac{\tau K^2+K}{\eta}A_\mu^{-\eta_1}((\tau K^2+K)r_0)\theta(r_0)\frac{r(\tau KB_0)}{\mu(\tau KB_0)}.\end{aligned}$$

Combining estimates, we obtain

$$|f_{B_1,\nu}-f_{B_0,\nu}|\le C\int_{\tau KB_0}|g(y)|\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,\mathrm{d}\mu(y),$$

with

$$C=2\frac{\tau-1}{\tau K^2+K}\frac{P(r_0)}{\theta(r_0)}\left[A_\nu((\tau K^2+K)r_0)A_\mu((\tau K^2+K)r_0)\right]^{\eta_1}.$$

For the first term on the right in (6), we can assume that $\lim_{s\to 0}f_{B(x,s)}=f(x)$ since this holds for ν-a.e. x. Let $\bar{z}$ be such that $\rho(x,\bar{z})=\eta r_0=r(B_1)=r_1$; because of the segment property, there exists a geodesic $\gamma:[0,r_1]\to\mathcal{S}$ connecting x and $\bar{z}$ (i.e. $\gamma(0)=x$, $\gamma(r_1)=\bar{z}$) so that $\rho(\gamma(t'),\gamma(t''))=|t'-t''|$ for all $t',t''\in[0,r_1]$. If $\theta\in(0,\frac{1}{2})$, we consider the sequence of points $x_k=\gamma(t_k)$ for $k\in\mathbb{N}$ defined as follows:

$$t_1=\frac{1}{2K}r_1,\quad t_{k+1}=t_k-\theta t_k\quad\text{for } k\ge 1,$$

and we put

$$Q_k = B(x_k, \theta t_k).$$

We note that $t_k = (1/2K)(1-\theta)^{k-1} r_1$ for $k \in \mathbb{N}$, so that if $y \in Q_k$ then

$$\begin{aligned}\rho(x,y) &\le K[\rho(x,x_k) + \rho(y,x_k)]\\ &\le K\left(\frac{1}{2K}(1-\theta)^{k-1} r_1 + \frac{\theta}{2K}(1-\theta)^{k-1} r_1\right)\\ &= \frac{1+\theta}{2}(1-\theta)^{k-1} r_1 < r_1,\end{aligned}$$

and hence $Q_k \subseteq B_1$ for $k \in \mathbb{N}$.

Now

$$|f(x) - f_{B_1,\nu}| \le |f(x) - f_{Q_1,\nu}| + |f_{Q_1,\nu} - f_{B_1,\nu}|. \tag{7}$$

But

$$\begin{aligned}|f_{Q_1,\nu} - f_{B_1,\nu}| &= \left| ⨍_{Q_1} (f(y) - f_{B_1,\nu})\, \mathrm{d}\nu(y)\right|\\ &\le \frac{1}{\nu(Q_1)} \int_{Q_1} |f(y) - f_{B_1,\nu}|\, \mathrm{d}\nu(y) \le \frac{1}{\nu(Q_1)} \int_{B_1} |f(y) - f_{B_1,\nu}|\, \mathrm{d}\nu(y).\end{aligned}$$

We now want to compare $\nu(Q_1)$ and $\nu(B_1)$ by using doubling. Since

$$\rho(x, x_1) = \rho(\gamma(0), \gamma(t_1)) = t_1 = \frac{r_1}{2K},$$

we have

$$B_1 = B(x, r_1) \subseteq B(x_1, (K+1) r_1) = \frac{2K(K+1)}{\theta} Q_1,$$

and hence

$$\nu(B_1) \le \nu\left(\frac{2K(K+1)}{\theta} Q_1\right).$$

Now let $m \in \mathbb{N}$ be such that

$$2^{m-1} < \frac{2K(K+1)}{\theta} \le 2^m, \text{ so that } m \le \log_2\left(\frac{2K(K+1)}{\theta}\right) + 1 := \eta_2.$$

We can again apply formula (2) m times to ν and $r=2^k r(Q_1)$, with k varying from $m-1$ to 0 and $R=(K+1)r_1$, and we obtain

$$\begin{aligned}\nu(B_1) &\leq \nu(2^m Q_1) \leq A_\nu(2^{m-1}r(Q_1))\nu(2^{m-1}Q_1)\\ &\leq A_\nu((K+1)r_1)\nu(2^{m-1}Q_1) \quad (\text{since } 2^{m-1}r(Q_1) < (K+1)r_1)\\ &\leq \cdots \leq A_\nu^m((K+1)r_1)\nu(Q_1)\\ &\leq A_\nu^{\eta_2}((K+1)r_1)\nu(Q_1).\end{aligned}$$

Thus, by (3) (since $B_1 \subseteq \tau K B_0$),

$$\begin{aligned}&\frac{1}{\nu(Q_1)}\int_{B_1}|f(y)-f_{B_1,\nu}|\,\mathrm{d}\nu(y)\\ &\leq A_\nu^{\eta_2}((K+1)r_1)\frac{1}{\nu(B_1)}\int_{B_1}|f(y)-f_{B_1,\nu}|\,\mathrm{d}\nu(y)\\ &\leq P(r_0)A_\nu^{\eta_2}((K+1)r_1)\frac{r_1}{\mu(B_1)}\int_{B_1}|g(y)|\,\mathrm{d}\mu(y)\\ &= \sum_{k=0}^{\infty}P(r_0)A_\nu^{\eta_2}((K+1)r_1)\frac{r_1}{\mu(B_1)}\int_{2^{-k-1}r_1\leq\rho(x,y)<2^{-k}r_1}|g(y)|\,\mathrm{d}\mu(y).\end{aligned}$$

On the other hand, if $2^{-k-1}\, r_1 \leq \rho(x,y) < 2^{-k}r_1$, then, choosing $\tilde{B}$ and B in (4) to be $\tilde{B}=B(x,2^{-k}r_1)$ and $B=B(x,r_1)=B_1\subseteq\tau K B_0$, we get

$$\frac{1}{2}\theta(r_0)\frac{r_1}{\mu(B_1)} \leq \frac{2^{-k-1}r_1}{\mu(B(x,2^{-k}r_1))} \leq \frac{\rho(y,x)}{\mu(B(x,\rho(x,y)))},$$

so that

$$\begin{aligned}&|f_{Q_1,\nu}-f_{B_1,\nu}|\\ &\leq \frac{2P(r_0)}{\theta(r_0)}A_\nu^{\eta_2}((K+1)r_1)\int_{B_1}|g(y)|\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,\mathrm{d}\mu(y)\\ &\leq \frac{2P(r_0)}{\theta(r_0)}A_\nu^{\eta_2}((K+1)r_1)\int_{\tau K B_0}|g(y)|\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,\mathrm{d}\mu(y).\end{aligned}$$

Thus, the second term in (7) is estimated as desired since $A_\mu, A_\nu \geq 1$ and $(K+1)r_1=(K+1)\eta r_0 \leq (\tau K^2+K)r_0$ assuming that $1<\tau\leq 2$.

In order to exhibit a bound for the first term in (7), we note that

$$\lim_{k\to\infty} \fint_{Q_k} f(y)\,\mathrm{d}\nu(y) = f(x),$$

since x is a Lebesgue point for f. Indeed

$$\left| \fint_{Q_k} f(y)\,\mathrm{d}\nu(y) - f(x) \right| \le \frac{1}{\nu(Q_k)} \int_{Q_k} |f(y) - f(x)|\,\mathrm{d}\nu(y);$$

on the other hand, as we showed above,

$$Q_k \subseteq B(x, \tfrac{1}{2}(1+\theta)(1-\theta)^{k-1} r_1) = \tilde{Q}_k,$$

and $\nu(\tilde{Q}_k) \approx \nu(Q_k)$ by doubling since $r(\tilde{Q}_k) \approx r(Q_k) \approx \rho(x, x_k)$, so that

$$\begin{aligned} &\frac{1}{\nu(Q_k)} \int_{\tilde{Q}_k} |f(y) - f(x)|\,\mathrm{d}\nu(y) \\ &\quad \le \text{const} \cdot \frac{1}{\nu(\tilde{Q}_k)} \int_{\tilde{Q}_k} |f(y) - f(x)|\,\mathrm{d}\nu(y) \to 0 \end{aligned}$$

as $k \to \infty$, by Lebesgue's differentiation theorem.

Thus we can write

$$|f(x) - f_{Q_1,\nu}| = \left| \sum_{k=1}^{\infty} (f_{Q_k,\nu} - f_{Q_{k+1},\nu}) \right| \le \sum_{k=1}^{\infty} |f_{Q_k,\nu} - f_{Q_{k+1},\nu}|.$$

If now $y \in Q_{k+1}$, then since $\rho(x_k, x_{k+1}) = |t_k - t_{k+1}|$, we have

$$\begin{aligned} \rho(y, x_k) &\le K(\rho(y, x_{k+1}) + \rho(x_k, x_{k+1})) \\ &< K\left(\frac{\theta}{2K}(1-\theta)^k r_1 + \frac{\theta}{2K}(1-\theta)^{k-1} r_1 \right) \\ &= \frac{2-\theta}{2}\theta(1-\theta)^{k-1} r_1 < \theta(1-\theta)^{k-1} r_1, \end{aligned}$$

so that if we set

$$Q_k^* = B(x_k, \theta(1-\theta)^{k-1} r_1),$$

then $Q_{k+1} \subset Q_k^*$, and in addition, $Q_k = (1/2K) Q_k^* \subset Q_k^*$ since

$$r(Q_k) = \frac{\theta}{2K}(1-\theta)^{k-1} r_1 = \frac{1}{2K} r(Q_k^*).$$

Moreover, by the triangle inequality,

$$Q_k^* \subseteq \frac{2K^2+K}{1-\theta} Q_{k+1},$$

so that we can now estimate $\nu(Q_k^*)$ in terms of $\nu(Q_k)$ and $\nu(Q_{k+1})$ by a doubling argument as above. Indeed, let $m \in \mathbb{N}$ be such that

$$2^{m-1} < \frac{2K^2+K}{1-\theta} \leq 2^m;$$

then

$$\nu(Q_k^*) \leq \nu\left(\frac{2K^2+K}{1-\theta} Q_{k+1}\right) \leq \nu(2^m Q_{k+1}).$$

Moreover, if $1 \leq \ell < m$, then

$$r(2^\ell Q_{k+1}) = 2^\ell \frac{\theta}{2K}(1-\theta)^k r_1 \leq \frac{2K+1}{2(1-\theta)}\theta(1-\theta)^k r_1 \leq \frac{2K+1}{2}\theta\eta r_0,$$

and hence, by applying (2) m times as above, we obtain

$$\begin{aligned}
\nu(Q_k^*) &\leq A_\nu^m\left(\frac{2K+1}{2}\theta\eta r_0\right)\nu(Q_{k+1}) \\
&\leq \left(A_\nu\left(\frac{2K+1}{2}\theta\eta r_0\right)\right)^{1+\log_2((2K^2+K)/(1-\theta))} \nu(Q_{k+1}).
\end{aligned}$$

Since $Q_{k+1} \subseteq Q_k$, also

$$\nu(Q_k^*) \leq \left(A_\nu\left(\frac{2K+1}{2}\theta\eta r_0\right)\right)^{1+\log_2((2K^2+K)/(1-\theta))} \nu(Q_k).$$

Then we have

$$\begin{aligned}
|f_{Q_k,\nu} - f_{Q_{k+1},\nu}| &\leq |f_{Q_k^*,\nu} - f_{Q_k,\nu}| + |f_{Q_k^*,\nu} - f_{Q_{k+1},\nu}| \\
&\leq \frac{1}{\nu(Q_k)}\int_{Q_k} |f(y) - f_{Q_k^*}|\,\mathrm{d}\nu(y) + \frac{1}{\nu(Q_{k+1})}\int_{Q_{k+1}} |f(y) - f_{Q_k^*}|\,\mathrm{d}\nu(y) \\
&\leq C(\nu, K, \theta)\frac{1}{\nu(Q_k^*)}\int_{Q_k^*} |f(y) - f_{Q_k^*}|\,\mathrm{d}\nu(y),
\end{aligned}$$

with

$$C(\nu, K, \theta) = 2A_\nu\left(\frac{2K+1}{2}\theta\eta r_0\right)^{1+\log_2((2K^2+K)/(1-\theta))}$$
$$\leq \text{ (by choosing } \theta \text{ sufficiently small) } 2A_\nu((\tau K^2+K)r_0)^{\eta_1}.$$

The next step will consist of applying the Poincaré inequality (3) to each term above. To do so, we have to first prove that $Q_k^* \subseteq \tau K B_0$ for all $k \geq 1$. Let y be any point in Q_k^*. If we choose θ so small that $K\theta < \frac{1}{2}$, then by the triangle inequality we get

$$\begin{aligned}\rho(y, x_{B_0}) &\leq K(\rho(y,x) + \rho(x, x_{B_0}))\\ &< K^2(\rho(y,x_k) + \rho(x_k, x)) + Kr_0\\ &< K^2\theta(1-\theta)^{k-1}\eta r_0 + \frac{K}{2}(1-\theta)^{k-1}\eta r_0 + Kr_0\\ &< (\eta+1)Kr_0 = \tau K r_0.\end{aligned}$$

Thus, by Poincaré's inequality (3),

$$|f(x) - f_{Q_1,\nu}| \leq A_\nu((\tau K^2+K)r_0)^{\eta_1} P(r_0) \sum_{k=1}^{\infty} \frac{r(Q_k^*)}{\mu(Q_k^*)}\int_{Q_k^*} |g(y)|\,\mathrm{d}\mu(y).$$

To finish the proof of Theorem 1, we need now to have the kernel

$$\rho(x,y)/\mu(B(x,\rho(x,y)))$$

appearing on the right-hand side. To accomplish this, we note first that if $y \in Q_k^*$ then

$$\begin{aligned}\rho(x,y) &\geq \frac{1}{K}\rho(x_k, x) - \rho(y, x_k)\\ &\geq \frac{1}{2K^2}(1-\theta)^{k-1}r_1 - \theta(1-\theta)^{k-1}r_1 = \left[\frac{1}{2K^2} - \theta\right](1-\theta)^{k-1}r_1\\ &> \frac{1}{4K^2}(1-\theta)^{k-1}r_1,\end{aligned}$$

if we choose θ sufficiently close to zero. On the other hand,

$$\begin{aligned}\rho(x,y) &\leq K(\rho(x_k,x)+\rho(y,x_k))\\ &\leq K\left(\frac{1}{2K}(1-\theta)^{k-1}r_1+\theta(1-\theta)^{k-1}r_1\right)\\ &=\left(\frac{1}{2}+\theta(1-\theta)K\right)(1-\theta)^{k-1}r_1\\ &<(1-\theta)^{k-1}r_1,\end{aligned}$$

if θ is sufficiently close to zero. Thus, if $y \in Q_k^*$, we have

$$r(Q_k^*)=\theta(1-\theta)^{k-1}r_1 \leq \theta 4K^2\rho(x,y) \leq \rho(x,y),$$

if θ is small enough. Moreover, if $z \in B(x,\rho(x,y))$, then by the triangle inequality,

$$\begin{aligned}\rho(z,x_k) &\leq K(\rho(z,x)+\rho(x,x_k))\\ &\leq K\left(\rho(x,y)+\frac{1}{2K}(1-\theta)^{k-1}r_1\right)\\ &\leq K\left((1-\theta)^{k-1}+\frac{1}{2K}(1-\theta)^{k-1}\right)r_1<(K+1)(1-\theta)^{k-1}r_1\\ &=\frac{K+1}{\theta}r(Q_k^*),\end{aligned}$$

so that, arguing as above,

$$\mu(B(x,\rho(x,y))) \leq \mu\left(\frac{K+1}{\theta}Q_k^*\right) \leq A_\mu((\tau K^2+K)r_0)^{1+\log_2((K+1)/\theta)}\mu(Q_k^*).$$

Indeed, let $m \in \mathbb{N}$ be such that $2^{m-1}<(K+1)/\theta \leq 2^m$; we have

$$\mu\left(\frac{K+1}{\theta}Q_k^*\right) \leq \mu(2^m Q_k^*).$$

On the other hand, if $1 \leq \ell < m$, then

$$r(2^\ell Q_k^*) \leq \frac{K+1}{\theta}\theta(1-\theta)^{k-1}\eta r_0=(K+1)\eta r_0<(\tau K^2+K)r_0$$

if $\tau \leq 2$ (say). Thus, for θ sufficiently close to zero, we get

$$\sum_{k=1}^{\infty} \frac{r(Q_k^*)}{\mu(Q_k^*)} \int_{Q_k^*} |g(y)| \,\mathrm{d}\mu(y)$$
$$\leq A_\mu((\tau K^2 + K)r_0)^{1+\log_2((K+1)/\theta)} \sum_{k=1}^{\infty} \int_{Q_k^*} |g(y)| \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))} \,\mathrm{d}\mu(y).$$

To complete the proof, we have only to show that each ball Q_k^* overlaps only a finite number of balls Q_i^*, the number being bounded by a constant depending only on θ. To this end, note that if $Q_k^* \cap Q_i^* \neq \emptyset$, then

$$\rho(x_k, x_i) < K(r(Q_k^*) + r(Q_i^*)),$$

so that

$$|t_k - t_i| = \left| \frac{1}{2K}(1-\theta)^{k-1} r_1 - \frac{1}{2K}(1-\theta)^{i-1} r_1 \right|$$
$$< K\theta((1-\theta)^{k-1} + (1-\theta)^{i-1}) r_1,$$

and hence

$$\left|1 - (1-\theta)^{i-k}\right| < 2K^2\theta\left(1 + (1-\theta)^{i-k}\right).$$

Without loss of generality we can assume that $2K^2\theta < \frac{1}{2}$, so that $\frac{1}{3} \leq (1-\theta)^{i-k} \leq 3$, and hence

$$k + \frac{\ln 3}{\ln(1-\theta)} \leq i \leq k - \frac{\ln 3}{\ln(1-\theta)}.$$

Thus Theorem 1 is completely proved, keeping in mind that $\theta(r_0) \leq 1$.

In [FLW] we listed several examples for which the assumptions of Theorem 1 of [FLW] (and hence a fortiori the assumptions of our Theorem 1 above) are fulfilled. We refer to Examples 1–6 there; it is easy to formulate them in the more general situation covered by our present results. Here we restrict ourselves to two remarks strictly related to our new assumptions. We thank S. Gallot and S. Chanillo for helpful discussions about the first of these remarks.

Remark 2 Let $(\mathcal{S}, \rho, m)$ be a complete, connected Riemannian manifold of dimension n endowed with its Riemannian distance ρ and its Riemannian volume m. We shall follow the arguments of [SC], Section 2. Denote by g the Riemannian metric tensor of $\mathcal{S}$, and assume that $Ric_{\mathcal{S}}$, the Ricci tensor of $\mathcal{S}$, satisfies

$$Ric_{\mathcal{S}} \geq -\kappa g,$$

for some positive constant $\kappa \geq 0$. Then, if we set $V(x,r) = m(B(x,r))$ for $x \in \mathcal{S}$ and $r > 0$, the Bishop–Gromov theorem ([GHL], Theorem 4.19) implies that the function $r \to V(x,r)/V_\kappa(r)$ is decreasing, where $V_\kappa(r)$ denotes the volume of the disk of radius $r > 0$ in $\mathbb{M}_{\kappa/(1-n)}$, the complete space of constant sectional curvature $\kappa/(1-n)$. Thus

$$\frac{V(x,2r)}{V(x,r)} \leq \frac{V_\kappa(2r)}{V_\kappa(r)},$$

and hence as in [SC] the doubling constant of m satisfies

$$A_m(R) \leq 2^n \exp\left(\sqrt{(n-1)\kappa} R\right).$$

Thus, Theorem 1 can be applied if the volume of balls $V(x,r)$ satisfies estimate (4), since the Poincaré inequality (3) holds for these manifolds with $g = |\nabla f|$ and

$$P(r) = C_n \exp\left(c\sqrt{\kappa} r\right)$$

(see [B]), and then we obtain that the representation formula (5) holds with $g = |\nabla f|$, $\mu = \nu = m$ and

$$C = \frac{c_1}{\theta(r_0)} \exp\left(c_2 \sqrt{\kappa} r_0\right),$$

where $c_1, c_2 > 0$ are geometric constants.

On the other hand, condition (4) holds in many relevant situations for which the constant $\theta(r_0)$ can be explicitly estimated, as we shall see in Examples 1 and 2 below. We first observe that if $\tilde{B} = B(\tilde{x}, \tilde{r})$, $B = B(x,r)$, $B_0 = B(x_0, r_0)$ are balls with $\tilde{B} \subseteq B \subseteq \tau B_0$ as in Theorem 1, and if $\tau B_0 \neq \mathcal{S}$, then $\tilde{r} \leq 2r$ and $r \leq 2\tau r_0$. Indeed, let us prove the second assertion; the first one is proved in a similar way. Consider a point

$y \notin \tau B_0$, and let γ be a geodesic connecting x and y (which exists by the Hopf–Rinow Theorem). Without loss of generality, we may assume that γ is parametrized by arclength so that $\rho(x,\gamma(t)) = t$. By continuity, there exists a point $z = \gamma(t_0)$ such that $\rho(z, x_0) = \tau r_0$. Clearly, $z \notin B$ since $B \subseteq \tau B_0$, and hence $r \le \rho(x,z) = t_0$. But $\rho(x,z) \le \rho(x,x_0) + \rho(x_0,z) < 2\tau r_0$, and we are done by combining inequalities.

Example 1 Let K denote the sectional curvature of $\mathcal{S}$ and assume that there exists $b \ge 0$ such that $K \le b$ on $\mathcal{S}$. If $\tilde{B}, B, B_0$ are as in Theorem 1, with $(\tau-1)r_0 \le r \le \tau r_0$ (see Remark 1(i) following Theorem 1), we have$B_0 \subseteq (2/(\tau-1))B$, so that, arguing as in the proof of Theorem 1, we obtain

$$m(B_0) \le A_m(c'_\tau r_0)^{\eta_3(\tau)} m(B) \le \exp(c''_{\tau,n}\sqrt{\kappa} r_0) m(B),$$

and hence

$$\frac{m(\tilde{B})}{m(B)} \le \exp(c''_{\tau,n}\sqrt{\kappa} r_0)\frac{m(\tilde{B})}{m(B_0)}.$$

Suppose now that B_0 does not meet the cut locus of x_0; then, if we apply Bishop's volume estimate to $\tilde{B}$ and Gunther's estimate to B_0 (see [GHL], Theorem 3.101), we get

$$\begin{aligned}\frac{m(\tilde{B})}{m(B_0)} &\le \frac{\exp(c''_{\tau,n}\sqrt{\kappa} r_0)\omega_n \tilde{r}^n \exp(\sqrt{(n-1)\kappa}\tilde{r})}{V_{(1-n)b}(r_0)}\\ &\le \frac{\omega_n \exp(c'''_{\tau,n}\sqrt{\kappa} r_0)\tilde{r}^n}{V_{(1-n)b}(r_0)},\end{aligned}$$

where ω_n is the volume of the unit Euclidean ball. But if $r_0 \le \pi/\sqrt{b}$, $V_{(1-n)b}(r_0)$ satisfies

$$V_{(1-n)b}(r_0) = \omega_n \int_0^{r_0} \frac{1}{\sqrt{b}}\{\sin(\sqrt{b}t)\}^{n-1}\mathrm{d}t \ge c(n,b) r_0^n,$$

so that since $\tilde{r} \le 2r \le 2\tau r_0$,

$$\frac{\tilde{r}^n}{V_{(1-n)b}(r_0)} \le c'(n,b,\tau)\frac{\tilde{r}^n}{r^n} \le c''(n,b,\tau)\frac{\tilde{r}}{r}.$$

Thus we get

$$\frac{m(\tilde{B})}{m(B)} \leq c(n,b,\tau)\exp(c(n,\tau)\sqrt{\kappa}r_0)\frac{\tilde{r}}{r},$$

and so the following representation formula holds:

$$|f(x) - f_B| \leq c_3 \exp(c_4\sqrt{\kappa}r_0) \int_{\tau B_0} |\nabla f(y)| \frac{\rho(x,y)}{m(B(x,\rho(x,y)))} \, \mathrm{d}m(y),$$

where $c_4 = c_4(n,\tau)$ and $c_3 = c_3(n,b,\tau)$.

Example 2 Suppose now that $\kappa = 0$ (i.e., that $Ric_{\mathcal{S}} \geq 0$) and let $\tau B_0 \neq \mathcal{S}$; in this case, $A_m(R) \leq 2^n$ for any $R > 0$. Let $\tilde{B}, B, B_0$ again be as in Theorem 1, and set $B^* = B(\tilde{x}, r)$. Since $B^* \subseteq 2B$, we have $m(B^*) \leq m(2B) \leq 2^n m(B)$, and hence

$$\frac{m(\tilde{B})}{m(B)} \leq 2^n \frac{m(\tilde{B})}{m(B^*)}.$$

In order to estimate the right side, suppose that $3\tilde{r} \leq r$, the reverse case being trivial by doubling. If we apply inequality (4.13) in [CGT], we obtain

$$\frac{m(\tilde{B})}{m(B)} \leq 2^n \frac{r^n - (r - 2\tilde{r})^n}{(r - 2\tilde{r})^n} \leq \sigma^n \left(1 - \left(1 - 2\frac{\tilde{r}}{r}\right)^n\right)$$
$$\leq c_n \frac{\tilde{r}}{r}.$$

Thus (4) is proved, $\theta(r_0)$ being a dimensional constant. Then the representation formula (5) holds with $g = |\nabla f|$ and a dimensional constant c.

Example 3 Let Ω be a bounded open subset of $\mathbb{R}^n$, and let $X_1, \ldots, X_m$ be Lipschitz continuous vector fields in Ω_0, where Ω_0 is an open neighborhood of $\bar{\Omega}$. Denote by ρ the Carnot–Carathéodory distance associated with $X_1, \dots, X_m$ in Ω_0 (see below), and assume that

(i) $\rho(x,y) < \infty$ for all $x, y \in \Omega_0$;
(ii) ρ is continuous with respect to the Euclidean topology;
(iii) $|B(x,2r)| \leq A|B(x,r)|$ for all $x \in \bar{\Omega}$ and $r \in (0, R_0)$, where $B(x,r)$ is a ball for the metric ρ and $|E|$ denotes the Lebesgue measure of E.

We recall that the Carnot–Carathéodory distance can be defined as follows. We say that an absolutely continuous curve $\gamma : [0,T] \to \Omega_0$ is a sub-unit curve (with respect to $X_1, \dots, X_m$) if

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^{n} \langle X_j(\gamma(t)), \xi \rangle^2$$

for any $\xi \in \mathbb{R}^n$ and for a.e. $t \in [0, \mathrm{T}]$. If $x, y \in \Omega_0$, we put

$$\rho(x,y) = \inf\{T > 0\text{: there exists a sub-unit curve } \gamma : [0,T] \to \mathbf{R}^n \text{ with } \gamma(0) = x \text{ and } \gamma(T) = y\}.$$

If we now set $\mathcal{S} = \bar{\Omega}$, endowed with the Carnot–Carathéodory distance ρ and Lebesgue measure, assumptions (1) and (2) hold.

We shall now prove that condition (4) is satisfied for any ball $B_0 = B(x_{B_0}, r_0)$ with $r_0 \leq R_0$. Since the metric space $(\bar{\Omega}, \rho)$ enjoys the segment property,[‡] Theorem 1 can then be applied, and hence the representation formula (5) is always equivalent to the Poincaré inequality (3) (we are assuming that $\nu = \mu = m$ is Lebesgue measure). We stress also the fact that in this case ρ is a metric and hence $K = 1$.

To prove (4), let us first note that it will be enough to show that for each point $\bar{x} \in \bar{\Omega}$ there exists a metric ball $B_{\bar{x}}$ centered at $\bar{x}$ such that for any $\sigma \in (0, 1)$ we have

$$|B(x, \sigma r)| \leq C(\bar{x})\sigma|B(x,r)| \quad \text{for each ball } B(x,r) \subset \frac{1}{2}B_{\bar{x}}. \tag{8}$$

[‡] The present result is basically a local result, and hence the segment property follows from the Arzelà–Ascoli theorem (see [FGW]). But this property – which implies that the space is a length space in the sense of Gromov – still holds in the large. This global property was kindly pointed out by N. Garofalo.

Indeed, suppose (8) true, and take a finite family $\{B_1 = B_{\bar{x}_1}, \ldots, B_p = B_{\bar{x}_p}\}$ of balls satisfying (8) such that $\{\frac{1}{4}B_j,\ j=1,\ldots,p\}$ is an open covering of $\bar{\Omega}$, and $\cup_{j=1}^{p} B_j \subset\subset \Omega_0$. Put $R_0 = \min\{\frac{1}{4}r(B_j),\ j=1,\ldots,p\}$, $C = \max\{C(\bar{x}_j), j=1,\ldots,p\}$, and let $B(x,r)$ be a metric ball with $x\in\Omega$ and $r<R_0$. Since x belongs to $\frac{1}{4}B_k$ for a suitable $k\in\{1,\ldots,p\}$, then $B(x,r)\subset\frac{1}{2}B_k$, and hence by (8)

$$|B(x,\sigma r)| \le C(\bar{x}_k)\sigma|B(x,r)| \le C\sigma|B(x,r)| \tag{9}$$

for $\sigma\in(0,1)$.

Now let $B, \tilde{B}, B_0$ be as in (4), i.e., $\tilde{B}\subseteq B\subseteq \tau B_0$. If $\tilde{B} = B(\tilde{x},\tilde{r})$, $B = B(x,r)$, and $\tilde{r}\le r$, then $B(\tilde{x},\tilde{r})\subseteq B(\tilde{x},r)$ and $\rho(x,\tilde{x}) < r$, so that by doubling $|B|\approx|B(\tilde{x},r)|$. Combining this estimate with (9), if $r(B_0)$ is sufficiently small we get

$$|\tilde{B}| = |B(\tilde{x},\tilde{r})| = |B(\tilde{x},\frac{\tilde{r}}{r}r)| \le C\frac{\tilde{r}}{r}|B(\tilde{x},r)| \le C'\frac{\tilde{r}}{r}|B|,$$

and (4) follows. On the other hand, if $\tilde{r}\ge r$, then $\tilde{r}\approx r$ since $\tilde{r}\le 2r$, so that $|\tilde{B}|\approx|B|$ and (4) is immediate.

Thus (4) will be proved by proving (8). Let us now prove (8).

Step 1 The first step will consist of proving that, up to a change of variable, we can assume that one of the vector fields is the derivative with respect to one of the variables. We begin by proving the following result.

For each point $\bar{x}\in\bar{\Omega}$, there exists a neighborhood $\mathcal{U}$ of the origin and a Lipschitz continuous map $\Phi:\mathcal{U}\to\mathbb{R}^n$ such that

(i) Φ *is* $1-1$ *and* $\Phi(0)=\bar{x}$;
(ii) $\mathcal{V} = \{y\in\mathbb{R}^n\colon\ |y-\bar{x}|<\delta\}\subseteq\Phi(\mathcal{U})$ *for some* $\delta>0$;
(iii) Φ *and* Φ^{-1} *are both Lipschitz continuous*;
(iv) *if* $u\in\mathbf{C}^1(\mathcal{V},\mathbb{R})$, *then* $(\partial/\partial y_1)(u\circ\Phi) = (X_k u)\circ\Phi$ *for a suitable* $k\in\{1,\ldots,m\}$.

Proof Since the Carnot–Carathéodory metric can be defined as the shortest time required to go from a point to another along piecewise integral curves of the vector fields $\pm X_1,\ldots,\pm X_m$, we can assume that there exists one of these vector fields, say X_1, such that $X_1(\bar{x})\ne 0$, since otherwise we would have $\rho(\bar{x},x)=\infty$ for all x. Moreover, without loss of generality, we can assume that $X_1 = (a_{11}, \dots, a_{1n})$ is such that $a_{11}(\bar{x}) > 0$, and that $\bar{x} = 0$. For $z \in \Omega$ we denote by $\gamma_z(\cdot)$ the integral curve of X_1 issuing from z at $t = 0$. If y belongs to a neighborhood $\mathcal{U}$ of the origin, we set $y' = (y_2, \dots, y_n)$ and define

$$\Phi(y) = \gamma_{(0,y')}(y_1).$$

If we replace the assumption '$X_1, \dots, X_m$ Lipschitz' by '$X_1, \dots, X_m$ continuously differentiable', then the assertion is more or less trivial, because of the local invertibility theorem. Otherwise, a few technicalities are in order.

Without loss of generality we can assume $0 < \lambda \le a_{11} \le \Lambda$ in $\mathcal{U}_1 = \{y \in \mathbb{R}^n, |y| < \delta_0\}$. Let us now prove (i)–(iv) in order.

(i) Obviously, $\Phi(0) = 0$. Moreover, if $\Phi(y) = \Phi(\eta) = x$, then $\gamma_x(-y_1) = (0, y')$ and $\gamma_x(-\eta_1) = (0, \eta')$. But the first component of $\gamma_x(t)$ is strictly increasing for small t, and hence $y_1 = \eta_1$, so that $y' = \eta'$ and then $y = \eta$.

(ii) By the local boundedness of X_1, $|\gamma_x(t)| \le |x| + C|t|$ for small t; choose now δ such that $(2C/\lambda + 1)\delta < \delta_0$, so that, if $|x| < \delta$ and $|t| \le (2/\lambda)|x_1|$, then $|\gamma_x(t)| \le |x| + (2C/\lambda)|x_1| < (2C/\lambda + 1)\delta < \delta_0$. This implies that $\lambda \le a_{11}(\gamma_x(t)) \le \Lambda$ if $|t| \le (2/\lambda)|x_1|$, and hence that there exists $t(x)$ with $|t(x)| \le (2/\lambda)|x_1|$ such that the first component of $\gamma_x(t(x))$ equals zero. Indeed, suppose $x_1 > 0$; then the first component of $\gamma_x(-2x_1/\lambda)$ is

$$x_1 + \int_0^{-2x_1/\lambda} a_{11}(\gamma_x(s))\, ds \le x_1 - \frac{2\lambda}{\lambda} x_1 \le 0,$$

so that the existence of $t(x)$ follows by continuity.

Now take $y = \gamma_x(t(x)) - t(x)e_1$. We have

$$\Phi(y) = \gamma_{(0,y')}(y_1) = \gamma_{\gamma_x(t(x))}(-t(x)) = x,$$

and then assertion (ii) is proved.

(iii) We have

$$\Phi(y) - \Phi(\eta) = [\Phi(y) - \Phi((y_1, \eta'))] + [\Phi((y_1, \eta')) - \Phi(\eta)] = I_1 + I_2.$$

Now

$$
\begin{aligned}
|I_1| &= |\gamma_{(0,y')}(y_1) - \gamma_{(0,\eta')}(y_1)| \\
&\leq \int_0^{|y_1|} |X_1(\gamma_{(0,y')}(s)) - X_1(\gamma_{(0,\eta')}(s))| \, \mathrm{d}s + |y' - \eta'| \\
&\leq L \int_0^{|y_1|} |\gamma_{(0,y')}(s) - \gamma_{(0,\eta')}(s)| \, \mathrm{d}s + |y' - \eta'|,
\end{aligned}
$$

so that, by Gronwall's inequality,

$$
\begin{aligned}
|I_1| &\leq |y' - \eta'| \left(1 + L \int_0^{|y_1|} \mathrm{e}^{L(|y_1| - \tau)} \, \mathrm{d}\tau \right) \\
&\leq C|y - \eta|.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|I_2| &= |\gamma_{(0,\eta')}(y_1) - \gamma_{(0,\eta')}(\eta_1)| \\
&= \left| \int_{\eta_1}^{y_1} X_1(\gamma_{(0,\eta')}(s)) \, \mathrm{d}s \right| \\
&\leq C|y_1 - \eta_1| \leq C|y - \eta|,
\end{aligned}
$$

since X_1 is bounded. This proves that Φ is Lipschitz continuous.

Let us now prove that Φ^{-1} is Lipschitz continuous. Arguing as above, it is easy to see that the map $x \to \gamma_x(t)$ is uniformly Lipschitz continuous for $|t| \leq 1$. Indeed, if $t > 0$, then

$$
\begin{aligned}
|\gamma_x(t) - \gamma_\xi(t)| &\leq |x - \xi| + \int_0^t |X_1(\gamma_x(s)) - X_1(\gamma_\xi(s))| \, \mathrm{d}s \\
&\leq |x - \xi| + L \int_0^t |\gamma_x(s) - \gamma_\xi(s)| \, \mathrm{d}s,
\end{aligned}
$$

and hence the assertion follows by Gronwall's inequality. Let us next show that the map $x \to t(x)$ is Lipschitz continuous for $|x| < \delta$. To this end, let x, ξ be such that $|x|$, $|\xi| < \delta$; without loss of generality we may suppose that $|x_1| \leq |\xi_1|$. By definition of $t(x)$,

$$
0 = x_1 + \int_0^{t(x)} a_{11}(\gamma_x(s)) \, \mathrm{d}s = \xi_1 + \int_0^{t(\xi)} a_{11}(\gamma_\xi(s)) \, \mathrm{d}s,
$$

so that

$$\begin{aligned} x_1 - \xi_1 &= \int_0^{t(\xi)} a_{11}(\gamma_\xi(s))\,\mathrm{d}s - \int_0^{t(x)} a_{11}(\gamma_x(s))\,\mathrm{d}s \\ &= \int_{t(x)}^{t(\xi)} a_{11}(\gamma_\xi(s))\,\mathrm{d}s + \int_0^{t(x)} [a_{11}(\gamma_\xi(s)) - a_{11}(\gamma_x(s))]\,\mathrm{d}s. \end{aligned}$$

Note now that $|t(\xi)| \le (2/\lambda)|\xi_1|$ and $|t(x)| \le (2/\lambda)|x_1| \le (2/\lambda)|\xi_1|$, so that, when s lies between $t(x)$ and $t(\xi)$, $|\gamma_\xi(s)| \le |\xi| + (2C/\lambda)|\xi_1| \le ((2C/\lambda)+1)\delta < \delta_0$ and hence $a_{11}(\gamma_\xi(s)) \ge \lambda$. Thus we have

$$\begin{aligned} \lambda|t(x) - t(\xi)| &\le \left| \int_{t(x)}^{t(\xi)} a_{11}(\gamma_\xi(s))\,\mathrm{d}s \right| \\ &\le |x_1 - \xi_1| + \int_0^{|t(x)|} |a_{11}(\gamma_\xi(s)) - a_{11}(\gamma_x(s))|\,\mathrm{d}s \\ &\le |x - \xi| + L\int_0^1 |\gamma_\xi(s) - \gamma_x(s)|\,\mathrm{d}s \le L_1|x - \xi|. \end{aligned}$$

Hence, since we showed above that $\Phi^{-1}(x) = \gamma_x(t(x)) - t(x)e_1$, we have

$$\begin{aligned} |\Phi^{-1}(x) - \Phi^{-1}(\xi)| &= |\gamma_x(t(x)) - t(x)e_1 - \gamma_\xi(t(\xi)) + t(\xi)e_1| \\ &\le |t(x) - t(\xi)| + |\gamma_\xi(t(x)) - \gamma_\xi(t(\xi))| + |\gamma_x(t(x)) - \gamma_\xi(t(x))|. \end{aligned}$$

Now, the first and the third terms are bounded by $\mathrm{const}\cdot|x - \xi|$ by what we proved above, whereas the second one equals

$$\left| \int_{t(x)}^{t(\xi)} a_{11}(\gamma_\xi(s))\,\mathrm{d}s \right| \le \Lambda|t(x) - t(\xi)| \le \mathrm{const}\cdot|x - \xi|,$$

and the third assertion is completely proved.

(iv) An easy calculation (together with Rademacher's theorem) shows that

$$\frac{\partial}{\partial y_1}(u \circ \Phi) = \left\langle (\nabla u) \circ \Phi, \frac{\partial \Phi}{\partial y_1} \right\rangle = \langle \nabla u, X_1 \rangle \circ \Phi = (X_1 u) \circ \Phi,$$

so that in the new variables y the vector field X_1 is the vector field $\partial/\partial y_1$.

This completes the proofs of (i)–(iv).

To complete Step 1, consider now the vector fields $\tilde{X}_1, \ldots, \tilde{X}_m$ in $\mathcal{U}$ defined by $\tilde{X}_j(u \circ \Phi) = (X_j u) \circ \Phi$. By part (iv) above, we can assume $\{\tilde{X}_1, \ldots, \tilde{X}_m\} = \{\partial/\partial y_1, \tilde{X}_2, \ldots, \tilde{X}_m\}$. In addition, a curve $\gamma: [a, b] \to \mathcal{U}$ is a sub-unit curve with respect to $\{\tilde{X}_1, \ldots, \tilde{X}_m\}$ if and only if $\Phi \circ \gamma$ is a sub-unit curve with respect to $\{X_1, \ldots, X_m\}$, so that, if we denote by $\tilde{\rho}$ the Carnot–Carathéodory metric associated with $\{\tilde{X}_1, \ldots, \tilde{X}_m\}$, then $\tilde{\rho}(y, \eta) = \rho(\Phi(y), \Phi(\eta))$. Therefore, if for awhile we denote by $B_{\tilde{\rho}}$ and B_ρ the metric balls with respect to $\tilde{\rho}$ and ρ respectively, then by shrinking $\mathcal{U}$ if necessary, we get $B_{\tilde{\rho}}(y, r) = \Phi^{-1}(B_\rho(\Phi(y), r)))$ for $y \in \mathcal{U}$ and r sufficiently small. Hence

$$\begin{aligned} |B_\rho(x, r)| &= \int_{B_\rho(x,r)} \mathrm{d}\xi = \int_{\Phi^{-1}(B_\rho(x,r))} |\det \mathcal{J}_\Phi(\eta)| \, \mathrm{d}\eta \\ &= \int_{B_{\tilde{\rho}}(\Phi^{-1}(x),r)} |\det \mathcal{J}_\Phi(\eta)| \, \mathrm{d}\eta \approx |B_{\tilde{\rho}}(\Phi^{-1}(x), r)|. \end{aligned}$$

Thus, to prove (8) it will be enough to prove the same assertion for $\partial/\partial y_1, \tilde{X}_2, \ldots, \tilde{X}_m$. To avoid cumbersome notation, from now on we will assume that $\{X_1, \ldots, X_m\} = \{\partial/\partial y_1, \tilde{X}_2, \ldots, \tilde{X}_m\}$.

Step 2 This step will consist of proving the following technical result:

If we put

$$Y_j = (a_{j1}, \ldots, a_{jn}) \quad \textit{and} \quad \tilde{Y}_j = (0, a_{j2}, \ldots, a_{jn}),$$

then there exists $M > 0$ such that the metric associated with $\{M\partial_1, \tilde{Y}_2, \ldots, \tilde{Y}_m\}$ is equivalent to the metric associated with $\{\partial_1, Y_2, \ldots, Y_m\}$.

Proof For all $\xi \in \mathbb{R}^n$ we have

$$\begin{aligned} (M\xi_1)^2 + \sum_{j=2}^m \langle \tilde{Y}_j, \xi \rangle^2 &= M^2 \xi_1^2 + \sum_{j=2}^m \big(\langle Y_j, \xi \rangle - a_{j1}\xi_1 \big)^2 \\ &\le M^2 \xi_1^2 + 2 \sum_{j=2}^m \langle Y_j, \xi \rangle^2 + 2m \max_j \sup_{\bar{\Omega}} |a_{j1}|^2 \xi_1^2 \\ &\le C \left(\xi_1^2 + \sum_{j=2}^m \langle Y_j, \xi \rangle^2 \right). \end{aligned}$$

On the other hand, if $a, b \in \mathbb{R}$, then

$$(a-b)^2 \geq \frac{3}{4}a^2 - 3b^2.$$

Therefore

$$(M\xi_1)^2 + \sum_{j=2}^{m} \langle \tilde{Y}_j, \xi \rangle^2 \geq M^2\xi_1^2 + \frac{3}{4}\sum_{j=2}^{m} \langle Y_j, \xi \rangle^2 - C\xi_1^2,$$

and then it is enough to choose $M^2 = \frac{3}{4} + C$. Hence, if $\gamma(t)$ is a sub-unit curve for one set of vector fields, then there exists $0 < c < \infty$ such that $\gamma(ct)$ is sub-unit for the other, and then the related distances are equivalent.

Step 3 It follows from Steps 1 and 2 that (8) will be proved by proving it for the vector fields $\partial_1, X_2, \ldots, X_m$ when X_j does not contain the first derivative for $j = 2, \ldots, m$.

Let $B(x, \sigma r)$ be a ball with $\sigma \in (0, 1)$ and let

$$\psi_\sigma(y) = (x_1 + \frac{1}{\sigma}(y_1 - x_1), y_2, \ldots, y_n).$$

We will show that

$$\psi_\sigma(B(x, \sigma r)) \subseteq B(x, r).$$

If so, then setting $\eta = \psi_\sigma(y)$, we will have

$$|B(x, \sigma r)| = \sigma \int_{\psi_\sigma(B(x,\sigma r))} d\eta \leq \sigma |B(x, r)|,$$

and the proof of (8) will be achieved.

Thus, let $y \in B(x, \sigma r)$ be given; by definition, there exists a sub-unit curve $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(T) = y$, and $T \leq \sigma r$. In particular, if we choose $\xi = (1, 0, \ldots, 0)$ in the definition of sub-unit curve, putting $\gamma = (\gamma_1, \ldots, \gamma_n)$, we have

$$|\dot{\gamma}_1(t)| \leq 1 \quad \text{for a.e. } t \in [0, T]$$

since $\langle X_j, \xi\rangle = 0$ for $j = 2, \ldots, m$. Then

$$|y_1 - x_1| = \Big|\int_0^T \dot\gamma(s)\,\mathrm{d}s\Big| \le \sigma r.$$

Thus

$$\rho(\psi_\sigma(y), x) \le \rho(\psi_\sigma(y), y) + \rho(y, x) < \rho(\psi_\sigma(y), y) + \sigma r.$$

Since y and $\psi_\sigma(y)$ differ only in their first components, a sub-unit curve connecting them is given by $t \to y_1 + t$, with t between 0 and $(1/\sigma - 1)(y_1 - x_1)$. Therefore

$$\rho(\psi_\sigma(y), y) \le \left(\frac{1}{\sigma} - 1\right)|y_1 - x_1| \le (1 - \sigma) r.$$

This proves that $\rho(\psi_\sigma(y), x) < r$, and then the assertion is completely proved.

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