J. of Inequal. & Appl., 1999, Vol. 3, pp. 161–181
Reprints available directly from the publisher
Photocopying permitted by license only

Published by license under
the Gordon and Breach Science
Publishers imprint.
Printed in Malaysia.

A Study of Variational Inequalities for Set-Valued Mappings

KOK-KEONG TAN[a], ENAYET TARAFDAR[b] and
GEORGE XIAN-ZHI YUAN[b,*]

[a] *Department of Mathematics, Statistics and CS, Dalhousie University, Halifax, Canada B3H 3J5;* [b] *Department of Mathematics, The University of Queensland, Brisbane, Australia 4072*

(Received 27 November 1997; Revised 14 February 1998)

In this paper, Ky Fan's KKM mapping principle is used to establish the existence of solutions for simultaneous variational inequalities. By applying our earlier results together with Fan–Glicksberg fixed point theorem, we prove some existence results for implicit variational inequalities and implicit quasi-variational inequalities for set-valued mappings which are either monotone or upper semi-continuous.

Keywords: Monotone pair; Simultaneous variational inequality; KKM mapping principle; Fan–Glicksberg fixed point theorem; Implicit quasi-variational inequality

1991 Mathematics Subject Classification: Primary 47H05, 47H10, 49J40; Secondary 52A07

1. INTRODUCTION

It is well known that variational inequality theory does not only have many important applications in partial differential equations such as free boundary problems and so on (e.g., see [2]), but it also has been successfully used in the study of operations research, mathematical programming and optimization theory (e.g., see [1]). Due to the development of set-valued analysis, the study of variational inequalities has been under much attention recently, for example, see Ding and Tan

---

* Corresponding author. E-mail: xzy@maths.uq.edu.au.

[3], Harker and Pang [8], Husain and Tarafdar [9], Granas [7], Karamolegos and Kravvaritis [11], Kravvaritis [12], Mosco [13], Shih and Tan [14–16], Tarafdar and Yuan [18] and many others whose names are not mentioned here. It is our purpose in this paper to study the existence of solutions for variational inequalities and quasi-variational inequalities of set-valued mappings either in simultaneous form or in implicit form as applications of Ky Fan's KKM-mapping principle in [5] and Fan–Glicksberg fixed point theorem (see [4,6]). Precisely, we shall establish the existence of solutions for simultaneous variational inequalities in Section 2. Then implicit variational inequality and implicit quasi-variational inequality in which set-valued mappings are monotone (resp., upper semicontinuous) will be investigated in Section 3 (resp., in Section 4). Our results either generalize or improve corresponding ones given in recent literature.

We shall denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of natural numbers, respectively. Let X be a set. We shall denote by 2^X the family of all non-empty subsets of X. If X is a topological space (resp., a non-empty subset of a topological vector space), we shall denote by $K(X)$ (resp., $KC(X)$) the family of all non-empty compact subsets of X (resp., the family of all non-empty compact and convex subsets of X). If X is a subset of a vector space E, then $\mathrm{co}X$ denotes the convex hull of X in E. Let $f: X \to 2^{\mathbb{R}}$ be a (set-valued) mapping. For each $x \in \mathrm{X}$, let $\inf f(x) := \inf\{z\colon z \in f(x)\}$. Let E^* be the dual space of a Hausdorff topological vector space E and X be a non-empty subset of E. We shall denote by $\langle w, x\rangle$ the dual pair between E^* and E for $w \in E^*$ and $x \in E$, and by $\mathrm{Re}\langle w, x\rangle$ the real part of the complex number $\langle w, x\rangle$. A mapping $T: X \to 2^{E^*}$ is said to be *monotone* if for each $x, y \in X$, $\mathrm{Re}\langle u - v, x - y\rangle \geq 0$ for all $u \in T(x)$ and $v \in T(y)$. Throughout this paper, E denotes a given Hausdorff topological vector space unless otherwise specified.

Let X be a non-empty convex subset of E, $f, g: X \times X \to 2^{\mathbb{R}}$, $f_1: X \to 2^{\mathbb{R}}$, $h: X \to \mathbb{R} \cup \{-\infty, +\infty\}$ and $H: X \to 2^{E^*}$. Then

(1) $\{f, g\}$ is said to be a *monotone pair* if for each $x, y \in X$, $u + w \geq 0$ for each $u \in f(x, y)$ and $w \in g(y, x)$; f is said to be *monotone* if the pair $\{f, f\}$ is monotone. In particular, when f is single-valued, we recover the notion of monotone pair reduces to that of a monotone mapping defined by Mosco [13] (see also [9,17]).

(2) f is said to be *hemicontinuous* if for each $x, y \in X$, the mapping $k : [0,1] \to 2^{\mathbb{R}}$ defined by $k(t) := f((1-t)x + ty, y)$ for all $t \in [0,1]$ is such that for each given $s \in \mathbb{R}$ with $f(x,y) \subset (s, +\infty)$, there exists $t_0 \in (0,1]$ such that $f((1-t)x + ty, y) \subset (s, +\infty)$ for all $t \in (0, t_0)$. We note that if f is single-valued, our definition of *hemicontinuity* reduces to the classical one given by Mosco [13], i.e., the function $t \mapsto f(x + t(y-x), y)$ from $[0,1]$ to $\mathbb{R}$ is lower semicontinuous as $t \downarrow 0$.

(3) f_1 is said to be *concave* if for each $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and non-negative $\lambda_1, \ldots, \lambda_n$ with $\sum_{i=1}^{n} \lambda_i = 1$ and for each $u \in f_1(\sum_{i=1}^{n} \lambda_i x_i)$, there exist $v_i \in f_1(x_i)$ for $i = 1, \ldots, n$ such that $u \geq \sum_{i=1}^{n} \lambda_i v_i$.

(4) h is said to be *lower semicontinuous* (resp., *upper semicontinuous*) if for each $\lambda \in \mathbb{R}$, the set $\{x \in X \colon h(x) \leq \lambda\}$ (resp., $\{x \in X \colon h(x) \geq \lambda\}$ is closed in X.

(5) H is said to be *w^*-demicontinuous* if for each $x \in X$, $\lambda \in \mathbb{R}$ and $z \in E$ with $H(x) \subset \{p \in E^* \colon \operatorname{Re}\langle p, z\rangle > \lambda\}$, there exists an open neighborhood N of x in X such that $H(y) \subset \{p \in E^* \colon \operatorname{Re}\langle p, z\rangle > \lambda\}$ for all $y \in N$.

Example 1.1 Let X be a non-empty convex subset of a Banach space $(E, \|\cdot\|)$ and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. We may assume its subdifferential $\partial\psi(x)$ exists for some $x \in X$ (e.g., if ψ is lower semicontinuous and convex by Theorem 5.4.3 of Aubin and Ekeland [1, p. 262]), i.e.,

$$\partial\psi(x) := \{p \in E^* \colon \psi(x) - \psi(z) \leq \operatorname{Re}\langle p, x - z\rangle \text{ for all } z \in X\}.$$

Then the mapping $A : X \to 2^{E^*}$ defined by $A(x) := \partial\psi(x)$ for each $x \in X$ is a monotone mapping. Define $f \colon X \times X \to 2^{\mathbb{R}}$ by $f(x,y) := \{\operatorname{Re}\langle u, x-y\rangle \colon u \in A(x)\}$ for each $x \in X$. It is clear that f is a monotone mapping. For each fixed positive real number β, define $g : X \times X \to 2^{\mathbb{R}}$ by

$$g_\beta(x,y) := \{\operatorname{Re}\langle u, x-y\rangle \colon u \in A(x)\} + \beta\|x-y\|\}$$

for each $(x,y) \in X \times X$. Then it is obvious that $\{f, g_\beta\}$ is a monotone pair.

Let X and Y be two topological spaces, $F \colon X \to 2^Y$ and $G \colon X \to 2^{\mathbb{R}}$. Then (a) F is said to be *upper semicontinuous* (in short, USC) (resp., *lower semicontinuous* (in short, LSC)) if for each $x \in X$ and for each open set U in Y with $F(x) \subset U$ (resp., $F(x) \cap U \neq \emptyset$), there is an open neighborhood N of x in X such that $F(y) \subset U$ (resp., $F(y) \cap U \neq \emptyset$) for all $y \in N$; (b) the graph of F is the set $\{(x,y) \in X \times Y : y \in F(x)\}$; and (c) G is lower (resp., upper) demicontinuous if for each $x \in X$ and $s \in \mathbb{R}$ with $G(x) \subset (s,\infty)$ (resp., $G(x) \subset (-\infty,s)$), there is an open neighborhood N of x in X such that $G(y) \subset (s,\infty)$ (resp., $G(y) \subset (-\infty,s)$) for all $y \in N$. We note that (i) if G is USC, then G is both lower demicontinuous and upper demicontinuous; (ii) when $X \subset E$, $Y = E^*$ and E^* is equipped with the w^*-topology, if F is USC, then F is w^*-demicontinuous; and (iii) when G is single-valued, the notions of lower demicontinuity (resp., upper demicontinuity) and LSC (resp., USC) coincide.

Example 1.2 Define $F:[0,\infty) \to 2^{\mathbb{R}}$ by $F(x) = \{x\}$ if $x \geq 1$ or $x = 0$ and $F(x) = [x, 1/x]$ if $0 < x < 1$. Define $G:(-\infty,0] \to 2^{\mathbb{R}}$ by $G(x) = \{x\}$ if $x \leq -1$ or $x = 0$ and $G(x) = [1/x, x]$ if $-1 < x < 0$. Then it is easy to see that (1) F is both lower demicontinuous and w^*-demicontinuous but not USC and not upper demicontinuous and (2) G is both upper demicontinuous and w^*-demicontinuous but not USC and not lower demicontinuous.

For each non-empty subset A of E and each $r > 0$, let $U(A;r) := \{w \in E^* \colon \sup_{x \in A} |\langle w, x\rangle| < r\}$. Let $\delta(E^*, E)$ be the topology on E^* generated by the family $\{U(A;r)$: A is a non-empty bounded subset of E and $r > 0\}$ as a base for the neighborhood system at 0. Then E^*, when equipped with the topology $\delta(E^*, E)$ becomes a locally convex topological vector space. The topology $\delta(E^*, E)$ is called the *strong topology* on E^*.

2. SIMULTANEOUS VARIATIONAL INEQUALITIES

Let X be a non-empty convex subset of E, $\psi: X \to \mathbb{R}$ and $f, g: X \times X \to \mathbb{R}$. One of the interesting problem is to find a point $x_0 \in X$ which simultaneously satisfies the following inequalities:

$$\psi(x_0) + f(x_0, y) \leq \psi(y) \quad \text{for all } y \in X \tag{I}$$

and

$$\psi(x_0) + g(x_0, y) \leq \psi(y) \quad \text{for all } y \in X; \tag{II}$$

i.e., to find a common solution for both variational inequalities (I) and (II) above. This is the so-called existence problem for solutions of simultaneous variational inequalities and this problem has been studied by Husain and Tarafdar [9]. In this section, we shall study the existence of solutions for the simultaneous variational inequality problem in the set-valued setting. We first need the following result.

LEMMA 2.1 *Let* $f, g\colon X \times X \to 2^{\mathbb{R}}$.

(1) *Suppose* $\{f, g\}$ *is a monotone pair and* $x, y \in X$. *If* $\inf f(x, y) \leq 0$, *then* $\inf g(y, x) \geq 0$.
(2) *Suppose* f *is hemicontinuous and for each* $x \in X$, $\inf f(x, x) \leq 0$ *and* $y \mapsto f(x, y)$ *is concave. If* $x_0 \in X$ *is such that* $\inf f(y, x_0) \geq 0$ *for all* $y \in X$, *then* $\inf f(x_0, y) \leq 0$ *for all* $y \in X$.

Proof (1) If $\inf f(x, y) \leq 0$, then for any $\epsilon > 0$, there exists $u \in f(x, y)$ such that $u < \epsilon$. As $\{f, g\}$ is a monotone pair, for each $w \in g(y, x)$, we have $u + w \geq 0$, so that $w \geq -u > -\varepsilon$. Thus $\inf g(y, x) \geq -\varepsilon$, which implies that $\inf g(y, x) \geq 0$ as $\varepsilon > 0$ is arbitrary.

(2) Assume that $\inf f(y, x_0) \geq 0$ for all $y \in X$, but $\inf f(x_0, y_0) > 0$ for some $y_0 \in X$. Let $s \in \mathbb{R}$ be such that $\inf f(x_0, y_0) > s > 0$. Let $U := (s, \infty)$. Then $f(x_0, y_0) \subset U$. Since f is hemicontinuous, there exists $t_0 \in (0, 1)$ such that $f(z_t, y_0) \subset U$ for all $t \in (0, t_0)$, where $z_t := (1 - t)x_0 + ty_0$ for each $t \in [0, 1]$. As $y \mapsto f(z_{t_0}, y)$ is concave, for each $u \in f(z_{t_0}, (1 - t)x_0 + ty_0)$, there exist $v_1 \in f(z_{t_0}, x_0)$ and $v_2 \in f(z_{t_0}, y_0)$ such that $u \geq (1 - t_0)v_1 + t_0v_2 > ((1 - t_0) \cdot 0 + t_0 \cdot 0s) = (1 - t_0)s$ as $\inf f(z_{t_0}, x_0) \geq 0$ by assumption. Hence $\inf f(z_{t_0}, z_{t_0}) = \inf f(z_{t_0}, (1 - t)x_0 + ty_0) \geq (1 - t_0)s > 0$, which contradicts the assumption that $\inf f(x, x) \leq 0$ for each $x \in X$.

As an application of Lemma 2.1, we have the following:

THEOREM 2.1 *Let* $f, g : X \times X \to 2^{\mathbb{R}}$ *be such that*

(i) $\{f, g\}$ *is a monotone pair*;
(ii) *for each* $x \in X$, $\inf f(x, x) \leq 0$ *and* $\inf g(x, x) \leq 0$;
(iii) f, g *are hemicontinuous*; *and*
(iv) *for each* $x \in X$, *the mappings* $y \mapsto f(x, y)$ *and* $y \mapsto g(x, y)$ *are concave.*

Then $x_0 \in X$ *is a solution of the following simultaneous variational inequalities*:

$$\begin{cases} \inf f(x_0, y) \leq 0 & \textit{for all } y \in X, \\ \inf g(x_0, y) \leq 0 & \textit{for all } y \in X, \end{cases}$$

if and only if that x_0 *is either a solution of the variational inequality*:

$$\inf f(x_0, y) \leq 0 \quad \textit{for all } y \in X \tag{III}$$

or, a solution of the following variational inequality:

$$\inf g(x_0, y) \leq 0 \quad \textit{for all } y \in X. \tag{IV}$$

Proof We only need to prove the sufficiency. Suppose $\inf f(x_0, y) \leq 0$ for all $y \in X$. By Lemma 2.1(1), $\inf g(y, x_0) \geq 0$ for all $y \in X$. By Lemma 2.1(2), $\inf g(x_0, y) \leq 0$ for all $y \in X$. Similarly, if $\inf g(x_0, y) \leq 0$ for all $y \in X$, then by Lemma 2.1, $\inf f(x_0, y) \leq 0$ for all $y \in X$.

As an immediate consequence of Theorem 2.1, we have the following result which is Theorem 2.1 of Husain and Tarafdar in [9]:

COROLLARY 2.1 *Let X be a non-empty convex subset of E and* $\psi: X \to \mathbb{R}$ *a convex function. Suppose that* $f, g: X \times X \to \mathbb{R}$ *satisfy*:

(1) $\{f, g\}$ *is a monotone pair*;
(2) *for each* $x \in X, f(x, x) = g(x, x) = 0$; *and*
(3) *for each fixed* $x \in X$, *both* $f(x, \cdot)$ *and* $g(x, \cdot)$ *are concave; and f and g are hemicontinuous.*

Then there exists $x_0 \in X$ *is a common solution of both* (I) *and* (II) *if and only if* x_0 *is either a solution of* (I) *or a solution of* (II).

Proof Define $f, g: X \times X \to \mathbb{R}$ by

$$\hat{f}(x, y) := \psi(x) + f(x, y) - \psi(y)$$

and

$$\hat{g}(x, y) := \psi(x) + g(x, y) - \psi(y)$$

for each $(x,y) \in X \times X$. Applying Theorem 2.1 to $\hat{f}$ and $\hat{g}$, the conclusion follows.

In what follows, we shall prove some sufficient conditions which guarantee that either inequality (III) or (IV) has a solution. In order to do so, we need the following:

LEMMA 2.2 *Let* $g: X \to 2^{\mathbb{R}}$ *be lower demicontinuous. Then the mapping* $G: X \to \mathbb{R} \cup \{-\infty\}$ *defined by* $G(x) := \inf g(x)$ *for each* $x \in X$ *is LSC.*

Proof Let $\lambda \in \mathbb{R}$ be given. Suppose $\{x_\alpha\}_{\alpha \in \Gamma}$ is a net in X and $x_0 \in X$ such that $\inf g(x_\alpha) \leq \lambda$ for all $\alpha \in \Gamma$ and $x_\alpha \to x_0$. Suppose $\inf g(x_0) > \lambda$. Choose any $s \in \mathbb{R}$ such that $\inf g(x_0) > s > \lambda$. Let $U := (s, \infty)$, then $g(x_0) \subset U$. Since g is lower demicontinuous, there exists an open neighborhood N of x_0 in X such that $g(x) \subset U$ for all $x \in N$. But then there exists $\alpha_0 \in \Gamma$ such that $x_\alpha \in N$ for all $\alpha \geq \alpha_0$. Hence $g(x_{\alpha_0}) \subset U$ so that $\inf g(x_{\alpha_0}) \geq s > \lambda$ which is a contradiction. Therefore we must have $\inf g(x_0) \leq \lambda$. This shows that the set $\{x \in X\colon \inf g(x) \leq \lambda\}$ is closed in X. Thus G is lower semicontinuous.

Let X be a non-empty subset of a vector space V and $F: X \to 2^V$. We recall that F is said to be a *KKM mapping* (e.g., see [5]) if $\mathrm{co}\{x_i\colon i = 1, \ldots, n\} \subset \bigcup_{i=1}^{n} F(x_i)$ for each $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

We shall also need the following simple observation:

LEMMA 2.3 *Let V be a vector space and X a non-empty convex subset of V. Suppose $f: X \times X \to 2^{\mathbb{R}}$ is such that*

(i) *for each $x \in X$, $\inf f(x, x) \leq 0$;*
(ii) *for eachx $\in X$, $y \mapsto f(x, y)$ is concave.*

Define $F: X \to 2^X$ by $F(w) = \{x \in X\colon \inf f(x, w) \leq 0\}$ for each $w \in X$. Then F is a KKM-mapping.

Proof Suppose not, then there exist $n \in \mathbb{N}$, $w_1, \ldots, w_n \in X$ and $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $\sum_{i=1}^{n} \lambda_i w_i \notin \bigcup_{j=1}^{n} F(w_j)$. It follows that $\inf f(\sum_{i=1}^{n} \lambda_i w_i, w_j) > 0$ for all $j = 1, \ldots, n$. Let $s \in \mathbb{R}$ be such that $\min_{1 \leq j \leq n} \inf f(\sum_{i=1}^{n} \lambda_i w_i, w_j) > s > 0$. Since $y \mapsto f(\sum_{i=1}^{n} \lambda_i w_i, y)$ is concave by (ii), for each $u \in f(\sum_{i=1}^{n} \lambda_i w_i, \sum_{j=1}^{n} \lambda_j w_j)$, there exist $v_j \in f(\sum_{i=1}^{n} \lambda_i w_i, w_j)$ for $j = 1, \ldots, n$ such that $u \geq \sum_{j=1}^{n} \lambda_j v_j > s$. Thus $\inf f(\sum_{i=1}^{n} \lambda_i w_i, \sum_{j=1}^{n} \lambda_j w_j) \geq s > 0$, which contradicts (i). Hence F must be a KKM-mapping.

THEOREM 2.2 *Let X be a non-empty closed convex subset of E. Suppose $f: X \times X \to 2^{\mathbb{R}}$ is such that*

(i) *for each $x \in X$, $\inf f(x, x) \leq 0$;*
(ii) *for each $x \in X$, $y \mapsto f(x, y)$ is concave;*
(iii) *for each $y \in X$, $x \mapsto f(x, y)$ is lower demicontinuous; and*
(iv) *there exist a non-empty compact subset B of X and $w_0 \in B$ such that*

$$\inf f(x, w_0) > 0 \quad \textit{for all } x \in X \setminus B.$$

Then the set $S := \{x \in X\colon \inf f(x,w) \le 0$ *for all* $w \in X\}$ *is a non-empty compact subset of* B.

Proof Define $F : X \to 2^X$ by

$$F(w) := \{x \in X\colon \inf f(x,w) \le 0\}$$

for each $w \in X$. By (i), $F(w) \neq \emptyset$ for all $w \in X$, so that F is well defined. By (iii) and Lemma 2.2, for each $w \in X$, the set $F(w)$ is closed in X. By (iv), $F(w_0)$ is a closed subset of B so that $F(w_0)$ is compact. By (i), (ii) and Lemma 2.3, F is a KKM-mapping. By Ky Fan's KKM-mapping principle [5, Lemma 1], $\bigcap_{w \in X} F(w) \neq \emptyset$. Thus $S = \bigcap_{w \in X} F(w)$ is a non-empty compact subset of B.

LEMMA 2.4 *Let* X *be a non-empty closed convex subset of* E. *Suppose* $g : X \to 2^{\mathbb{R}}$ *and let* $W := \{x \in X\colon \inf g(x) \ge 0\}$. *Then* (a) W *is closed in* X *if* g *is LSC and* (b) W *is convex if* g *is concave.*

Proof (a) If W were not closed in X, then there would exist a net $\{x_\alpha\}_{\alpha \in \Gamma}$ in X and $x_0 \in X$ such that $x_\alpha \to x_0$, and $\inf g(x_\alpha) \ge 0$ for all $\alpha \in \Gamma$ but $\inf g(x_0) < 0$. Let $s \in \mathbb{R}$ be such that $\inf g(x_0) < s < 0$ and $U := (-\infty, s)$. Then $g(x_0) \cap U \neq \emptyset$. Since g is LSC, there exists an open neighborhood N of $x_0 \in X$ such that $g(x) \cap U \neq \emptyset$ for all $x \in N$. As $x_\alpha \to x$, there exists $\alpha_0 \in \Gamma$ such that $x_\alpha \in N$ for all $\alpha \ge \alpha_0$. Thus $g(x_{\alpha_0}) \cap U \neq \emptyset$ so that $\inf g(x_{\alpha_0}) < s < 0$, which is a contradiction. Thus W is closed in X.

(b) Suppose $x, y \in W$ and $\lambda \in (0,1)$, then $\inf g(x) \ge 0$ and $\inf g(y) \ge 0$. Since g is concave, for each $u \in g(\lambda x + (1-\lambda)y)$, there exist $v_1 \in g(x)$ and $v_2 \in g(y)$ such that $u \ge \lambda v_1 + (1-\lambda)v_2 \ge 0$. Thus $\inf g(\lambda x + (1-\lambda)y) \ge 0$ and we have $\lambda x + (1-\lambda)y \in W$. Therefore W is convex.

THEOREM 2.3 *Let* X *be a non-empty closed convex subset of* E *and* $f : X \times X \to 2^{\mathbb{R}}$ *be such that*

(i) *for each* $x \in X$, $\inf f(x,x) \le 0$;
(ii) *for each* $x \in X$, $y \mapsto f(x,y)$ *is concave and LSC*;
(iii) f *is hemicontinuous*;
(iv) *there exist a non-empty compact* $B \subset X$ *and* $w_0 \in B$ *such that*

$$\inf f(x, w_0) > 0 \quad \textit{for all } x \in X \setminus B;$$

(v) f *is monotone.*

Then the set $S := \{x \in X\colon \inf f(x,w) \leq 0$ *for all* $w \in X\}$ *is a non-empty compact convex subset of* B.

Proof Define $F, G, H\colon X \to 2^X$ by

$$\begin{aligned} F(w) &= \{x \in X\colon\ \inf f(x,w) \leq 0\}, \\ G(w) &= cl_X F(w), \\ H(w) &= \{x \in X\colon\ \inf f(w,x) \geq 0\}, \end{aligned}$$

for each $w \in X$. Then by (i), (ii) and Lemma 2.3, F is a KKM-mapping so that G is also a KKM-mapping. Note that by (iv), $F(w_0) \subset B$ so that $G(w_0) \subset B$ and $G(w_0)$ is compact.

Again by Ky Fan's KKM-mapping principle, $\bigcap_{w \in X} G(w) \neq \emptyset$. By (ii) and Lemma 2.4(a), for each $w \in X$, $H(w)$ is closed and convex.

To complete the proof, it is sufficient to show that

$$S = \bigcap_{w \in X} F(w) = \bigcap_{w \in X} G(w) = \bigcap_{w \in X} H(w).$$

Indeed, if $w \in X$ and $x \in F(w)$, then $\inf f(x,w) \leq 0$ so that by (v) and Lemma 2.1(1), $\inf f(w,x) \geq 0$. It follows that $x \in H(w)$. Hence $F(w) \subset H(w)$ so that $G(w) \subset H(w)$. Therefore $\bigcap_{w \in X} F(w) \subset \bigcap_{w \in X} G(w) \subset \bigcap_{w \in X} H(w)$.

On the other hand, if $x \in \bigcap_{w \in X} H(w)$, then $\inf f(w,x) \geq 0$ for all $w \in X$. Thus by (i)–(iii), (v) and Lemma 2.1(2), we have $\inf f(x,w) \leq 0$ for all $w \in X$. Thus $x \in \bigcap_{w \in X} F(w)$. Therefore $\bigcap_{w \in X} H(w) \subset \bigcap_{w \in X} F(w)$. Hence we have $S = \bigcap_{w \in X} F(w) = \bigcap_{w \in X} G(w) = \bigcap_{w \in X} H(w)$.

3. IMPLICIT VARIATIONAL INEQUALITIES – THE MONOTONE CASE

Let C be a non-empty subset of E and C_1 a non-empty subset of C. Suppose $f\colon C_1 \times C \times C \to \mathbb{R}$ and $g\colon C_1 \times C \to \mathbb{R}$ are such that $f(u,v,v) \geq 0$ for all $u \in C_1$ and $v \in C$. Mosco [13] had investigated the following so called *implicit variational inequality problems*: Find a vector $v \in C_1$ such that

$$g(v,v) \leq f(v,v,w) + g(v,w) \quad \text{for all } w \in C. \tag{V}$$

In this section, it is our goal to study the existence of solutions for implicit variational inequality and implicit quasi-variational inequalities which are variant forms of the implicit variational inequality (V) above. Indeed, as applications of Theorem 2.3 and by combining Fan–Glicksberg fixed point theorem, we shall provide some sufficient conditions to guarantee the existence of variational and quasi-variational inequalities in their implicit forms, and in which the set-valued mappings are monotone.

As an application of Theorem 2.3, we have the following variational inequality:

THEOREM 3.1 *Let* X *be a non-empty closed convex subset of* E *and* $T: X \to 2^{E^*}$ *be monotone such that*

(i) *for each* $x \in X$, $T(x)$ *is* w^**-compact;*

(ii) T *is* w^**-USC from line segments in* X *to the weak***-topology* $\sigma(E^*, E)$ *on* E^**; and*

(iii) *there exists a non-empty weakly compact subset* B *of* X *and* $w_0 \in B$ *such that*

$$\inf_{u \in T(x)} \operatorname{Re}\langle u, x - w_0\rangle > 0 \quad \textit{for all } x \in X \setminus B.$$

Then the set $S := \{y \in X: \inf_{w \in Ty} \operatorname{Re}\langle w, y - x\rangle \leq 0$ *for all* $x \in X\}$ *is a non-empty weakly compact convex subset of* B.

Proof Define $f: X \times X \to 2^{\mathbb{R}}$ by

$$f(x, y) = \{\operatorname{Re}\langle u, x - y\rangle : \; u \in Tx\}$$

for each $x, y \in X$. Then we have

(1) f is monotone as T is monotone.

(2) For each $x, y \in X$, $f(x, y)$ is a non-empty compact subset of $\mathbb{R}$.

(3) For each $x \in X$, $f(x, x) = \{0\}$ so that $\inf f(x, x) \leq 0$.

(4) For each $x \in X$, the mapping $y \mapsto f(x, y)$ is concave. Indeed, for each $n \in \mathbb{N}$, $y_1, \ldots, y_n \in X$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$ and for each $s \in f(x, \sum_{i=1}^n \lambda_i y_i)$, there exists $u \in Tx$ such that $s = \operatorname{Re}\langle u, x - \sum_{i=1}^n \lambda_i y_i\rangle$. But then $\operatorname{Re}\langle u, x - y_i\rangle \in f(x, y_i)$ for each $i = 1, 2, \ldots, n$ and

$$s = \operatorname{Re}\left\langle u, x - \sum_{i=1}^{n} \lambda_i y_i \right\rangle = \sum_{i=1}^{n} \operatorname{Re}\langle u, x - y_i\rangle.$$

Therefore $y \mapsto f(x, y)$ is concave.

(5) For each $x \in X$, the mapping $y \mapsto f(x,y)$ is weakly LSC; i.e., the mapping $y \mapsto f(x,y)$ is LSC when X is equipped with the relative weak topology. Indeed, let $y_0 \in X$ and $U \subset \mathbb{R}$ be open such that $f(x,y_0) \cap U \neq \emptyset$. Then there exists $u \in Tx$ such that $\mathrm{Re}\langle u, x-y_0\rangle \in U$. For each fixed $x \in X$ and $u \in T(x)$, as $y \mapsto \mathrm{Re}\langle u, x-y\rangle$ is weakly continuous, there exists a weakly open neighborhood N of y_0 in X such that $\mathrm{Re}\langle u, x-y\rangle \in U$ for all $y \in N$, so that $f(x,y) \cap U \neq \emptyset$ for all $y \in N$. Thus $y \mapsto f(x,y)$ is weakly LSC.

(6) f is hemicontinuous. Indeed, fix any $x, y \in X$ and define $k: [0,1] \to X$ by $k(t) = f((1-t)x + ty, y)$ for each $t \in [0,1]$. Let $U = (s, \infty)$ where $s \in \mathbb{R}$ be such that $f(x,y) \subset U$. Note that $f(x,y)$ is compact as Tx is weak*-compact. Let $r_0 = \inf f(x,y)$. Then $r_0 > s$. Set $r := (r_0 + s)/2$, $t_1 := (r-s)/r$ and $V := (r, \infty)$. Then $t_1 \in (0,1)$, $f(x,y) \subset V$ and $(1-t)V \subset U$ for all $t \in (0, t_1)$. Let $W = \{w \in E^*: \mathrm{Re}\langle w, x-y\rangle > r\}$, then W is w^*-open and $T(x) \subset W$. By (ii), there exists $t_0 \in (0, t_1)$ such that $T((1-t)x + ty) \subset W$ for all $t \in (0, t_0)$. Thus for each $u \in T((1-t)x + ty)$ and $t \in (0, t_0)$, we have

$$U \supset (1-t)V \supset (1-t)\ \mathrm{Re}\langle u, x-y\rangle = \mathrm{Re}\langle u, ((1-t)x + ty) - y\rangle.$$

Therefore $U \supset f((1-t)x + ty, y)$ for all $t \in (0, t_0)$. Hence f is hemicontinuous.

(7) By (iii), there exists a non-empty weakly compact subset B of X and $w_0 \in B$ such that

$$\inf f(x, w_0) = \inf_{u \in Tx} \mathrm{Re}\langle u, x - w_0\rangle > 0,$$

for all $x \in X \setminus B$.

Now equip E with weak topology, then all hypotheses of Theorem 2.3 are satisfied. Thus

$$\begin{aligned} S &= \{x \in X: \inf f(x,w) \le 0 \ \text{ for all } w \in X\} \\ &= \left\{x \in X: \inf_{u \in Tx} \mathrm{Re}\langle u, x - w\rangle \le 0 \text{ for all } w \in X\right\} \end{aligned}$$

is a non-empty weakly compact convex subset of B.

As an application of Theorem 3.1, we have the following result which is Theorem 1 of Shih and Tan [16].

COROLLARY 3.1 *Let $(E, \|\cdot\|)$ be a reflexive Banach space and X a non-empty closed convex subset of E. Suppose $T: X \to 2^{E^*}$ is monotone such that each $T(x)$ is a weakly compact subset of E^* and T is upper semicontinuous from line segments in X to the weak topology of E^*. Assume that there exists $x_0 \in X$ such that*

$$\lim_{\substack{\|y\|\to\infty \\ y\in X}} \inf_{w\in Ty} \operatorname{Re}\langle w, y - x_0\rangle > 0. \tag{VI}$$

Then there exists $\hat{y} \in X$ such that

$$\inf_{w\in T\hat{y}} \operatorname{Re}\langle w, \hat{y} - x\rangle \leq 0 \quad \text{for all } x \in X.$$

Proof By (VI), there exist $M > 0$ and $R > 0$ with $\|x_0\| \leq R$ such that $\inf_{w\in Ty} \operatorname{Re}\langle w, y - x_0\rangle > M$ for all $y \in X$ with $\|y\| > R$. Let $B := \{x \in X: \|x\| \leq R\}$. Then B is a non-empty weakly compact (and convex) subset of X such that $\inf_{w\in Ty}\operatorname{Re}\langle w, y - x_0\rangle > 0$ for all $x \in X \setminus B$. It is easy to see that all hypotheses of Theorem 3.1 are satisfied so that the conclusion follows.

We note that under the assumptions in Corollary 3.1, the conditions "T is USC from line segments in X to the weak topology of E" and "T is w^*-demicontinuous from line segments in X to the w^*-topology of E" are equivalent (see e.g., [1, Theorem 10, p. 128]).

As a second application of Theorem 2.3, we have the following implicit variational inequality:

THEOREM 3.2 *Let E be locally convex, X be a non-empty compact convex subset of E and $g: X \times X \times X \to K(\mathbb{R})$ be such that*

- (i) *For each $u, x \in X$, $\inf g(u, x, x) \leq 0$.*
- (ii) *For each $u, x \in X$, the mapping $y \mapsto g(u, x, y)$ is concave.*
- (iii) *For each $u \in X$, the mapping $(x, y) \mapsto g(u, x, y)$ is monotone and hemicontinuous.*
- (iv) *For each $x \in X$, the mapping $(u, y) \mapsto g(u, x, y)$ is LSC.*

Then the set $W := \{u \in X: \inf f(u, u, w) \leq 0$ for all $w \in X\}$ is a non-empty compact subset of X.

Proof For each fixed $u \in X$, define $f_u: X \times X \to 2^{\mathbb{R}}$ by

$$f_u(x, y) = g(u, x, y)$$

for each $x, y \in X$. Then f_u satisfies all hypotheses in Theorem 2.3 so that the set

$$\begin{aligned} S(u) &= \{x \in X\colon \inf f_u(x, w) \leq 0 \text{ for all } w \in X\} \\ &= \{x \in X\colon \inf g(u, x, w) \leq 0 \text{ for all } w \in X\} \end{aligned}$$

is a non-empty compact convex subset of X and S is thus a mapping from X to $K(X)$. We shall show that S has a closed graph. Indeed, let $(x_\alpha)_{\alpha \in \Gamma}$ be a net in X and $y_\alpha \in S(x_\alpha)$ for all $\alpha \in \Gamma$ such that $x_\alpha \rightarrow x_0 \in X$ and $y_\alpha \rightarrow y_0 \in X$. Note that for each $\alpha \in \Gamma$, $\inf g(x_\alpha, y_\alpha, w) \leq 0$ for all $w \in X$. Let $w \in X$ be given and fix an arbitrary $\alpha \in \Gamma$. Since $g(x_\alpha, y_\alpha, w)$ is compact, there exists $u_\alpha \in g(x_\alpha, y_\alpha, w)$ such that $u_\alpha = \inf g(x_\alpha, y_\alpha, w) \leq 0$. Since $(y, z) \mapsto g(x_\alpha, y, z)$ is monotone, for each $v \in g(x_\alpha, w, y_\alpha)$, we have $u_\alpha + v \geq 0$ so that $v \geq -u_\alpha \geq 0$. Thus $\inf g(x_\alpha, w, y_\alpha) \geq 0$. As $w \in X$ is arbitrarily given, $\inf g(x_\alpha, w, y_\alpha) \geq 0$ for all $w \in X$. By (iv) and Lemma 2.4, for each $w \in X$, the set $\{(x, y) \in X \times X\colon \inf g(x, w, y) \geq 0\}$ is closed. It follows that $\inf g(x_0, w, y_0) \geq 0$ for all $w \in X$. By Lemma 2.1(2), $\inf g(x_0, y_0, w) \leq 0$ for all $w \in X$ which shows that $y_0 \in S(x_0)$. Hence S has a closed graph so that S is upper semicontinuous. Now by Fan–Glicksberg fixed point theorem (e.g., see [4] or [6]), there exists $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$, i.e., $\inf g(\hat{x}, \hat{x}, w) \leq 0$ for all $w \in X$ so that $W \neq \emptyset$. To complete the proof, it remains to show that W is a closed subset of X. Suppose $\{u_\alpha\}_{\alpha \in \Gamma}$ is a net in W such that $u_\alpha \rightarrow u_0 \in X$. Then $\inf g(u_\alpha, u_\alpha, w) \leq 0$ for all $w \in X$. Now by the same argument as above (with $y_\alpha = x_\alpha = u_\alpha$ for all $\alpha \in \Gamma$ and $x_0 = y_0 = u_0$), $\inf g(u_0, u_0, w) \leq 0$ for all $w \in X$. Thus $u_0 \in S(u_0)$ so that $u_0 \in W$. Therefore W is closed in X.

As an application of Theorem 3.2, we have the following implicit quasi-variational inequality:

THEOREM 3.3 *Let E be locally convex, X be a non-empty compact convex subset of E, $S: X \rightarrow KC(X)$ be continuous and $g: X \times X \times X \rightarrow 2^{\mathbb{R}}$ be such that*

- (i) *For each $u, x \in X$, $\inf g(u, x, x) \leq 0$.*
- (ii) *For each $u, x \in X$, the mapping $y \mapsto g(u, x, y)$ is concave and for each $y \in X$, the mapping $u \mapsto g(u, y, u)$ is concave.*
- (iii) *For each $u \in X$, the mapping $(x, y) \mapsto g(u, x, y)$ is monotone and hemicontinuous.*

(iv) *For each* $x \in X$, *the mapping* $(u, y) \mapsto g(u, x, y)$ *is LSC.*
(v) *The mapping* $(u, x) \mapsto g(u, x, u)$ is *LSC.*

Then (a) *there exists* $\hat{y} \in X$ *such that*

$$\begin{cases} \hat{y} \in S(\hat{y}) \\ \inf g(\hat{y}, \hat{y}, w) \leq 0 \quad \textit{for all } w \in S(\hat{y}) \end{cases}$$

and (b) *the set*

$$\{y \in X\colon y \in S(y) \textit{ and } \inf g(y, y, w) \leq 0 \textit{ for all } w \in S(y)\}$$

is a non-empty compact subset of X.

Proof (a) Define $F: X \to KC(X)$ by

$$F(u) = \{y \in S(u)\colon \inf g(y, y, w) \leq 0 \text{ for all } w \in S(u)\}$$

for each $u \in X$. Let $u \in X$ be given. By Theorem 3.2, $F(u)$ is non-empty and compact. We shall now show that $F(u)$ is also convex. Let $x, y \in F(u)$ and $\lambda \in (0, 1)$ be given. As $x, y \in S(u)$ and $S(u)$ is convex, $\lambda x + (1 - \lambda)y \in S(u)$. Since $\inf g(x, x, w) \leq 0$ and $\inf g(y, y, w) \leq 0$ for all $w \in S(u)$, $\inf g(x, w, x) \geq 0$ and $\inf g(y, w, y) \geq 0$ for all $w \in S(u)$ by (iii) and Lemma 2.1(1). It follows that $\inf g(\lambda x + (1 - \lambda)y, w, \lambda x + (1 - \lambda)y) \geq 0$ for all $w \in S(u)$ by (ii) and Lemma 2.4. By Lemma 2.1(2), $\inf g(\lambda x + (1 - \lambda)y, \lambda x + (1 - \lambda)y, w) \leq 0$ for all $w \in S(u)$. Thus $\lambda x + (1 - \lambda)y \in F(u)$. Hence $F(u)$ is also convex. This shows that F is well defined.

Now we shall show that F has a closed graph. Indeed, let $((x_\alpha, y_\alpha))_{\alpha \in \Gamma}$ be a net in $X \times X$ and $(x_0, y_0) \in X \times X$ be such that $(x_\alpha, y_\alpha) \to (x_0, y_0)$ and $y_\alpha \in F(x_\alpha)$ for all $\alpha \in \Gamma$. Since $y_\alpha \in S(x_\alpha)$ for each $\alpha \in \Gamma$, $y_0 \in S(x_0)$ as S is USC. Now fix an arbitrary $w_0 \in S(x_0)$. Since S is LSC, there is a net $(w_\alpha)_{\alpha \in \Gamma}$ in X with $w_\alpha \in S(x_\alpha)$ for all $\alpha \in \Gamma$ such that $w_\alpha \to w_0$. Since $\inf g(y_\alpha, y_\alpha, w_\alpha) \leq 0$ for all $\alpha \in \Gamma$, by (iii) and Lemma 2.1(1), we have $\inf g(y_\alpha, w_\alpha, y_\alpha) \geq 0$ for all $\alpha \in \Gamma$. By (v) and Lemma 2.4, $\inf g(y_0, w_0, y_0) \geq 0$. Since $w_0 \in S(x_0)$ is arbitrary, we have $\inf g\ (y_0, w, y_0) \geq 0$ for all $w \in S(x_0)$. By (ii), (iii) and Lemma 2.1(2), it follows that $\inf g(y_0, y_0, w) \leq 0$ for all $w \in S(x_0)$ so that $y_0 \in F(x_0)$. Thus F has a closed graph and hence F is USC.

By Fan–Glicksberg fixed point theorem again, there exists $\hat{y} \in X$ such that $\hat{y} \in F(\hat{y})$; i.e.,

$$\begin{cases} \hat{y} \in S(\hat{y}), \\ \inf g(\hat{y}, \hat{y}, w) \leq 0 \quad \text{for all } w \in S(\hat{y}). \end{cases}$$

(b) By (a), the set $\{y \in \mathrm{X}: \ y \in S(y)$ and $\inf g(y, y, w) \leq 0$ for all $w \in S(y)\}$ is non-empty; it is also compact by following the same argument as in the proof of Theorem 3.2.

We would like to remark that our results in this section unify and generalize corresponding results in the literature given by Aubin and Ekeland [1], Baiocchi and Capelo [2], Harker and Pang [8], Husain and Tarafdar [9], Mosco [13], and Shih and Tan [14,16].

4. IMPLICIT VARIATIONAL INEQUALITIES – THE USC CASE

Parallel to the ideas used in Section 3 and as application of Theorem 2.2 instead of Theorem 2.3, we can also study the existence of solutions for implicit variational and implicit quasi-variational inequalities in which real set-valued mappings are USC instead of being monotone. First we have the following implicit variational inequality:

THEOREM 4.1 *Let E be locally convex, X be a non-empty compact convex subset of E and $g: X \times X \times X \to K(\mathbb{R})$ be such that*

(i) *For each $u \in X$, $\inf g(u, x, x) \leq 0$.*
(ii) *For each $u, x \in X$, the mapping $y \mapsto g(u, x, y)$ is concave.*
(iii) *For each $y \in X$, the mapping $(u, x) \mapsto g(u, x, y)$ is lower demicontinuous.*

Then the set $W := \{u \in X: \inf f(u, u, w) \leq 0$ for all $w \in X\}$ is a non-empty compact subset of X.

Proof For each fixed $u \in X$, define $f_u: X \times X \to 2^{\mathbb{R}}$ by

$$f_u(x, y) = g(u, x, y)$$

for each $x, y \in X$. Then f_u satisfies all hypotheses in Theorem 2.2 so that the set

$$\begin{aligned} S(u) &= \{x \in X: \ \inf f_u(x, w) \leq 0 \text{ for all } w \in X\} \\ &= \{x \in X: \ \inf g(u, x, w) \leq 0 \text{ for all } w \in X\} \end{aligned}$$

is a non-empty compact convex subset of X and S is thus a mapping from X to $K(X)$. We shall now show that S has a closed graph. Indeed, let $(x_\alpha)_{\alpha\in\Gamma}$ be a net in X and $y_\alpha \in S(x_\alpha)$ for each $\alpha\in\Gamma$ such that $x_\alpha \to x_0 \in X$ and $y_\alpha \to y_0 \in X$. Note that for each $\alpha \in T$, $\inf g(x_\alpha, y_\alpha, w) \leq 0$ for all $w \in X$. By (iii) and Lemma 2.2, for each $w\in X$, the mapping $(x,y)\mapsto \inf g(x,y,w)$ is LSC. It follows that $\inf g(x_0, y_0, w) \leq 0$ for all $w \in X$ so that $y_0 \in S(x_0)$. Thus S has a closed graph and hence is USC. Now by Fan–Glicksberg fixed point theorem, there exists $\hat{x}\in X$ such that $\hat{x}\in S(\hat{x})$, i.e., $\inf g(\hat{x},\hat{x},w)\leq 0$ for all $w\in X$. This shows that $\hat{x}\in W$ so that the set W is non-empty. Moreover, by (iii) and Lemma 2.2, the set W is closed in X and is hence compact.

So far, we have established some existence theorems of solutions for implicit variational inequalities and quasi-variational inequalities as applications of Fan–Glicksberg fixed point theorem. However, we can also study variational inequalities as applications of existence theorems of equilibria for generalized games (resp., abstract economics). Some results in this direction have been given by Tarafdar and Yuan [18]. In what follows, we shall use that method to prove an implicit quasi-variational inequality (Theorem 4.2 below). We need the following result which is a special case of Theorem 5 of Tulcea [19] (See also Yuan [20]):

LEMMA 4.1 *Let E be locally convex, X be a non-empty compact convex subset of E, $A: X\to KC(X)$ be* USC *and $P: X\to 2^X\cup\{\emptyset\}$ be such that*

(i) *For each $y\in X$, the set $P^{-1}(y) := \{x\in X: y\in P(x)\}$ is open in X.*
(ii) *For each $x\in X$, $x\notin \mathrm{co}P(x)$.*
(iii) *The set $\{x\in X: A(x)\cap P(x)\neq\emptyset\}$ is open in X.*

Then there exists $\hat{x}\in X$ such that $\hat{x}\in A(\hat{x})$ and $A(\hat{x})\cap P(\hat{x})=\emptyset$.

We shall now apply Lemma 4.1 to prove the following implicit quasi-variational inequality:

THEOREM 4.2 *Let E be locally convex, X be a non-empty compact convex subset of E, $S: X\to KC(X)$ be continuous (i.e., S is both LSC and USC on X) and $f: X\times X\to 2^{\mathbb{R}}$ be lower demicontinuous such that*

(i) *For each $x\in X$, $\inf f(x,x)\leq 0$.*
(ii) *For each $x\in X$, $y\mapsto f(x,y)$ is concave.*

Then there exists $u \in X$ *such that*

$$\begin{cases} u \in S(u), \\ \inf f(u, w) \leq 0 \quad \textit{for all } w \in S(u). \end{cases}$$

Proof Define $P: X \to 2^X \cup \{\emptyset\}$ by

$$P(x) = \{y \in X\colon \ \inf f(x, y) > 0\}$$

for each $x \in X$. We then have:

(1) For each $y \in X$, the set $P^{-1}(y)$ is open in X by Lemma 2.2 as $x \mapsto f(x, y)$ is lower demicontinuous.

(2) For each $x \in X$, $x \notin \operatorname{co} P(x)$. Indeed, suppose there exists $x_0 \in X$ such that $x_0 \in \operatorname{co} P(x_0)$. Let $y_1, \ldots, y_n \in P(x_0)$, $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ be such that $x_0 = \sum_{i=1}^n \lambda_i y_i$. As $y \mapsto f(x_0, y)$ is concave, for each $u \in f(x_0, x_0) = f(x_0, \sum_{i=1}^n \lambda_i y_i)$, there exist $u_i \in f(x_0, y_i)$ for $i = 1, \ldots, n$ such that $u \geq \sum_{i=1}^n \lambda_i u_i \geq \sum_{i=1}^n \lambda_i \inf f(x_0, y_i)$. Then $\inf f(x_0, x_0) \geq \sum_{i=1}^n \lambda_i \inf f(x_0, y_i) > 0$, which contradicts (i). Hence $x \notin \operatorname{co} P(x)$ for all $x \in X$.

(3) The set $\{x \in X\colon S(x) \cap P(x) \neq \emptyset\}$ is open in X. Indeed, suppose $S(x_0) \cap P(x_0) \neq \emptyset$. Let $y_0 \in S(x_0) \cap P(x_0)$. Then $y_0 \in S(x_0)$ and $\inf f(x_0, y_0) > 0$. Let $s \in \mathbb{R}$ be such that $\inf f(x_0, y_0) > s > 0$ and $U := (s, \infty)$. Since f is lower demicontinuous and $f(x_0, y_0) \subset U$, there exist open neighborhoods N_1 of x_0 in X and V of y_0 in X such that $f(x, y) \subset U$ for all $(x, y) \in N_1 \times V$. Since $V \cap S(x_0) \neq \emptyset$ and S is LSC, there exists an open neighborhood N_2 of x_0 in X such that $V \cap S(x) \neq \emptyset$ for all $x \in N_2$. Let $N := N_1 \cap N_2$. Then N is an open neighborhood of x_0 in X. Suppose $x \in N$ is given. As $V \cap S(x) \neq \emptyset$, we may take any $y \in V \cap S(x)$; then $f(x, y) \subset U$ so that $\inf f(x, y) \geq s > 0$ and hence $y \in P(x) \cap S(x)$. Thus $S(x) \cap P(x) \neq \emptyset$ for all $x \in N$. Therefore the set $\{x \in X\colon S(x) \cap P(x) \neq \emptyset\}$ is open in X.

Now by Lemma 4.1, there exists $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and $S(\hat{y}) \cap P(\hat{y}) = \emptyset$, i.e.,

$$\begin{cases} \hat{y} \in S(\hat{y}), \\ \inf f(\hat{y}, w) \leq 0 \quad \text{for all } w \in S(\hat{y}). \end{cases}$$

LEMMA 4.2 *Let X be a non-empty and bounded subset of E and T: $X \to K(E^*)$ be USC, where E^* is equipped with the strong topology. Define $f: X \times X \to 2^{\mathbb{R}}$ by*

$$f(x,y) = \{\mathrm{Re}\langle u, x-y\rangle:\ u \in Tx\} \quad \textit{for all } x, y \in X.$$

Then f is USC.

Proof Let $x_0, y_0 \in X$ and $U \subset \mathbb{R}$ be open such that

$$\{\mathrm{Re}\langle u, x_0 - y_0\rangle:\ u \in Tx_0\} = f(x_0, y_0) \subset U.$$

Note that the mapping $(u,z) \mapsto \langle u,z\rangle$ is (jointly) continuous on $(X-X) \times E^*$. Thus for each $u \in Tx_0$, there exist a strongly open neighborhood V_u of u and an open neighborhood M_u of x_0 in X and an open neighborhood N_u of y_0 in X such that

$$\{\mathrm{Re}\langle v, w-z\rangle:\ v \in V_u, w \in M_u, z \in N_u\} \subset U.$$

Since $Tx_0 \subset \bigcup_{w \in Tx_0} V_{u_i}$ and Tx_0 is strongly compact, there exist $u_1, \ldots, u_n \in Tx_0$ such that $Tx_0 \subset \bigcup_{i=1}^n V_{u_i}$. Since T is USC, there exists an open neighborhood M_1 of x_0 in X such that $Tx \subset \bigcup_{i=1}^n V_{u_i}$ for all $x \in M_1$. Let $M_{x_0} := M_1 \cap \bigcap_{i=1}^n M_{u_i}$ and $N_{y_0} := \bigcap_{i=1}^n N_{u_i}$. Then M_{x_0} and N_y are open neighborhoods of x_0 and y_0 in X, respectively. Now suppose $x \in M_{x_0}, y \in N_{y_0}$, and $u \in Tx$ are given. Let $i_0 \in \{1, \ldots, n\}$ be such that $u \in V_{u_{i_0}}$. As $x \in M_1 \cap M_{u_{i_0}}$ and $y \in N_{u_{i_0}}$, $\mathrm{Re}\langle u, x-y\rangle \in U$. It follows that $f(x,y) \subset U$ for all $x \in M_{x_0}$ and $y \in N_{y_0}$. Therefore f is USC.

By combining both Theorem 4.2 and Lemma 4.2, we have the following result which is Theorem 4 of Shih and Tan [14]:

COROLLARY 4.1 *Let E be locally convex, X be a non-empty compact convex subset of E, $S: X \to KC(X)$ be continuous and $T: X \to K(E^*)$ be USC, where E^* is equipped with the strong topology. Then exists $\hat{y} \in X$ such that*

$$\begin{cases} \hat{y} \in S(\hat{y}), \\ \inf_{w \in T\hat{y}} \mathrm{Re}\langle w, \hat{y} - x\rangle \le 0 \quad \textit{for all } x \in S(\hat{y}). \end{cases}$$

Proof Define $f: X \times X \to 2^{\mathbb{R}}$ by

$$f(x,y) = \{\mathrm{Re}\langle u, x-y\rangle:\ u \in Tx\}$$

for each $x, y \in X$. By Lemma 4.2, f is USC. Now the conclusion follows from Theorem 4.2.

Finally, we have the following implicit quasi-variational inequality:

THEOREM 4.3 *Let E be locally convex, X be a non-empty compact convex subset of E, $S: X \to KC(X)$ be continuous and $g: X \times X \times X \to 2^{\mathbb{R}}$ be such that*

(i) *For each $u, x \in X$, $\inf g(u, x, x) \leq 0$.*
(ii) *For each $u, y \in X$, the mapping $w \mapsto g(u, y, w)$ is concave.*
(iii) *g is lower demicontinuous on $X \times X \times X$.*
(iv) *For each $(u, w) \in X \times X$, the mapping $y \mapsto \inf g(u, y, w)$ is convex.*

Then (a) *there exists $\hat{y} \in X$ such that*

$$\begin{cases} \hat{y} \in S(\hat{y}), \\ \inf g(\hat{y}, \hat{y}, w) \leq 0 \quad \textit{for all } w \in S(\hat{y}) \end{cases}$$

and (b) *the set*

$$\{y \in X\colon\ y \in S(y)\{\textit{and } \inf g(y, y, w) \leq 0 \textit{ for all } w \in S(y)\}$$

is a (*non-empty*) *compact subset of X.*

Proof Define $F: X \to KC(X)$ by

$$F(u) = \{y \in S(u)\colon\ \inf g(u, y, w) \leq 0 \quad \text{for all } w \in S(u)\}$$

for each $u \in X$. By Theorem 2.2, F is non-empty valued. Now we shall show that F has a closed graph. Indeed, let $((x_\alpha, y_\alpha))_{\alpha \in \Gamma}$ be a net in $X \times X$, $(x_0, y_0) \in X \times X$ such that $(x_\alpha, y_\alpha) \to (x_0, y_0)$ and $y_\alpha \in S(x_\alpha)$ for each $\alpha \in \Gamma$. Then $y_0 \in S(x_0)$ since S is USC. Now fix an arbitrary $w_0 \in S(x_0)$. Since S is LSC, there is a net $(w_\alpha)_{\alpha \in \Gamma}$ in X with $w_\alpha \in S(x_\alpha)$ for all $\alpha \in \Gamma$ such that $w_\alpha \to w_0$. Note that $\inf g(x_\alpha, y_\alpha, w_\alpha) \leq 0$ for all $\alpha \in \Gamma$. By (iii) and Lemma 2.2, $\inf g$ is jointly lower semicontinuous. It follows that $\inf g(x_0, y_0, w_0) \leq 0$. As $w_0 \in S(x_0)$ is arbitrary, $y_0 \in F(x_0)$. Thus F has a closed graph. It follows that for each $u \in X$, $F(u)$ is closed in X and is therefore compact, and is also convex by (iv). Therefore F is well-defined. Moreover, as X is compact and F has a closed graph, F is USC. By Fan–Glicksberg fixed point theorem again, there exists $\hat{y} \in X$ such that $\hat{y} \in F(\hat{y})$; i.e.,

$$\begin{cases} \hat{y} \in S(\hat{y}), \\ \inf g(\hat{y}, \hat{y}, w) \leq 0 \quad \text{for all } w \in S(\hat{y}). \end{cases}$$

Thus the proof is completed.

Before we conclude this section, we would like to note that the results established in this paper can be applied to study many nonlinear problems such as nonlinear operators, nonlinear optimization, complementarity problems and so on by using those ideas which have been illustrated by Aubin and Ekeland [1], Baiocchi and Capelo [2], Granas [7], Harker and Pang [8], Husain and Tarafdar [9], Karamolegos and Kravvaritis [11], Kravvaritis [12], Mosco [13] and references therein.

References

[1] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analyis*, Wiley, New York, 1984.

[2] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, Wiley, New York, 1984.

[3] X.P. Ding and K.K. Tan, Generalized variational inequalities and generalized quasi-variational inequalities *J. Math. Anal. Appl.* **148** (1990), 497–508.

[4] K. Fan, Fixed points and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 131–136.

[5] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* **142** (1961), 305–310.

[6] I.L. Glicksberg, A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points, *Proc. Amer. Math. Soc.* **3** (1952), 170–174.

[7] A. Granas, Méthodes Topologiques en Analyse Convexe, Séminaire de Mathématiques Supérieures, Vol. 110, Les Presses de I'Université Montreal, Canada, 1990.

[8] P.T. Harker and J.S. Pang, Finite dimensional variational inequalities and nonlinear complementarity problems: a survey of theory, algorithms and applications, *Math. Programming* **48** (1990), 161–220.

[9] T. Husain and E. Tarafdar Simultaneous variational inequalities, minimization problems and related results *Math. Japonica* **39** (1994), 221–231.

[10] H. Kneser Sur une théorème fondamental de la théorie des jeux, *C. R. Acad. Sci. Paris* **234** (1952), 2418–2420.

[11] A. Karamolegos and D. Kravvaritis Nonlinear random operator equations and inequalities in Banach spaces *Internat. J. Math. & Math. Sci.* **15** (1992), 111–118.

[12] D. Kravvaritis Nonlinear equations and inequalities in Banach spaces *J. Math. Anal. Appl.* **67** (1979), 205–214.

[13] U. Mosco Implicit variational problems and quasi variational inequalities, in: *Nonlinear Operators and the Calculus of Variations* (eds. J.P. Gossez, E.J. Lami Dozo, J. Mawhin and L. Waelbrook), 1976, pp. 83–156, Springer-Verlag, Berlin.

[14] M.H. Shih and K.K. Tan Generalized quasi-variational inequalities in locally convex topological vector spaces, *J. Math. Anal. Appl.* **108** (1985), 333–343.

[15] M.H. Shih and K.K. Tan A minimax inequality and Browder-Hartman-Stampacchia variational inequalities for multi-valued monotone operators *Proceedings of Fourth FRANCO-SEAMS Joint Conference*, Chiang Mai, Thailand 1988.

[16] M.H. Shih and K.K. Tan Browder–Hartman–Stampacchia variation inequalities for multi-valued monotone operators *J. Math. Anal. Appl.* **134** (1988), 431–440.

[17] E. Tarafdar Nonlinear variational inequality with application to the boundary value problem for quasi-linear operator in generalized divergence form *Funkcialaj Ekvacioj* **33** (1991), 441–453.

[18] E. Tarafdar and X.Z. Yuan Non-compact generalized quasi-variational inequalities in locally convex topological vector spaces, *Nonlinear World* **1** (1994), 273–283.

[19] C.I. Tulcea, On the approximation of upper semi-continuous correspondences and the equilibriums of generalized games *J. Math. Anal. Appl.* **136** (1988), 267–289.

[20] G.X.Z. Yuan, the study of minimax inequalities and applications to economics and variational inequalities, *Mem. Amer. Math. Soc.* No. 625, **132** (1998), 1–140.