Differential Inequalities with Initial Time Difference and Applications

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The investigation of initial value problems of differential equations where the initial time differs with each solution is initiated in this paper. Basic preliminary results in qualitative theory are discussed to understand possible ramifications.

Keywords: Differential equations; Change of initial time

Subject index: 34A40, 34D20

1. INTRODUCTION

In the investigation of initial value problems of differential equations, we have been partial to initial time all along in the sense that we only perturb or change the dependent variable or the space variable and keep the initial time unchanged [1,2]. However, it appears important to vary the initial time as well because it is impossible not to make errors in the starting time [3]. For example, the solutions of the unperturbed differential system may start at some initial time and the solutions of perturbed differential system may have to be started at a different initial time. Moreover, in considering the solution of a given system, it may be impossible, in real situations, to keep the initial time the same

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and only perturb the space variable as we have been continuously doing. If we do change the initial time for each solution, then we are faced with the problem of comparing any two solutions which differ in the initial starting time. There may be several ways of comparing and to each choice of measuring the difference, we may end up with a different result.

   In this paper, we initiate the investigation in this direction and make an attempt to study some preliminary results to understand the possible ramifications.

2. COMPARISON RESULTS

We begin with the basic result in the theory of differential inequalities parallel to the well known results [1].

THEOREM 2.1 *Assume that*

   (i) $\alpha, \beta \in C^1[R_+, R]$, $f \in C[R_+ \times R, R]$ *and* $\alpha' \leq f(t, \alpha)$, $\alpha(t_0) \leq x_0$,
         $t_0 \geq 0$, $\beta' \geq f(t, \beta)$, $\beta(\tau_0) \geq x_0$, $\tau_0 \geq 0$;
   (ii) $f(t, x) - f(t, y) \leq L(x-y)$, $x \geq y$, $L > 0$;
   (iii) $\tau_0 > t_0$ *and* $f(t, x)$ *is nondecreasing in* t *for each* x.

   Then (a) $\alpha(t) \leq \beta(t + \eta)$, $t \geq t_0$ *and* (b) $\alpha(t - \eta) \leq \beta(t)$, $t \geq \tau_0$, *where* $\eta = \tau_0 - t_0$.

Proof   Define $\beta_0(t) = \beta(t + \eta)$ so that $\beta_0(t_0) = \beta(t_0 + \eta) = \beta(\tau_0) \geq x_0 \geq \alpha(t_0)$. Also,

$$\beta_0'(t) = \beta'(t + \eta) \geq f(t + \eta, \beta_0(t)), \quad t \geq t_0.$$

Let $\tilde{\beta}_0(t) = \beta_0(t) + \epsilon e^{2Lt}$ for some $\epsilon > 0$ small. Then

$$\tilde{\beta}_0(t) \geq \beta_0(t) \quad \text{and} \quad \tilde{\beta}_0(t_0) > \beta_0(t_0) \geq \alpha(t_0). \qquad (2.1)$$

We shall show that $\alpha(t) < \tilde{\beta}_0(t_0)$ for $t \geq t_0$. If this is not true, because of (2.1), there would exist a $t_1 > t_0$ such that

$$\alpha(t_1) = \tilde{\beta}_0(t_1) \quad \text{and} \quad \alpha(t) < \tilde{\beta}_0(t), \qquad t_0 \leq t < t_1. \qquad (2.2)$$

We get from (2.2), the relation $\alpha'(t_1) \geq \tilde{\beta}_0'(t_1)$ which implies in view of (ii),

$$f(t_1, \alpha(t_1)) \geq \alpha'(t_1) \geq \tilde{\beta}_0'(t_1) = \beta_0'(t_1) + 2L\epsilon e^{2Lt_1}$$

$$\geq f(t_1 + \eta, \beta_0(t_1)) + 2L\epsilon e^{2Lt_1}.$$

Using (ii) and (2.1), we then get

$$f(t_1, \alpha(t_1)) \geq f(t_1 + \eta, \tilde{\beta}_0(t_1)) - L\epsilon e^{2Lt_1} + 2L\epsilon e^{2Lt_1}$$

$$> f(t_1 + \eta, \tilde{\beta}_0(t_1)).$$

Since $\tau_0 > t_0$, condition (iii) now leads to the contradiction because of (2.2), namely,

$$f(t_1, \alpha(t_1)) > f(t_1, \tilde{\beta}_0(t_1)),$$

which proves $\alpha(t) < \tilde{\beta}_0(t)$, $t \geq t_0$. Making $\epsilon \to 0$, we conclude that $\alpha(t) \leq \beta(t + \eta)$, $t \geq t_0$ which proves (a).

To prove (b), we set $\alpha_0(t) = \alpha(t - \eta)$, $t \geq \tau_0$ and note that $\alpha_0(\tau_0) = \alpha(\tau_0 - \eta) = \alpha(t_0) \leq \beta(\tau_0)$. Then letting $\tilde{\alpha}_0(t) = \alpha_0(t) - \epsilon e^{2Lt}$ for small $\epsilon > 0$ and proceeding similarly, we derive the estimate $\alpha(t - \eta) \leq \beta(t)$, $t \geq \tau_0$. The proof is therefore complete.

In case $t_0 > \tau_0$, to prove the conclusion of Theorem 2.1, assumption (iii) needs to be replaced by

(iii*) $t_0 > \tau_0$ and $f(t, x)$ is nonincreasing in t for each x.

Then the dual result is valid.

THEOREM 2.2 *Assume that conditions* (i), (ii) *and* (iii*) *hold. Then the conclusion of Theorem 2.1 is valid.*

We can immediately deduce the following comparison result.

THEOREM 2.3 *Assume that*

(a) $m \in C[R_+, R_+]$, $g \in C[R_+^2, R]$ *and* $D_- m(t) \leq g(t, m(t))$, $m(t_0) \leq w_0$, $t_0 \geq 0$, *where*

$$D_- m(t) = \liminf_{h \to 0^-} \frac{1}{h} [m(t + h) - m(t)];$$

(b) *the maximal solution $r(t) = r(t, \tau_0, w_0)$ of $w' = g(t, w)$, $w(\tau_0) = w_0 \geq 0$,*
   $\tau_0 \geq 0$, exists for $t \geq \tau_0$;
(c) *$g(t, w)$ is nondecreasing in t for each w and $\tau_0 > t_0$.*

Then $m(t) \leq r(t + \eta)$, $t \geq t_0$ and $m(t - \eta) \leq r(t)$, $t \geq \tau_0$.

Proof   It is well known [1] that if $w(t, \epsilon)$ is any solution of

$$w' = g(t, w) + \epsilon, \quad w(\tau_0) = w_0 + \epsilon,$$

for $\epsilon > 0$ sufficiently small, then $\lim_{\epsilon \to 0} w(t, \epsilon) = r(t, \tau_0, w_0)$ on every compact subset $[\tau_0, \tau_0 + T]$. Hence setting $w_0(t, \epsilon) = w(t + \eta, \epsilon)$, we have $w_0(t_0, \epsilon) = w(t_0 + \eta, \epsilon) = w(\tau_0, \epsilon) = w_0 + \epsilon > w_0 \geq m(t_0)$, and

$$w_0'(t, \epsilon) = g(t + \eta, w_0(t, \epsilon)) + \epsilon > g(t + \eta, w_0(t, \epsilon)), \quad t \geq t_0.$$

By Theorem 2.1, we then get $m(t) < w_0(t, \epsilon)$, $t \geq \tau_0$ and hence it follows that $m(t) \leq r(t + \eta, \tau_0, w_0)$, $t \geq t_0$.
   To prove the second part of the conclusion, we set $m_0(t) = m(t - \eta)$ so that $m_0(\tau_0) = m(t_0) \leq w_0$ and note that

$$D_- m_0(t) \leq g(t - \eta, m_0(t)), \quad t \geq \tau_0.$$

We now prove, following the earlier arguments, that $m_0(t) < w(t, \epsilon)$, $t \geq \tau_0$ and the conclusion follows taking the limit as $\epsilon \to 0$. The proof of theorem is complete.

   One can have a dual of Theorem 2.3 on the basis of Theorem 2.2. We merely state such a result.

THEOREM 2.4   *Assume that conditions (a) and (b) hold. Suppose further that condition (c) is replaced by*

(c*) *$g(t, w)$ is nonincreasing in t for each w and $t_0 > \tau_0$.*

Then the conclusion of Theorem 2.3 remains valid.

   In this framework, Gronwall Lemma takes the following form.

GRONWALL'S LEMMA 2.5   *Let $m, \lambda \in C[R_+, R_+]$ and*

$$m(t) \leq m(t_0) + \int_{t_0}^{t} \lambda(s) m(s) \, ds, \quad t_0 \geq 0.$$

If $\lambda(t)$ is nondecreasing in t and $\tau_0 > t_0$, then

$$m(t) \leq m(t_0) \exp\left[\int_{t_0}^{t} \lambda(s + \eta)\, ds\right], \quad t \geq t_0,\ \eta = \tau_0 - t_0,$$

and

$$m(t - \eta) \leq m(t_0) \exp\left[\int_{t_0}^{t} \lambda(s)\, ds\right], \quad t \geq \tau_0.$$

If $\lambda(t)$ is nonincreasing in t and $t_0 > \tau_0$, then

$$m(t - \eta) \leq m(t_0) \exp\left[\int_{\tau_0}^{t} \lambda(s - \eta)\, ds\right], \quad t \geq \tau_0,$$

and

$$m(t) \leq m(t_0) \exp\left[\int_{\tau_0}^{t} \lambda(s)\, ds\right], \quad t \geq t_0.$$

Remark 2.1   Let us note that if the functions f and g in the foregoing discussions are autonomous, the monotone assumptions imposed in the results are automatically satisfied.

3. METHOD OF VARIATION OF PARAMETERS

Consider the two differential systems

$$y' = F(t, y), \quad y(t_0) = y_0,\ t_0 \geq 0, \tag{3.1}$$
$$x' = f(t, x), \quad x(\tau_0) = x_0,\ \tau_0 \geq 0, \tag{3.2}$$

where $F, f \in C[R_+ \times R^n, R^n]$. Assume that $\partial F(t, y)/\partial y$ exists and is continuous on $R_+ \times R^n$, then we know [1] that the solution $y(t, t_0, y_0)$ is unique for each (t_0, y_0), $\partial y(t, t_0, y_0)/\partial t_0$ and $\partial y(t, t_0, y_0)/\partial y_0$ are the solutions of the variational system

$$z' = F_y(t, y(t, t_0, y_0))z \tag{3.3}$$

satisfying the initial conditions

$$\frac{\partial y}{\partial t_0}(t_0, t_0, y_0) = -F(t_0, y_0), \qquad \frac{\partial y}{\partial y_0}(t_0, t_0, y_0) = I, \quad \text{(Identity matrix)}$$

$$(3.4)$$

and the identity

$$\frac{\partial y}{\partial t_0}(t, t_0, y_0) + \frac{\partial y}{\partial y_0}(t, t_0, y_0)F(t_0, y_0) \equiv 0, \qquad (3.5)$$

holds for $t_0 \leq t < \infty$.

Let $y(t, t_0, x_0)$ be the solution of (3.1) through (t_0, x_0) and let $\tilde{x}(t) = x(t + \eta, \tau_0, x_0)$, where $x(t, \tau_0, x_0)$ is any solution of (3.2) existing for $t \geq \tau_0$ and $\eta = \tau_0 - t_0$. Set $p(s) = y(t, s, \tilde{x}(s))$ for $t_0 \leq s \leq t$. Note that $\tilde{x}(t_0) = x(t_0 + \eta, \tau_0, x_0) = x_0$. Then we see that

$$\frac{dp(s)}{ds} = \frac{\partial y}{\partial t_0}(t, s, \tilde{x}(s)) + \frac{\partial y}{\partial y_0}(t, s, \tilde{x}(s))f(s + \eta, \tilde{x}(s))$$

$$\equiv \tilde{f}(s, \tilde{x}(s); \eta), \quad \text{say.} \qquad (3.6)$$

Integrating (3.6) from t_0 to t, we get

$$\tilde{x}(t) = y(t, t_0, x_0) + \int_{t_0}^{t} \tilde{f}(s, \tilde{x}(s); \eta)\, ds; \quad t \geq t_0. \qquad (3.7)$$

Now let $q(s) = y(t, t_0, \sigma(s))$ where $\sigma(s) = x_0 s + (1 - s)y_0$, $0 \leq s \leq 1$, so that we have

$$\frac{dq(s)}{ds} = \frac{\partial y}{\partial y_0}(t, t_0, \sigma(s))(x_0 - y_0).$$

Integrating from 0 to 1, we arrive at

$$y(t, t_0, x_0) = y(t, t_0, y_0) + \int_{0}^{1} \frac{\partial y}{\partial y_0}(t, t_0, \sigma(s))\, ds(x_0 - y_0). \qquad (3.8)$$

Combining (3.7) and (3.8) yields

$$x(t + \eta, \tau_0, x_0) = y(t, t_0, y_0) + \int_{0}^{1} \frac{\partial y}{\partial y_0}(t, t_0, \sigma(s))\, ds(x_0 - y_0)$$

$$+ \int_{t_0}^{t} \tilde{f}(s, x(s + \eta, \tau_0, x_0); \eta)\, ds, \quad t \geq t_0, \qquad (3.9)$$

which gives the relation between the solution of (3.1) and (3.2) starting at different initial data in time and space.

If $f(t+\eta, x) = F(t, x) + R(t+\eta, x)$, then it follows in view of (3.5), (3.6) and (3.9) that

$$x(t+\eta, \tau_0, x_0) = y(t, t_0, y_0) + \int_0^1 \frac{\partial y}{\partial y_0}(t, t_0, \sigma(s)) \, ds(x_0 - y_0)$$

$$+ \int_{t_0}^t \frac{\partial y}{\partial y_0}(t, s, x(s+\eta, \tau_0, x_0)) R(s+\eta, x(s+\eta, \tau_0, x_0)) \, ds$$

$$(3.10)$$

for $t \geq t_0$.

This is the nonlinear variation of parameters formula connecting the solutions of perturbed and unperturbed differential systems.

If F in (3.1) is linear and autonomous, namely, $F(t, y) = Ay$ where A is $n \times n$ constant matrix and f in (3.2) is taken as $f(t, x) = Ax + R(t, x)$, then (3.10) reduces to

$$x(t+\eta, \tau_0, x_0) = e^{A(t-t_0)} y_0 + e^{A(t-t_0)}(x_0 - y_0)$$

$$+ \int_{t_0}^t e^{A(t-s)} R(s+\eta, x(s+\eta, \tau_0, x_0)) \, ds. \quad (3.11)$$

The relation (3.9) which offers an expression for the difference of two solutions $x(t+\eta, \tau_0, x_0)$ and $y(t, t_0, y_0)$ can be obtained in a different form as follows.

Let $w(t) = x(t+\eta, \tau_0, x_0) - y(t, t_0, y_0) \equiv \tilde{x}(t) - y(t)$. Then $w(t_0) = x_0 - y_0$ and

$$w'(t) = f(t+\eta, w(t) + y(t)) - F(t, y(t)) \equiv H(t, w(t); \eta).$$

Setting $p(s) = y(t, s, w(s))$, we find

$$\frac{dp(s)}{ds} = \frac{\partial y}{\partial t_0}(t, s, w(s)) + \frac{\partial y}{\partial y_0}(t, s, w(s)) H(s, w(s); \eta)$$

$$= \tilde{H}(t, s, w(s); \eta).$$

Integrating from t_0 to t, we have

$$w(t) = y(t, t_0, x_0 - y_0) + \int_{t_0}^t \tilde{H}(t, s, w(s); \eta) \, ds, \quad t \geq t_0,$$

or equivalently, for $t \geq t_0$,

$$x(t + \eta, \tau_0, x_0) - y(t, t_0, y_0) = y(t, t_0, x_0 - y_0) + \int_{t_0}^{t} \tilde{H}(t, s, w(s); \eta) \, ds.$$

$$(3.9^*)$$

One can also obtain the expression (3.9^*) for $t \geq \tau_0$. For this purpose, we need to set $w(t) = x(t, \tau_0, x_0) - y(t - \eta, t_0, y_0) = x(t) - \tilde{y}(t)$ so that $w(\tau_0) = x_0 - y_0$ and

$$w'(t) = f(t, w(t) + \tilde{y}(t)) - F(t - \eta, \tilde{y}(t)) \equiv H(t, w(t); \eta).$$

Proceeding as above, we arrive at

$$x(t, \tau_0, x_0) - y(t - \eta, t_0, y_0) = y(t, t_0, x_0 - y_0) + \int_{\tau_0}^{t} \tilde{H}(t, s, w(s), \eta) \, ds$$

$$(3.9^{**})$$

for $t \geq \tau_0$.

Let us next consider another differential system

$$z' = G(t, z), \quad z(\tau_0) = z_0, \tag{3.12}$$

where $G \in C[R_+ \times R^n, R^n]$. Define

$$v(t) = z(t + \eta, \tau_0, z_0) - x(t, t_0, x_0) \quad \text{for } t \geq t_0, \tag{3.13}$$

where $z(t, \tau_0, z_0)$ and $x(t, t_0, x_0)$ are the solutions of (3.12) and (3.2) through (τ_0, z_0) and (t_0, x_0) respectively. Then $v(t_0) = z_0 - x_0$ and

$$\begin{aligned}
v'(t) &= G(t + \eta, z(t + \eta, \tau_0, z_0)) - f(t, x(t, t_0, x_0)) \\
&= G(t + \eta, v(t) + x(t, t_0, x_0)) - f(t, x(t, t_0, x_0)) \\
&\equiv f_0(t, v(t); \eta), \quad \text{say.}
\end{aligned} \tag{3.14}$$

We set $p(s) = y(t, s, v(s))$ where $y(t, t_0, y_0)$ is the solution of (3.1) so that

$$\frac{dp(s)}{ds} = \frac{\partial y}{\partial t_0}(t, s, v(s)) + \frac{\partial y}{\partial y_0}(t, s, v(s)) f_0(s, v(s); \eta)$$

$$\equiv \tilde{G}(t, s, v(s); \eta), \quad \text{say.} \tag{3.15}$$

Integrating from t_0 to t, we get

$$v(t) = y(t, t_0, z - x_0) + \int_{t_0}^{t} \tilde{G}(t, s, v(s); \eta) \, ds, \quad t \geq t_0, \qquad (3.16)$$

which implies for $t \geq t_0$,

$$z(t + \eta, \tau_0, z_0) - x(t, t_0, x_0) = y(t, t_0, z_0 - x_0) + \int_{t_0}^{t} \tilde{G}(t, s, v(s); \eta) \, ds. \qquad (3.17)$$

This relation connects the solutions of three differential equations and is the nonlinear variation of parameters formula as well.

Let us consider some special cases of (3.17).

(a) Suppose that $F(t, y) \equiv 0$ so that $y(t, t_0, y_0) \equiv y_0$. Then (3.17) yields

$$z(t + \eta, \tau_0, z_0) - x(t, t_0, x_0) = z_0 - x_0 + \int_{t_0}^{t} f_0(s, v(s); \eta) \, ds, \quad t \geq t_0. \qquad (3.18)$$

(b) Suppose that $G(t, x) \equiv f(t, x)$. Then (3.17) reduces to

$$x(t + \eta, \tau_0, z_0) - x(t, t_0, x_0) = y(t, t_0, z_0 - x_0) + \int_{t_0}^{t} \tilde{G}(t, s, v(s); \eta) \, ds, \qquad (3.19)$$

where

$$\tilde{G}(t, s, v(s); \eta) = \frac{\partial y}{\partial t_0}(t, s, v(s))$$

$$+ \frac{\partial y}{\partial y_0}(t, s, v(s))[f(t + \eta, x(t) + v(t)) - f(t, x(t))].$$

(c) If $F(t, y) \equiv 0$ and $G(t, x) \equiv f(t, x)$, we get from (3.17) the relation

$$x(t + \eta, \tau_0, z_0) - x(t, t_0, x_0)$$

$$= z_0 - x_0 + \int_{t_0}^{t} [f(s + \eta, x(s) + v(s)) - f(s, x(s))] \, ds. \qquad (3.20)$$

(d) If $F(t, y) \equiv 0$ and $G(t, x) = f(t, x) + R(t, x)$, then (3.17) reduces to

$$z(t + \eta, \tau_0, z_0) - x(t, t_0, x_0) = z_0 - x_0 + \int_{t_0}^{t} f_0(s, v(s); \eta) \, ds, \quad (3.21)$$

where

$$f_0(t, v(t); \eta) = f(t + \eta, v(t) + x(t)) - f(t, x(t)) + R(t + \eta, z(t + \eta)).$$

Other possibilities exist and we omit, to avoid monotony.

4. STABILITY CRITERIA

Consider the differential system

$$x' = f(t, x), \quad (4.1)$$

where $f \in C[R_+ \times R^n, R^n]$. Let $x(t, \tau_0, x_0)$ and $x(t, t_0, y_0)$ be the solutions of (4.1) through (τ_0, x_0) and (t_0, y_0), $t_0, \tau_0 > 0$, $\eta = \tau_0 - t_0 > 0$.

In order to discuss stability of the difference of these two solutions, the notion of practical stability is more convenient which we define below in a suitable way.

DEFINITION   The system (4.1) is said to be

(1) practically stable, if given (λ, A) with $(\lambda < A)$, there exists a $\sigma = \sigma(\lambda, A) > 0$ such that

$$|x_0 - y_0| < \lambda, \quad |\eta| < \sigma \quad \Rightarrow \quad |x(t + \eta, \tau_0, x_0) - x(t, t_0, y_0)| < A$$

for $t \geq t_0$;

(2) strongly practically stable if (1) holds and given $(\lambda, B, T) > 0$ there exists a $\sigma_0 = \sigma_0(\lambda, B, T)$ such that

$$|x_0 - y_0| < \lambda, \quad |\eta| < \sigma_0 \quad \Rightarrow \quad |x(t + \eta, \tau_0, x_0) - x(t, t_0, y_0)| < B$$

for $t \geq t_0 + T$.

Now we can prove the following typical result concerning practical stability of (4.1) relative to the two solutions starting at different initial data.

THEOREM 4.1   *Suppose that*

$$\liminf_{h \to 0^-} \frac{1}{h} \left[|v + h\, \tilde{f}(t, v, \eta)| - |v| \right] \le -\alpha|v| + \beta|\eta|, \qquad (4.2)$$

$\alpha, \beta > 0$, *where* $\tilde{f}(t, v, \eta) = f(t + \eta, x(t) + v) - f(t, x(t))$ *and* $x(t) = x(t, t_0, y_0)$. *Then the system* (4.1) *is strongly practically stable.*

Proof   Set $v(t) = x(t + \eta, \tau_0, x_0) - x(t, t_0, y_0) = \tilde{x}(t) - x(t)$ for $t \ge t_0$, so that $v(t_0) = x_0 - y_0$ and

$$v'(t) = f(t + \eta, x(t) + v(t)) - f(t, x(t)) \equiv \tilde{f}(t, v; \eta). \qquad (4.3)$$

It is now easy to get the linear differential inequality

$$D_- m(t) \le -\alpha m(t) + \beta|\eta|, \quad t \ge t_0, \qquad (4.4)$$

where $m(t) = |v(t)| = |x(t + \eta, \tau_0, x_0) - x(t, t_0, y_0)|$, using (4.2). We then get from (4.4) the estimate

$$m(t) \le m(t_0)e^{-\alpha(t - t_0)} + |\eta| \frac{\beta}{\alpha}, \quad t \ge t_0. \qquad (4.5)$$

If $0 < \lambda < A$ are given, then choosing $\sigma = (A - \lambda)(\alpha/\beta)$, we see that (4.5) yields

$$|x_0 - y_0| < \lambda, \ |\eta| < \sigma \ \Rightarrow \ |x(t + \eta, \tau_0, x_0) - x(t, t_0, y_0)| < A$$

for $t \ge t_0$, proving practical stability.

Now let $(\lambda, B, T) > 0$ be given. Choose $\sigma_0 = \min(\sigma, \tilde{\sigma})$ where $\tilde{\sigma} = (B - \lambda e^{-\alpha T})(\alpha/\beta)$.

Then it follows from (4.5) that whenever $|x_0 - y_0| < \lambda$ and $|\eta| < \sigma_0$, we have

$$|x(t + \eta, \tau_0, x_0) - x(t, t_0, y_0)| < B, \quad t \ge t_0 + T.$$

This proves that the system (4.1) is strongly practically stable and the proof is complete.

Remark 4.1   One can also formulate the foregoing considerations to hold for $t \ge \tau_0$ by defining $v(t) = |x(t, \tau_0, x_0) - x(t - \eta, t_0, y_0)|$ and proceeding as before.

For more stability results in the present framework via nonlinear variation of parameters see [4].

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