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Asymptotic Behaviour of Fixed-order Error Constants of Modified Quadrature Formulae for Cauchy Principal Value Integrals

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We consider quadrature formulae for Cauchy principal value integrals

$$I_{w,\xi}[f] = \int_a^b \frac{f(x)}{x-\xi} w(x)\,\mathrm{d}x, \quad a < \xi < b.$$

The quadrature formulae considered here are so-called modified formulae, which are obtained by first subtracting the singularity, and then applying some standard quadrature formula Q_n. The aim of this paper is to determine the asymptotic behaviour of the constants $\kappa_{i,n}$ in error estimates of the form $|R_n^{\mathrm{mod}}[f;\xi]| < \kappa_{i,n}(\xi)\|f^{(i)}\|_\infty$ for fixed i and $n \to \infty$, where $R_n^{\mathrm{mod}}[f;\xi]$ is the quadrature error. This is done for quadrature formulae Q_n for which the Peano kernels $K_{i,n}$ of fixed order i behave in a certain regular way, including, e.g., many interpolatory quadrature formulae as Gauss–Legendre and Clenshaw–Curtis formulae, as well as compound quadrature formulae. It turns out that essentially all the interpolatory formulae behave in a very similar way.

Keywords: Cauchy principal value integral; Finite Hilbert transform; Modified quadrature formula; Peano kernel; Error estimate

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1 INTRODUCTION

We consider the numerical evaluation of the Cauchy principal value integral

$$I_{w,\xi}[f] = \int_a^b \frac{f(x)}{x-\xi} w(x)\,\mathrm{d}x,$$

for arbitrary, but fixed $\xi \in (a, b)$, where w is a given weight function. This expression is also known as the *finite Hilbert transform* of wf at the point ξ. The numerical methods we investigate are based on the principle of the subtraction of the singularity [4, p. 184]. We note that

$$I_{w,\xi}[f] = \int_a^b \frac{f(x)-f(\xi)}{x-\xi} w(x)\,\mathrm{d}x + f(\xi) \int_a^b \frac{w(x)}{x-\xi}\,\mathrm{d}x. \tag{1}$$

Then, we let $Q_n[g] = \sum_{i=1}^n a_i g(x_i)$ be a quadrature formula for the integral

$$I_w[g] = \int_a^b g(x)w(x)\,\mathrm{d}x, \tag{2}$$

with remainder term $R_n[g] = I_w[g] - Q_n[g]$. We now apply this quadrature formula to the first integral in (1), which yields

$$\begin{aligned} I_{w,\xi}[f] &= Q_n\left[\frac{f(\cdot)-f(\xi)}{\cdot-\xi}\right] + R_n\left[\frac{f(\cdot)-f(\xi)}{\cdot-\xi}\right] + f(\xi)I_{w,\xi}[1] \\ &= Q_{n+1}^{\mathrm{mod}}[f;\xi] + R_{n+1}^{\mathrm{mod}}[f;\xi], \end{aligned}$$

where

$$\begin{aligned} Q_{n+1}^{\mathrm{mod}}[f;\xi] &= Q_n\left[\frac{f(\cdot)-f(\xi)}{\cdot-\xi}\right] + f(\xi)I_{w,\xi}[1], \\ R_{n+1}^{\mathrm{mod}}[f;\xi] &= R_n\left[\frac{f(\cdot)-f(\xi)}{\cdot-\xi}\right]. \end{aligned} \tag{3}$$

A quadrature formula Q_{n+1}^{mod} obtained in this way is called a *modified quadrature formula*. It uses one node more than Q_n, namely ξ; if ξ is already a node of Q_n, then it is a double node of Q_{n+1}^{mod}. This method has frequently been considered, see, e.g., Davis and Rabinowitz [4, p. 184], Diethelm [5,6,9], Elliott and Paget [11], Gautschi [13], Köhler [16], Monegato [18], and Stolle and Strauß [23].

Whereas it is sufficient for (2) that $w \in L_1[a,b]$, more restrictive assumptions are necessary for the consideration of Cauchy principal value integrals. We assume that $w = w_{\alpha,\beta}\psi$, where $w_{\alpha,\beta}(x) = (x-a)^\alpha(b-x)^\beta$ $(\alpha, \beta > -1)$ is a Jacobi weight function, and ψ satisfies the Dini-type condition

$$\int_0^1 \frac{\omega(\psi, t)}{t}\,\mathrm{d}t < \infty$$

(ω denotes the modulus of continuity).

The modified quadrature formulae have got a certain weakness because they cannot be applied very well to functions of rather low smoothness properties. In particular, it is known [3,22] that divergence may happen if the function f does not fulfil a Lipschitz condition of order 1. On the positive side, we note that convergent subsequences exist under very weak assumptions on f [2]. Furthermore, as soon as we assume that f fulfils a Lipschitz condition of order 1, we can immediately deduce convergence for almost every reasonable choice of the quadrature formula Q_n [9]. A very attractive feature of modified methods is that, under this Lipschitz assumption, the convergence is automatically uniform for all $\xi \in (a,b)$ [9]. It is known that many other classes of quadrature methods for the Cauchy integral do not give uniform convergence at all [7]. Moreover, it is known that the Gaussian formulae for the Cauchy principal value integral, i.e. the formulae having the highest possible algebraic degree of exactness among all formulae having a prescribed number of nodes (a property that is heuristically used to argue that the error is likely to be very small for smooth functions), are methods of this special type.

Our main interest are error bounds of the form

$$|R_{n+1}^{\text{mod}}[f;\xi]| \le \kappa_{i,n}(\xi)\|f^{(i)}\|_\infty,$$

with best possible constants $\kappa_{i,n}(\xi)$, the so-called *error constants*. Obviously, in order to give bounds on the error, it is useful to have sharp bounds for these error constants. In particular, we shall look at the behaviour of the $\kappa_{i,n}$ as i is fixed and $n \to \infty$. For a bounded linear functional $L: C^i[a,b] \to \mathbb{R}$ with $L[\mathcal{P}_{i-1}] = 0$ (where $\mathcal{P}_{i-1}$ denotes the polynomials of degree less than i), let

$$\|L\|_i = \sup\left\{\frac{|L[f]|}{\|f^{(i)}\|_\infty}: f \in C^i[a,b],\ \|f^{(i)}\|_\infty \neq 0\right\}.$$

Then, it is well known that $\kappa_i = \|L\|_i$ is the best possible constant in the estimate $|L[f]| \leq \kappa_i \|f^{(i)}\|_\infty$. If $R_n[\mathcal{P}_{r-1}] = 0$ (but $R_n[p] \neq 0$ for some $p \in \mathcal{P}_r$), then $R_{n+1}^{\text{mod}}[\mathcal{P}_r; \xi] = 0$ for all ξ, and $\|R_{n+1}^{\text{mod}}[\cdot; \xi]\|_i$ is finite for all ξ and $i = 1, \ldots, r+1$. Therefore, in our case we have to consider $\|R_{n+1}^{\text{mod}}[\cdot; \xi]\|_i$.

Let us now recollect the most important previously known results about these expressions. Noble and Beighton [19] and Stolle and Strauß [23] have discussed the case of modified trapezoidal and modified Simpson methods for the weight function $w \equiv 1$. They have stated some upper bounds for special cases of i, but (except for one special case) these bounds did not display the correct order of magnitude of the constants as $n \to \infty$. Later, for the modified trapezoidal and midpoint methods, Diethelm [5] has shown

$$c_i^* n^{-i} \ln n \leq \|R_{n+1}^{\text{mod}}[\cdot; \xi]\|_i \leq \tilde{c}_i n^{-i} \ln n \quad \text{for } i = 1, 2,$$

with some constants c_i^* and $\tilde{c}_i$ independent of ξ. Using different methods, this has been extended to arbitrary modified compound methods [6] for all i satisfying $R_n[\mathcal{P}_{i-1}] = 0$, but still only for $w \equiv 1$, and with some significant gaps between the constants c_i^* in the lower bound and $\tilde{c}_i$ in the upper bound. Yet another approach [9] gave the upper bound for all i, but no lower bounds, for more general weight functions and modified interpolatory methods. Note that the results described up to this point were all uniform results for all $\xi \in (a,b)$. A particular consequence of this uniformity is uniform convergence results of the form

$$\sup_{\xi \in (a,b)} |R_{n+1}^{\text{mod}}[f; \xi]| \leq c_i \|f^{(i)}\|_\infty n^{-i} \ln n, \tag{4}$$

with c_i independent of f, n, and ξ in all the situations mentioned above.

Now, it is clear that a potential user of a quadrature formula would like to prefer a method having a small coefficient c_i in (4). Therefore, it is of interest to determine the coefficients corresponding to the various methods as precisely as possible in order to allow a comparison. The first step in this direction was done by Köhler [16] who, using a method similar to that of [9], determined the precise asymptotic constants for modified compound methods under the assumption $w \equiv 1$. Some results of this type concerning quadrature methods not based on the subtraction of the singularity have also been established recently [8,17]. In the present paper, we want to continue the work in this direction and generalize the results to more general weight functions and, in particular, to include the class of modified interpolatory quadrature formulae. Since, as we mentioned above, for all these methods the uniform and optimal order convergence of the error terms has already been established, we may look at the pointwise behaviour of the error constants $\|R_{n+1}^{\text{mod}}[\cdot;\xi]\|_i$ as ξ is fixed and $n \to \infty$. It turns out that we can determine this behaviour precisely. The main result in this context is (cf. Theorem 2.1) that asymptotically all the modified interpolatory formulae based on the classical choices of nodes behave in the same way, i.e. the coefficients c_i are identical. This gives the user the freedom to choose any one of these formulae, the choice possibly being based on the simplicity of the nodes or the availability of function values of f, without having to worry about the quality of the approximation.

In Section 2, we shall state our main results, i.e. the theorems describing the asymptotic behaviour of the constants $\|R_{n+1}^{\text{mod}}[\cdot;\xi]\|_i$ for various types of quadrature formulae. The proofs are based on a general theorem stated in Section 3. In Section 4, we shall prove the theorems of Section 2, and Section 5 finally contains the proof of the general result described in Section 3.

2 ASYMPTOTIC BEHAVIOUR OF THE ERROR CONSTANTS

2.1 Interpolatory Quadrature Formulae

Let nodes $a \leq x_1 < \cdots < x_n \leq b$ be given, and let $F[f] \in \mathcal{P}_{n-1}$ be the polynomial of degree $n-1$ interpolating a given function f defined on $[a,b]$ at $x_1, \ldots, x_n$. Then an interpolatory quadrature formula for

$I_w[f] = \int_a^b f(x)w(x)\,\mathrm{d}x$ is defined by

$$Q_n[f] = I_w[F[f]].$$

Various choices of interpolatory quadrature formulae have been suggested as basic methods for the construction of modified formulae, such as Clenshaw–Curtis formulae (Chawla and Jayarajan [1]), Gaussian formulae (Criscuolo and Mastroianni [2], Elliott and Paget [11], Gautschi [13], Hunter [14]), Gauss–Kronrod formulae (Rabinowitz [21]), or, more generally, positive interpolatory quadrature formulae (Diethelm [9]).

In this subsection, let $[a, b] = [-1, 1]$, let

$$w_\lambda(x) = (1 - x^2)^{\lambda - 1/2}$$

denote the ultraspherical weight functions and p_n^λ the corresponding orthogonal polynomials. Let the weight function w and the nodes $x_1, \ldots, x_n$ be chosen according to one of the following four cases.

(1) Let $w = w_\lambda$ for some $\lambda \geq 0$, and let the nodes x_i be the zeros of p_n^λ; i.e., Q_n is the Gauss formula for the weight function w_λ (this includes the Gauss–Legendre formulae for $\lambda = 1/2$).
(2) Let $w \equiv 1$, and let $x_1, \ldots, x_n$ be the zeros of p_n^λ for some $\lambda \in (-1/2, 2)$.
(3) Let $w \equiv 1$, let $x_1 = -1, x_n = 1$, and let $x_2, \ldots, x_{n-1}$ be the zeros of p_{n-2}^λ for some $\lambda \in [1/2, 4]$ (this includes the Clenshaw–Curtis formulae for $\lambda = 1$).
(4) Let $w \equiv 1$, let n be odd, and let $x_2, x_4, \ldots, x_{n-1}$ be the zeros of the Legendre polynomial of degree $(n-1)/2$, and $x_1, x_3, \ldots, x_n$ the zeros of the Stieltjes polynomial of degree $(n+1)/2$, so that the corresponding interpolatory quadrature formulae are the Gauss–Kronrod formulae.

It is known that in all four cases, the weights $a_1, \ldots, a_n$ are positive.

Under these assumptions, we can determine the asymptotic behaviour of the error constants precisely. In the following result, B_i denotes the ith Bernoulli polynomial with main coefficient $1/i!$, defined on $[0, 1]$.

THEOREM 2.1 *Let the Q_n, $n = 1, 2, \ldots$ be interpolatory quadrature formulae according to one of the four cases described above. Then,*

$$\lim_{n\to\infty} \frac{n^i}{\ln n} \|R_{n+1}^{\text{mod}}[\cdot;\xi]\|_i = 2\pi^i \|B_i\|_1 |w(\xi)|(1-\xi^2)^{i/2}$$

for all $i \geq 1$ and every $\xi \in (-1, 1)$.

The expression for the limit in Theorem 2.1 depends on i, w and ξ, but not on the special quadrature formula Q_n (i.e., not on the nodes x_i), so that, for functions of low smoothness (which means for non-analytical functions), it is likely that the performance of all quadrature formulae admitted in this subsection is very similar; other criteria like simplicity of the nodes x_i may therefore be preferred.

Remark 2.1 In [9, Theorem 2.3], the uniform upper bounds

$$\|R_{n+1}^{\text{mod}}[\cdot;\xi]\|_i \leq 2^{i+2}\pi^i \|B_i\|_\infty \left\| w\sqrt{(1-\cdot^2)} \right\|_\infty n^{-i} \ln n + C_i n^{-i}, \tag{5}$$

with an explicitly stated constant C_i, have been derived. A comparison with our Theorem 2.1 reveals that the bound of (5) overestimates the correct value significantly, especially if i is large and/or ξ is close to one of the end points of the interval of integration.

2.2 Compound Quadrature Formulae

Let nodes $0 \leq x_1^{\text{el}} < \cdots < x_r^{\text{el}} \leq 1$ be given, let $F[f] \in \mathcal{P}_{r-1}$ be the polynomial of degree $r-1$ interpolating a given function f defined on $[0, 1]$ at $x_1^{\text{el}}, \ldots, x_r^{\text{el}}$, and let $E[f] = f - F[f]$ denote the interpolation error.

Using F, a piecewise polynomial interpolant $F_{(m)}$ is defined in the following way. Let, for f defined on $[a, b]$,

$$F^j[f] = F[f \circ \Phi_j] \circ \Phi_j^{-1}, \quad \text{with } \Phi_j(t) = z_{j-1} + th_j,$$

and

$$F_{(m)}[f](x) = F^j[f](x) \quad \text{for } x \in (z_{j-1}, z_j),\ j = 1, \ldots, m.$$

Then a quadrature formula of compound type for $I_w[f] = \int_a^b f(x)w(x)\,\mathrm{d}x$ is defined by

$$Q_{(m)}[f] = I_w[F_{(m)}[f]].$$

This approach has been investigated, e.g., in Diethelm [5,6], Köhler [16], Noble and Beighton [19] and Stolle and Strauß [23].

In the notation of the previous sections, $Q_{(m)} = Q_n$, where $n = km + c$ for some $k \in \{r-1, r\}$ and $c \in \{0, 1\}$ but for convenience, we will use the notation $Q_{(m)}$ here.

Let the *elementary* quadrature formula Q^{el} be defined by $Q^{\mathrm{el}} = Q_{(1)}$ for $w \equiv 1$ and $[a, b] = [0, 1]$, and let

$$\mathcal{B}_i = K_i^{\mathrm{el}}$$

be the Peano kernels of Q^{el}, i.e.,

$$R^{\mathrm{el}}[f] = \int_0^1 f(x)\,\mathrm{d}x - Q^{\mathrm{el}}[f] = \int_0^1 f^{(i)}(t) K_i^{\mathrm{el}}(t)\,\mathrm{d}t.$$

Then, the following result describes the behaviour of the error constants for the modified quadrature formulae based on these compound methods.

THEOREM 2.2 *Assume that there exists a strictly increasing function* $z \in C^1[0, 1]$ *such that* $z(0) = a$, $z(1) = b$, *and*

$$mh_j = z'(z^{-1}(z_l)) + o(1)$$

uniformly for $l \in \{j-1, j\}$ *and* $z_{j-1}, z_j \in [\bar{a}, \bar{b}] \subset (a, b)$ *for all closed subintervals* $[\bar{a}, \bar{b}] \subset (a, b)$ *where* $h_j = z_j - z_{j-1}$. *Moreover, let*

$$\max_{1 \le j \le m} \left(h_j^{i-1} \int_{z_{j-1}}^{z_j} |w(x)|\,\mathrm{d}x \right) = O\left(\frac{1}{m^i}\right). \tag{6}$$

Then the error functional $R_{(m)}^{\mathrm{mod}}[\cdot;\xi] = I_{w,\xi} - Q_{(m)}^{\mathrm{mod}}[\cdot;\xi]$ *satisfies*

$$\lim_{m\to\infty} \frac{m^i}{\ln m} \|R_{(m)}^{\mathrm{mod}}[\cdot;\xi]\|_i = 2\|\mathcal{B}_i - b_i\|_1 |w(\xi)| (z'(z^{-1}(\xi)))^i$$

for $i = 1, \ldots, r$ *and* $\xi \in (a, b)$, *where* $b_i = \int_0^1 \mathcal{B}_i(x)\,\mathrm{d}x$, $\mathcal{B}_i = K_i^{\mathrm{el}}$.

Example 2.1 Let

$$z(x) = \begin{cases} a + (2x)^{\gamma}(b-a)/2 & \text{for } x \in [0, 1/2] \\ b - (2(1-x))^{\gamma}(b-a)/2 & \text{for } x \in [1/2, 1], \end{cases}$$

$$z_j = z\left(\frac{j}{m}\right) \quad \text{for } j = 0, \ldots, m,$$

and let

$$w(x) = (x-a)^{\alpha}(b-x)^{\beta}\psi(x),$$

where ψ is of Dini-type and $\alpha, \beta > -1$. Then (6) holds if

$$\gamma \geq \max\left(\frac{1}{\alpha+1}, \frac{1}{\beta+1}\right).$$

For $\gamma = 1$ and $w \equiv 1$ the $Q_{(m)}$ are the classical compound quadrature formulae with constant weight function and equidistant partition points (this case was treated in [16]). However, our new results now allow us to apply compound-type methods also for other weight functions. Moreover, by a suitable choice of the function z (that determines the mesh spacing), we can adapt the location of the nodes. Doing so, we may reduce adverse effects due to an irregular behaviour of the weight function.

Example 2.2 Let $[a, b] = [-1, 1]$,

$$z(x) = -\cos(\pi x),$$

and let z_j and w as in Example 2.1. Then (6) holds if

$$\alpha, \beta \geq -\tfrac{1}{2}.$$

Remark 2.2 Instead of defining F by polynomial interpolation, one can also use other linear approximation methods, cf. [15].

Remark 2.3 If only $w \equiv 1$ is considered, then it is not necessary that Q^{el} is defined by interpolation as above. Instead, one can choose for Q^{el} any quadrature formula defined on [0, 1] which is exact for polynomials of degree $r - 1 \geq 0$.

Remark 2.4 For $i = r + 1$, $K_{r+1,(m)}$ does not exist, so that the estimates developed here cannot be applied, though $K^{\mathrm{mod}}_{r+1,(m)}$ exists. As earlier results for $w \equiv 1$ and equidistant z_i show [6,16], it can be expected that $\|R^{\mathrm{mod}}_{(m)}[\cdot;\xi]\|_{r+1} = O(m^{-r})$.

3 A GENERAL RESULT ON THE ERROR CONSTANTS

For the error functional of the classical quadrature formulae R_n, we have Peano kernel representations

$$R_n[g] = \int_a^b g^{(i)}(x) K_{i,n}(x)\,\mathrm{d}x \quad \text{for } i = 1, \dots, r,$$

where the Peano kernels $K_{i,n}$ are defined by

$$K_{i,n}(x) = (-1)^i R_n\left[\frac{(x-\cdot)_+^{i-1}}{(i-1)!}\right] \quad \text{for } i = 1, \dots, r.$$

The behaviour of these Peano kernels is the crucial point in our considerations. Indeed, one can say that we require a certain regular behaviour of these Peano kernels. As we shall see in the proofs, most of the important types of classical quadrature formulae do have this regularity. In particular, this holds for compound methods and many positive interpolatory methods (Gauss, Gauss–Kronrod, Clenshaw–Curtis, etc.).

To be precise, we assume that there exist points

$$a = z_0 < z_1 < \cdots < z_m = b,$$

where n and m are related by

$$n = km + c \tag{7}$$

for some fixed values k and c, and that there exists a strictly increasing function $z \in C^1[0, 1]$ such that $z(0) = a$, $z(1) = b$, and

$$mh_j = z'(z^{-1}(z_l)) + o(1)$$
$$\text{uniformly for } l \in \{j-1, j\} \text{ and } z_{j-1}, z_j \in [\bar{a}, \bar{b}] \subset (a, b) \tag{8}$$

for all closed subintervals $[\bar{a}, \bar{b}] \subset (a, b)$, where $h_j = z_j - z_{j-1}$, and that there exist constants γ_i such that

$$\|K_{i,n}\|_\infty \leq \frac{\gamma_i}{n^i} \quad \text{for all } n. \tag{9}$$

We further need that $K_{i,n}$ behaves asymptotically like a function $K^{\mathcal{B}}_{i,n}$ of the form

$$K^{\mathcal{B}}_{i,n}(x) = w_j h_j^i \mathcal{B}_i\left(\frac{x - z_{j-1}}{h_j}\right) \quad \text{for } x \in [z_{j-1}, z_j], \tag{10}$$

where $\mathcal{B}_i$ is a function defined on [0, 1] which depends on $K_{i,n}$, and where

$$w_j = w(\eta_j) \quad \text{for some } \eta_j \in [z_{j-1}, z_j].$$

More precisely, this means that we assume the existence of constants $\gamma_{i,n} = \gamma_{i,n}(\bar{a}, \bar{b})$ (depending on n and the interval $[\bar{a}, \bar{b}]$) with $\gamma_{i,n} = o(1)$ for fixed i and $n \to \infty$, such that

$$\|K_{i,n} - K^{\mathcal{B}}_{i,n}\|_{L_\infty[\bar{a}, \bar{b}]} \leq \frac{\gamma_{i,n}}{n^i} = o\left(\frac{1}{n^i}\right) \tag{11}$$

for all $[\bar{a}, \bar{b}] \subset (a, b)$.

Having collected these hypotheses, we now state the fundamental theorem that we shall later use for the proofs of Theorems 2.1 and 2.2.

THEOREM 3.1 *Under the assumptions made above, there holds*

$$\lim_{n \to \infty} \frac{m^i}{\ln m} \|R^{\text{mod}}_{n+1}[\cdot; \xi]\|_i = 2\|\mathcal{B}_i - b_i\|_1 |w(\xi)| (z'(z^{-1}(\xi)))^i,$$

where $b_i = \int_0^1 \mathcal{B}_i(x)\,dx$.

Note that in statements where mainly the order is concerned, e.g., $\|K_{i,n}\|_\infty = O(n^{-i})$, it is of no importance if n or m is used (because of (7)); one could also write $\|K_{i,n}\|_\infty = O(m^{-i})$. However, in statements where the precise value of the constants is of interest, as in Theorem 3.1, one has to be more careful.

4 THE PROOFS OF THEOREMS 2.1 AND 2.2

To apply Theorem 3.1 to the case that Q_n is an interpolatory quadrature formula, it is necessary to show that the assumptions made in the previous section hold; especially, that there exists a function $K_{i,n}^{\mathcal{B}}$ of the form given in (10), such that (11) holds. For certain classes of interpolatory quadrature formulae, Petras [20] has shown that this is the case, with $\mathcal{B}_i$ being the Bernoulli polynomials. More precisely, let

$$\mathcal{B}_i = (-1)^i B_i$$

for the part dealing with the interpolatory formulae, where B_i is the Bernoulli polynomial of degree i, with main coefficient $1/i!$. The Bernoulli polynomials have the property that $\int_0^1 B_i(x)\,\mathrm{d}x = 0$ for $i \geq 1$, so that $b_i = 0$. Petras has considered rather general classes of weight functions and nodes, but for simplicity, we will restrict to the standard cases mentioned in the statement of Theorem 2.1 here.

Proof of Theorem 2.1 For open quadrature formulae (i.e., for $a < x_1$ and $x_n < b$), we define $m = n + 1$, $z_0 = -1$, $z_j = x_j$ for $j = 1, \ldots, m-1$, and $z_m = 1$, and for closed formulae (i.e., for $a = x_1$ and $x_n = \mathrm{b}$) we set $m = n - 1$ and $z_j = x_{j+1}$ for $j = 0, \ldots, m$. Further, let

$$z(x) = -\cos \pi x \quad \text{and} \quad w_j = w\left(\frac{z_{j-1} + z_j}{2}\right).$$

We consider the cases 1–3, using results of Petras [20]; the proof of case 4 is essentially the same, using results of Ehrich [10]. Equation (8) follows from the remark on page 218 of Freud [12] concerning the distance $\theta_{i+1} - \theta_i$, where $x_i = \cos \theta_i$. Further, there holds $\|K_{i,n}\|_\infty = O(n^{-i})$ (cf. [20, Eq. (1.2)]), so that (9) is satisfied. The relation $\|K_{i,n} - K_{i,n}^{\mathcal{B}}\|_{L_\infty[\bar{a},\bar{b}]} = o(n^{-i})$ for every $[\bar{a}, \bar{b}] \subset (-1, 1)$ follows from the results of Petras [20], so that also (11) holds. Therefore, we can apply Theorem 3.1, which, since $z'(z^{-1}(\xi)) = \pi(1 - \xi^2)^{1/2}$, completes the proof.

The proof of the corresponding theorem in the case of compound quadrature formulae relies on the construction principle of these formulae. In particular, our regularity assumptions on the Peano kernels are almost immediate consequences of this construction principle.

Proof of Theorem 2.2 The interpolation error E has Peano kernel representations

$$E[f](x) = \int_0^1 f^{(i)}(t) H_i(x,t)\,\mathrm{d}t$$

for $i = 1, \ldots, r$. We obtain

$$\begin{aligned} R_{(m)}[f] &= \sum_{j=1}^{m} \int_{z_{j-1}}^{z_j} w(x)(f(x) - F^i[f](x))\,\mathrm{d}x \\ &= \sum_{j=1}^{m} \int_{z_{j-1}}^{z_j} w(x) E[f \circ \Phi_j](\Phi_j^{-1}(x))\,\mathrm{d}x \\ &= \sum_{j=1}^{m} h_j^i \int_{z_{j-1}}^{z_j} w(x) \int_0^1 f^{(i)}(\Phi_j(t)) H_i(\Phi_j^{-1}(x), t)\,\mathrm{d}t\,\mathrm{d}x \\ &= \sum_{j=1}^{m} h_j^{i-1} \int_{z_{j-1}}^{z_j} f^{(i)}(u) \int_{z_{j-1}}^{z_j} w(x) H_i(\Phi_j^{-1}(x), \Phi_j^{-1}(u))\,\mathrm{d}x\,\mathrm{d}u \\ &= \int_a^b f^{(i)}(u) K_{i,(m)}(u)\,\mathrm{d}u, \end{aligned}$$

where

$$K_{i,(m)}(u) = h_j^{i-1} \int_{z_{j-1}}^{z_j} w(x) H_i(\Phi_j^{-1}(x), \Phi_j^{-1}(u))\,\mathrm{d}x \quad \text{for } u \in (z_{j-1}, z_j).$$

Let $w_j = \int_{z_{j-1}}^{z_j} w(x)\,\mathrm{d}x / h_j$, and let

$$\begin{aligned} K_{i,(m)}^{\mathcal{B}}(u) &= w_j h_j^{i-1} \int_{z_{j-1}}^{z_j} H_i(\Phi_j^{-1}(x), \Phi_j^{-1}(u))\,\mathrm{d}x \\ &= w_j h_j^i \mathcal{B}_i(\Phi_j^{-1}(u)). \end{aligned}$$

Then

$$K_{i,(m)}(u) - K_{i,(m)}^{\mathcal{B}}(u) = h_j^{i-1} \int_{z_{j-1}}^{z_j} (w(x) - w_j) H_i(\Phi_j^{-1}(x), \Phi_j^{-1}(u))\,\mathrm{d}x,$$

and

$$|K_{i,(m)}(u) - K^{\mathcal{B}}_{i,(m)}(u)| \leq h_j^i \|H_i\|_\infty \sup_{x,t\in[z_{j-1},z_j]} |w(x) - w(t)|$$
$$\text{for } u \in [z_{j-1}, z_j],$$

$2 \leq j \leq m-1$. Since $w \in C(a,b)$, this yields

$$\|K_{i,(m)} - K^{\mathcal{B}}_{i,(m)}\|_{L_\infty[\bar{a},\bar{b}]} = o\left(\frac{1}{m^i}\right)$$

for $[\bar{a},\bar{b}] \subset (a,b)$. Further,

$$\|K_{i,(m)}\|_\infty \leq \|H_i\|_\infty \max_{1\leq j\leq m} \left(h_j^{i-1} \int_{z_{j-1}}^{z_j} |w(x)|\, \mathrm{d}x \right) = O\left(\frac{1}{m^i}\right)$$

by assumption.

5 THE PROOF OF THEOREM 3.1

By [16], we have the following Peano kernel representations for the error functionals of the modified quadrature formula and the underlying classical formula, respectively:

$$R^{\text{mod}}_{n+1}[f;\xi] = \int_a^b f^{(i)}(x) K^{\text{mod}}_{i,n+1}(x;\xi)\, \mathrm{d}x - \delta_{i,1}\beta(\xi) f'(\xi)$$
$$\text{for } i = 1,\ldots,r+1,$$
$$R_n[g] = \int_a^b g^{(i)}(x) K_{i,n}(x)\, \mathrm{d}x \quad \text{for } i = 1,\ldots,r,$$
$$R_n[g] = -\int_a^b g^{(i)}(x)\, \mathrm{d}K_{i+1,n}(x) \quad \text{for } i = 0,\ldots,r-1,$$

where

$$\beta(\xi) = \begin{cases} 0 & \text{if } \xi \notin \{x_1,\ldots,x_n\}, \\ a_k & \text{if } \xi = x_k. \end{cases}$$

Therefore, by Hölder's inequality, we have that

$$\|R_{n+1}^{\mathrm{mod}}[\cdot;\xi]\|_i = \|K_{i,n+1}^{\mathrm{mod}}(\cdot;\xi)\|_{L_1(a,b)} + \delta_{i,1}|\beta(\xi)|.$$

Thus, in order to determine the behaviour of $\|R_{n+1}^{\mathrm{mod}}[\cdot;\xi]\|_i$, it will be useful to investigate the Peano kernels occurring here. These Peano kernels are defined by

$$K_{i,n+1}^{\mathrm{mod}}(x;\xi) = (-1)^i R_{n+1}^{\mathrm{mod}}\left[\frac{(x-\cdot)_+^{i-1}}{(i-1)!};\xi\right] \quad \text{for } i = 1,\dots,r+1, \tag{12}$$

and

$$K_{i,n}(x) = (-1)^i R_n\left[\frac{(x-\cdot)_+^{i-1}}{(i-1)!}\right] \quad \text{for } i = 1,\dots,r.$$

A key element in the proof is the following relation between the Peano kernel of the modified formula on the one hand and the Peano kernel of the classical formula on the other hand [9,16].

LEMMA 5.1 *Let the intervals $J_{\xi,x}$ be defined by*

$$J_{\xi,x} = \begin{cases} [a,x] & \text{for } x < \xi, \\ [x,b] & \text{for } x > \xi. \end{cases}$$

Then, for $x \in [a,b]\backslash\{\xi\}$, there holds

$$K_{i,n+1}^{\mathrm{mod}}(x;\xi) = |\xi - x|^{i-1}\int_{J_{\xi,x}} \frac{K_{i-1,n}(t)}{|\xi - t|^i}\,\mathrm{d}t \quad \text{for } i = 2,\dots,r+1, \tag{13}$$

$$K_{i,n+1}^{\mathrm{mod}}(x;\xi) = -|\xi - x|^{i-1}\int_{J_{\xi,x}} \frac{1}{|\xi - t|^i}\,\mathrm{d}K_{i,n}(t) \quad \text{for } i = 1,\dots,r. \tag{14}$$

This relation essentially allows us to express the relevant quantities in terms of Peano kernels of the classical formulas and use some facts about these kernels that have been established already.

Further, the following splitting of $K_{i,n+1}^{\mathrm{mod}}$ is used, which is obtained by partial integration of (14).

LEMMA 5.2 *Let* $1 \leq i \leq r$ *and* $x \in [a,b]\backslash\{\xi\}$. *Then*

$$K_{i,n+1}^{\mathrm{mod}}(x;\xi) = \mathcal{K}_i^1(x;\xi) + \mathcal{K}_i^2(x;\xi),$$

where

$$\mathcal{K}_i^1(x;\xi) = \frac{K_{i,n}(x)}{x-\xi},$$

$$\mathcal{K}_i^2(x;\xi) = i\,\mathrm{sign}(\xi - x)|\xi - x|^{i-1}\int_{J_{\xi,x}} \frac{K_{i,n}(t)}{|\xi - t|^{i+1}}\,\mathrm{d}t.$$

Now, we are in a position to give the proof of our fundamental result.

Proof of Theorem 3.1 Let $\xi \in (a,b)$ be fixed. In the following, we consider only the interval $[a,\xi]$, since the interval $[\xi,b]$ can be treated in a completely analogous way.

Choose $\bar{a}$ (not depending on n) and j_1, j_2 (depending on n) such that for n sufficiently large, the following holds:

$$a < \bar{a} \leq z_{j_1} < z_{j_2} < z_{j_2+1} \leq \xi.$$

For the proof, the interval $[a,\xi]$ is splitted into the three parts $[a, z_{j_1}]$, $[z_{j_1}, z_{j_2}]$ and $[z_{j_2}, \xi]$, and the Peano kernels are treated on each part separately:

$$\|K_{i,n+1}^{\mathrm{mod}}(\cdot;\xi)\|_{L_1[a,\xi]} = A + B + C, \tag{15}$$

where

$$A = \int_a^{z_{j_1}} |K_{i,n+1}^{\mathrm{mod}}(x;\xi)|\,\mathrm{d}x,$$

$$B = \int_{z_{j_1}}^{z_{j_2}} |K_{i,n+1}^{\mathrm{mod}}(x;\xi)|\,\mathrm{d}x,$$

$$C = \int_{z_{j_2}}^{\xi} |K_{i,n+1}^{\mathrm{mod}}(x;\xi)|\,\mathrm{d}x.$$

It is assumed that z_{j_1} and z_{j_2} have been chosen such that

$$\frac{1}{\xi - z_{j_1}} = O(\ln n)$$

and

$$\xi - z_{j_2} = O\left(\frac{1}{n}\right), \quad \frac{1}{\xi - z_{j_2}} = O(n).$$

Lemma 5.2 yields, for $a \le x < \xi$, that

$$\begin{aligned}
|K_{i,n+1}^{\mathrm{mod}}(x;\xi)| &\le \|K_{i,n}\|_\infty \left(\frac{1}{\xi - x} + i(\xi - x)^{i-1} \int_a^x \frac{1}{(\xi - t)^{i+1}} \, \mathrm{d}t \right) \\
&= \|K_{i,n}\|_\infty \left(\frac{1}{\xi - x} + (\xi - x)^{i-1} \left(\frac{1}{(\xi - x)^i} - \frac{1}{(\xi - a)^i} \right) \right) \\
&\le \frac{2}{\xi - x} \|K_{i,n}\|_\infty .
\end{aligned}$$

Recalling assumption (9) and the fact that $1/(\xi - z_{j_1}) = O(\ln n)$, this implies

$$A = \int_a^{z_{j_1}} |K_{i,n+1}^{\mathrm{mod}}(x;\xi)| \, \mathrm{d}x = O\left(\frac{\ln \ln n}{n^i}\right). \tag{16}$$

Note that this estimate holds uniformly for all $\xi \in (a, b)$ because $\bar{a}$ is not involved in the calculations.

Next, we consider the interval $[z_{j_2}, \xi]$. Using the representation (14), we derive $C \le C_1 + C_2$, where

$$\begin{aligned}
C_1 &= \int_{z_{j_2}}^{\xi} (\xi - x)^{i-1} \left| \int_a^{z_{j_2}} \frac{1}{(\xi - t)^i} \, \mathrm{d}K_{i,n}(t) \right| \mathrm{d}x \\
&= \frac{(\xi - z_{j_2})^i}{i} \left| \int_a^{z_{j_2}} \frac{1}{(\xi - t)^i} \, \mathrm{d}K_{i,n}(t) \right| \\
&= \frac{(\xi - z_{j_2})^i}{i} \left| \frac{K_{i,n}(z_{j_2})}{(\xi - z_{j_2})^i} - \int_a^{z_{j_2}} i \frac{K_{i,n}(t)}{(\xi - t)^{i+1}} \, \mathrm{d}t \right| \\
&\le \frac{2}{i} \|K_{i,n}\|_\infty = O\left(\frac{1}{n^i}\right),
\end{aligned}$$

and

$$C_2 = \int_{z_{j_2}}^{\xi} (\xi - x)^{i-1} \left| \int_{z_{j_2}}^{x} \frac{1}{(\xi - t)^i} \, \mathrm{d}K_{i,n}(t) \right| \mathrm{d}x$$
$$\leq \int_{z_{j_2}}^{\xi} (\xi - x)^{i-1} \int_{z_{j_2}}^{x} \frac{1}{(\xi - t)^i} \, \mathrm{d}V_{i,n}(t) \, \mathrm{d}x,$$

where $V_{i,n}(t) = \mathrm{Var}(K_{i,n}, [a, t])$ is the total variation of $K_{i,n}$ on the interval $[a, t]$. Changing the order of integration yields

$$C_2 \leq \int_{z_{j_2}}^{\xi} \frac{1}{(\xi - t)^i} \int_{t}^{\xi} (\xi - x)^{i-1} \, \mathrm{d}x \, \mathrm{d}V_{i,n}(t)$$
$$= \frac{1}{i} \int_{z_{j_2}}^{\xi} \mathrm{d}V_{i,n}(t)$$
$$= \frac{1}{i} \mathrm{Var}(K_{i,n}, [z_{j_2}, \xi]) \quad \text{for } 1 \leq i \leq r$$
$$\leq \frac{\xi - z_{j_2}}{i} \|K_{i-1,n}\|_\infty = O\left(\frac{1}{n^i}\right) \quad \text{for } 2 \leq i \leq r.$$

For $i = 1$, we use the fact that $K_{1,n}(x) = \bar{\kappa}(x) - \kappa(x)$, where

$$\bar{\kappa}(x) = \int_x^b w(t) \, \mathrm{d}t \quad \text{and} \quad \kappa(x) = \sum_{\mu=1}^{n} a_\mu (x_\mu - x)_+^0.$$

Then,

$$\mathrm{Var}(\bar{\kappa}, [z_{j_2}, \xi]) = \int_{z_{j_2}}^{\xi} |w(t)| \, \mathrm{d}t \leq \|w\|_{L_\infty[\bar{a},\xi]} (\xi - z_{j_2}) = O(1/n),$$

and

$$\mathrm{Var}(\kappa, [z_{j_2}, \xi]) \leq \sum_{z_{j_2} \leq x_\mu \leq \xi} |a_\mu| = O(1/n)$$

(since, as a consequence of $\|K_{1,n}\| = O(1/n)$, there holds $\max_{1 \leq \mu \leq n} |a_\mu| = O(1/n)$), and thus we have $C_2 = O(1/n^i)$ for $i = 1$, too.

Therefore,

$$C = O(n^{-i}) \tag{17}$$

for $1 \leq i \leq r$, and again this bound holds uniformly for all $\xi \in (a, b)$.

Combining (15), (16), and (17), we have

$$\|K_{i,n+1}^{\text{mod}}(\cdot;\xi)\|_{L_1[a,\xi]} = B + O(n^{-i}\ln\ln n)$$

uniformly for $\xi \in (a,b)$. It now remains to estimate B. Therefore, we now consider the interval $[z_{j_1}, z_{j_2}]$.

Let $\bar{K}_{i,n}$ be defined by

$$\bar{K}_{i,n}(x) = \begin{cases} K_{i,n}(x) & \text{for } x \in [a, z_{j_1}], \\ K_{i,n}^{\mathcal{B}}(x) & \text{for } x \in (z_{j_1}, \xi], \end{cases}$$

and $\bar{K}_{i,n+1}^{\text{mod}}$ by

$$\bar{K}_{i,n+1}^{\text{mod}}(x;\xi) = \frac{\bar{K}_{i,n}(x)}{x-\xi} + i\,\text{sign}(\xi - x)|\xi - x|^{i-1} \int_{J_{\xi,x}} |\xi - t|^{-i-1} \bar{K}_{i,n}(t)\,\mathrm{d}t\,.$$

For $z_{j_1} < x < \xi$, we obtain

$$\begin{aligned} &|K_{i,n+1}^{\text{mod}}(x;\xi) - \bar{K}_{i,n+1}^{\text{mod}}(x;\xi)| \\ &\quad\le \frac{|K_{i,n}(x) - \bar{K}_{i,n}(x)|}{\xi - x} + i(\xi - x)^{i-1}\left(\int_a^{z_{j_1}} + \int_{z_{j_1}}^{x}\right) \frac{|K_{i,n}(t) - \bar{K}_{i,n}(t)|}{(\xi - t)^{i+1}}\,\mathrm{d}t \\ &\quad\le \frac{\gamma_{i,n}}{n^i}\left(\frac{1}{\xi - x} + i(\xi - x)^{i-1}\int_{z_{j_1}}^{x} \frac{1}{(\xi - t)^{i+1}}\,\mathrm{d}t\right) \\ &\quad\le \frac{2\gamma_{i,n}}{n^i(\xi - x)}, \end{aligned}$$

since $K_{i,n}(t) - \bar{K}_{i,n}(t) = 0$ for $t \in [a, z_{j_1}]$. This implies

$$\int_{z_{j_1}}^{z_{j_2}} |K_{i,n+1}^{\text{mod}}(x;\xi) - \bar{K}_{i,n+1}^{\text{mod}}(x;\xi)|\,\mathrm{d}x = O\left(\frac{\gamma_{i,n}\ln n}{n^i}\right) = o\left(\frac{\ln n}{n^i}\right).$$

Let

$$\hat{K}_{i,n}(x) = \begin{cases} K_{i,n}(x) & \text{for } x \in [a, z_{j_1}], \\ w_j h_j^i b_i & \text{for } x \in (z_{j-1}, z_j],\ j = j_1 + 1, \ldots, j_2. \end{cases}$$

We split up $\bar{K}_{i,n+1}^{\mathrm{mod}}$ into four parts similar to Lemma 5.2, so that $\bar{K}_{i,n+1}^{\mathrm{mod}}(x;\xi) = \bar{\mathcal{K}}_i^1(x;\xi) + \bar{\mathcal{K}}_i^2(x;\xi) + \bar{\mathcal{K}}_i^3(x;\xi) + \bar{\mathcal{K}}_i^4(x;\xi)$ for $x \in [a,\xi)$, where

$$\begin{aligned}
\bar{\mathcal{K}}_i^1(x;\xi) &= \frac{\bar{K}_{i,n}(x) - \hat{K}_{i,n}(x)}{x-\xi},\\
\bar{\mathcal{K}}_i^2(x;\xi) &= i(\xi - x)^{i-1} \int_a^x \frac{\bar{K}_{i,n}(t) - \hat{K}_{i,n}(t)}{(\xi - t)^{i+1}}\, \mathrm{d}t,\\
\bar{\mathcal{K}}_i^3(x;\xi) &= i(\xi - x)^{i-1} \int_a^x \frac{\hat{K}_{i,n}(t) - \hat{K}_{i,n}(x)}{(\xi - t)^{i+1}}\, \mathrm{d}t,\\
\bar{\mathcal{K}}_i^4(x;\xi) &= -\hat{K}_{i,n}(x)\frac{(\xi - x)^{i-1}}{(\xi - a)^i}.
\end{aligned}$$

These four parts are now treated separately. Let $B_\mu = \int_{z_{j_1}}^{z_{j_2}} |\bar{\mathcal{K}}_i^\mu(x;\xi)|\, \mathrm{d}x$. Then

$$\begin{aligned}
B_1 &= \sum_{j=j_1+1}^{j_2} \int_{z_{j-1}}^{z_j} \left| \frac{w_j h_j^i \big(\mathcal{B}_i\big((x - z_{j-1})/h_j\big) - b_i\big)}{x-\xi} \right| \mathrm{d}x\\
&\le \sum_{j=j_1+1}^{j_2} \frac{|w_j| h_j^i}{\xi - z_j} \int_{z_{j-1}}^{z_j} \left| \mathcal{B}_i\left(\frac{x - z_{j-1}}{h_j}\right) - b_i)\right| \mathrm{d}x\\
&= \frac{1}{m^i} \|\mathcal{B}_i - b_i\|_1 \sum_{j=j_1+1}^{j_2} h_j \frac{|w_j|(mh_j)^i}{\xi - z_j},
\end{aligned}$$

and analogously, using $1/(\xi - x) \ge 1/(\xi - z_{j-1})$, one obtains

$$B_1 \ge \frac{1}{m^i} \|\mathcal{B}_i - b_i\|_1 \sum_{j=j_1+1}^{j_2} h_j \frac{|w_j|(mh_j)^i}{\xi - z_{j-1}},$$

which yields

$$B_1 = \|\mathcal{B}_i - b_i\|_1 |w(\xi)| (z'(z^{-1}(\xi)))^i \frac{\ln m}{m^i} + o\left(\frac{\ln m}{m^i}\right),$$

since $\lim_{n\to\infty} z_{j_1} = \lim_{n\to\infty} z_{j_2} = \xi$.

Let $z_{j-1} \le x \le z_j < \xi, j_1 + 1 \le j \le j_2$. Then

$$|\bar{\mathcal{K}}_i^2(x;\xi)| = i(\xi - x)^{i-1}\left|\int_{z_{j_1}}^{x} \frac{\bar{K}_{i,n}(t) - \hat{K}_{i,n}(t)}{(\xi - t)^{i+1}}\,\mathrm{d}t\right|$$
$$= i(\xi - x)^{i-1}\left|\sum_{\nu=j_1+1}^{j-1} w_\nu h_\nu^i \int_{z_{\nu-1}}^{z_\nu}\left(\mathcal{B}_i\left(\frac{t - z_{\nu-1}}{h_\nu}\right) - b_i\right)\phi(t)\,\mathrm{d}t\right.$$
$$\left. + w_j h_j^i \int_{z_{j-1}}^{x}\left(\mathcal{B}_i\left(\frac{t - z_{j-1}}{h_j}\right) - b_i\right)\phi(t)\,\mathrm{d}t\right|,$$

where $\phi(t) = 1/(\xi - t)^{i+1}$. Since $\int_0^1 (\mathcal{B}_i(t) - b_i)\,\mathrm{d}t = 0$, we obtain

$$\left|\int_{z_{\nu-1}}^{z_\nu}\left(\mathcal{B}_i\left(\frac{t - z_{\nu-1}}{h_\nu}\right) - b_i\right)\phi(t)\,\mathrm{d}t\right|$$
$$= \left|\int_{z_{\nu-1}}^{z_\nu}\left(\mathcal{B}_i\left(\frac{t - z_{\nu-1}}{h_\nu}\right) - b_i\right)(\phi(t) - \phi(z_\nu))\,\mathrm{d}t\right|$$
$$\le h_\nu(\phi(z_{\nu-1}) - \phi(z_\nu))\|\mathcal{B}_i - b_i\|_1.$$

Together with

$$\left|\int_{z_{j-1}}^{x}\left(\mathcal{B}_i\left(\frac{t - z_{j-1}}{h_j}\right) - b_i\right)\phi(t)\,\mathrm{d}t\right| \le h_j\phi(x)\|\mathcal{B}_i - b_i\|_1,$$

this yields

$$|\bar{\mathcal{K}}_i^2(x;\xi)| \le i\|\mathcal{B}_i - b_i\|_1(\xi - x)^{i-1}$$
$$\times\left(\sum_{\nu=j_1+1}^{j-1} |w_\nu| h_\nu^{i+1}(\phi(z_{\nu-1}) - \phi(z_\nu)) + |w_j| h_j^{i+1}\phi(x)\right)$$
$$\le 2i\|\mathcal{B}_i - b_i\|_1 \max_{j_1+1\le\nu\le j}(|w_\nu| h_\nu^{i+1})\frac{1}{(\xi - x)^2},$$

so that

$$B_2 \le 2i\|\mathcal{B}_i - b_i\|_1\|w\|_{L_\infty[\bar{a},\xi]} \max_{j_1+1\le\nu\le j_2} h_\nu^{i+1}\frac{1}{\xi - z_{j_2}} = O\left(\frac{1}{n^i}\right).$$

Obviously, $B_3 \leq B_{3,1} + B_{3,2}$, where

$$\begin{aligned} B_{3,1} &= \int_{z_{j_1}}^{z_{j_2}} i(\xi - x)^{i-1} \left| \int_a^{z_{j_1}} \frac{\hat{K}_{i,n}(t) - \hat{K}_{i,n}(x)}{(\xi - t)^{i+1}} \,\mathrm{d}t \right| \mathrm{d}x \\ &\leq \left(\|K_{i,n}\|_\infty + \max_{j_1+1 \leq \nu \leq j} (|w_\nu| h_\nu^i) |b_i| \right) \\ &\quad \times \int_{z_{j_1}}^{z_{j_2}} i(\xi - x)^{i-1} \int_a^{z_{j_1}} \frac{1}{(\xi - t)^{i+1}} \,\mathrm{d}t \,\mathrm{d}x \\ &\leq \left(\|K_{i,n}\|_\infty + \|w\|_{L_\infty[\bar{a},\xi]} \max_{j_1+1 \leq \nu \leq j_2} h_\nu^i |b_i| \right) \frac{1}{i} = O\left(\frac{1}{n^i}\right), \end{aligned}$$

and

$$\begin{aligned} B_{3,2} &= \int_{z_{j_1}}^{z_{j_2}} i(\xi - x)^{i-1} \left| \int_{z_{j_1}}^{x} \frac{\hat{K}_{i,n}(t) - \hat{K}_{i,n}(x)}{(\xi - t)^{i+1}} \,\mathrm{d}t \right| \mathrm{d}x \\ &\leq \sup_{x,t \in [z_{j_1}, z_{j_2}]} |\hat{K}_{i,n}(t) - \hat{K}_{i,n}(x)| \ln \frac{\xi - z_{j_1}}{\xi - z_{j_2}} \\ &= \sup_{x,t \in [z_{j_1}, z_{j_2}]} |\hat{K}_{i,n}(t) - \hat{K}_{i,n}(x)| O(\ln n) = o\left(\frac{\ln n}{n^i}\right), \end{aligned}$$

since

$$\sup_{x,t \in [z_{j_1}, z_{j_2}]} |\hat{K}_{i,n}(t) - \hat{K}_{i,n}(x)| = \frac{|b_i|}{n^i} \max_{\mu,\nu \in \{j_1+1,\dots,j_2\}} |w_\mu (nh_\mu)^i - w_\nu (nh_\nu)^i|$$

and $w_\mu \to w(\xi)$, $nh_\mu \to z'(z^{-1}(\xi))$ uniformly for all $\mu \in \{j_1 + 1, \dots, j_2\}$. Further,

$$\begin{aligned} B_4 &\leq \|\hat{K}_{i,n}\|_\infty \int_{z_{j_1}}^{z_{j_2}} \frac{(\xi - x)^{i-1}}{(\xi - a)^i} \,\mathrm{d}x \\ &\leq \|\hat{K}_{i,n}\|_\infty \frac{(\xi - z_{j_1})^i}{i(\xi - a)^i} = o\left(\frac{1}{n^i}\right). \end{aligned}$$

Collecting the estimates obtained above, yields

$$B = \int_{z_{j_1}}^{z_{j_2}} |K_{i,n+1}^{\mathrm{mod}}(x;\xi)|\,\mathrm{d}x = \int_{z_{j_1}}^{z_{j_2}} |\bar{K}_{i,n+1}^{\mathrm{mod}}(x;\xi)|\,\mathrm{d}x + o\left(\frac{\ln m}{m^i}\right)$$

$$= \|\mathcal{B}_i - b_i\|_1 |w(\xi)| (z'(z^{-1}(\xi)))^i \frac{\ln m}{m^i} + o\left(\frac{\ln m}{m^i}\right).$$

Therefore,

$$\|K_{i,n+1}^{\mathrm{mod}}(\cdot;\xi)\|_{L_1[a,\xi]} = \|\mathcal{B}_i - b_i\|_1 |w(\xi)| (z'(z^{-1}(\xi)))^i \frac{\ln m}{m^i} + o\left(\frac{\ln m}{m^i}\right).$$

An analogous reasoning can be applied to the interval $[\xi, b]$, completing the proof of Theorem 3.1.

References

[1] M.M. Chawla and N. Jayarajan, Quadrature formulas for Cauchy principal value integrals. *Computing*, **15** (1975), 347–355.

[2] G. Criscuolo and G. Mastroianni, On the convergence of the Gauss quadrature rules for the Cauchy principal value integrals. *Ricerche Mat.*, **35** (1986), 45–60.

[3] G. Criscuolo and G. Mastroianni, On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals. *Numer. Math.*, **54** (1989), 445–461.

[4] P.J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, 2nd edn., Academic Press, Orlando, FL, 1984.

[5] K. Diethelm, Error estimates for a quadrature rule for Cauchy principal value integrals. *Proc. Sympos. Appl. Math.*, **48** (1994), 287–291.

[6] K. Diethelm, Modified compound quadrature rules for strongly singular integrals. *Computing*, **52** (1994), 337–354.

[7] K. Diethelm, Uniform convergence of optimal order quadrature rules for Cauchy principal value integrals. *J. Comput. Appl. Math.*, **56** (1994), 321–329.

[8] K. Diethelm, Asymptotically sharp error bounds for a quadrature rule for Cauchy principal value integrals based on piecewise linear interpolation. *Approx. Theory Appl. (N.S.)*, **11**(4) (1995), 78–89.

[9] K. Diethelm, Peano kernels and bounds for the error constants of Gaussian and related quadrature rules for Cauchy principal value integrals. *Numer. Math.*, **73** (1996), 53–63.

[10] S. Ehrich, A note on Peano constants of Gauss–Kronrod quadrature schemes. *J. Comput. Appl. Math.*, **66** (1996), 177–183.

[11] D. Elliott and D.F. Paget, Gauss type quadrature rules for Cauchy principal value integrals. *Math. Comp.*, **33** (1979), 301–309.

[12] G. Freud, Orthogonale Polynome, VEB Deutscher Verlag der Wissenschaften, Berlin, 1969.

[13] W. Gautschi, A survey of Gauss–Christoffel quadrature formulae. In *E.B. Christoffel – The Influence of his Works on Mathematics and the Physical Sciences*, P.L. Butzer and F. Fehér, Eds., Birkhäuser, Basel, 1981, pp. 72–147.

[14] D.B. Hunter, Some Gauss-type formulae for the evaluation of Cauchy principal value integrals. *Numer. Math.*, **19** (1972), 419–424.

[15] P. Köhler, Error estimates for generalized compound quadrature formulas. *IMA J. Numer. Anal.*, **13** (1993), 477–491.

[16] P. Köhler, Asymptotically sharp error estimates for modified compound quadrature formulae for Cauchy principal value integrals. *Computing*, **55** (1995), 255–269.

[17] P. Köhler, On the error of quadrature formulae for Cauchy principal value integrals based on piecewise interpolation. *Approx. Theory Appl.* (*N.S.*), **13**(3) (1997), 58–69.

[18] G. Monegato, The numerical evaluation of one-dimensional Cauchy principal value integrals. *Computing*, **29** (1982), 337–354.

[19] B. Noble and S. Beighton, Error estimates for three methods of evaluating Cauchy principal value integrals. *J. Inst. Maths. Applics.*, **26** (1980), 431–446.

[20] K. Petras, Asymptotic behaviour of Peanokernels of fixed order. In *Numerical Integration III*, H. Braß and G. Hämmerlin, Eds., ISNM 85, Birkhäuser, Basel, 1988, pp. 186–198.

[21] P. Rabinowitz, Gauss–Kronrod integration rules for Cauchy principal value integrals. *Math. Comp.*, **41** (1983), 63–78; Corrigendum: *Math. Comp.*, **45** (1985), 277.

[22] P. Rabinowitz, On the convergence of Hunter's method for Cauchy principal value integrals. In *Numerical Solution of Singular Integral Equations*, A. Gerasoulis and R. Vichnevetsky, Eds., IMACS, 1984, pp. 86–88.

[23] H.W. Stolle and R. Strauß, On the numerical integration of certain singular integrals. *Computing*, **48** (1992), 177–189.