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A Generalized 2-D Poincaré Inequality

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Two 1-D Poincaré-like inequalities are proved under the mild assumption that the integrand function is zero at just one point. These results are used to derive a 2-D generalized Poincaré inequality in which the integrand function is zero on a suitable arc contained in the domain (instead of the whole boundary). As an application, it is shown that a set of boundary conditions for the quasi geostrophic equation of order four are compatible with general physical constraints dictated by the dissipation of kinetic energy.

Keywords: Inequalities; Quasi-geostrophic equations; Boundary conditions

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1 INTRODUCTION

Very roughly speaking, Poincaré inequality states that the L^2-norm of a function is less than the L^2-norm of its gradient, provided that the function fulfills some general requirements. Usually, Poincaré inequality is formulated in terms of a class of functions that are equal to zero on the boundary of a finite domain, and is typically applied – outside functional analysis – to stability problems in continuum mechanics (e.g., [1]).

In this work, by focusing on one- and two-dimensional situations, we give sharper results in that the involved functions are assumed to be zero only on a part of the boundary (or on a topologically equivalent part of the interior). Moreover, we motivate and illustrate the relevance of these results with an application to an important problem in (geophysical)

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fluid dynamics, namely, that of finding physically admissible boundary conditions for the quasi-geostrophic equations when the turbulence is parametrized by means of lateral diffusion of vorticity, thus giving rise to fourth-order spatial derivatives.

2 POINCARÉ INEQUALITIES WITH NONZERO BOUNDARY CONDITIONS

In the following lemmata, a prime (′) denotes differentiation.

LEMMA 1 *If* $\phi:[a,b]\to\mathbb{R}$ *is a smooth function such that* $\phi[c]=0$ *for a given* $c\in[a,b]$, *then*

$$\int_a^b \phi^2[x]\,\mathrm{d}x \le 4(b-a)^2\int_a^b (\phi'[x])^2\,\mathrm{d}x. \tag{1}$$

Proof Using the assumption $\phi[c]=0$, we get the identity

$$\phi^2[x]=2\int_c^x \phi\phi' \quad \forall x\in[a,b].$$

Hence, because of Schwarz inequality,

$$\phi^2[x]\le 2\left|\int_c^x \phi^2\right|^{1/2}\left|\int_c^x (\phi')^2\right|^{1/2}$$

and, a fortiori,

$$\phi^2[x]\le 2\left(\int_a^b \phi^2\right)^{1/2}\left(\int_a^b (\phi')^2\right)^{1/2}.$$

Integrating with respect to x, we obtain

$$\int_a^b \phi^2\le 2(b-a)\left(\int_a^b \phi^2\right)^{1/2}\left(\int_a^b (\phi')^2\right)^{1/2}.$$

Simplifying and squaring yields inequality (1).

LEMMA 2 *If* $0 < a$, *and* $\phi : [a, b] \to \mathbb{R}$ *is a smooth function such that* $\phi[c] = 0$ *for a given* $c \in [a, b]$, *then*

$$\int_a^b \phi^2[x] x \, dx \le 4 \frac{b}{a} (b - a)^2 \int_a^b (\phi'[x])^2 x \, dx. \tag{2}$$

Proof From the obvious inequality

$$\int_a^b \phi^2[x] x \, dx \le b \int_a^b \phi^2[x] \, dx$$

we get, using Lemma 1,

$$\int_a^b \phi^2[x] x \, dx \le 4b(b - a)^2 \int_a^b (\phi'[x])^2 \, dx.$$

The integral in the right-hand side of this inequality may be estimated as

$$\int_a^b (\phi'[x])^2 \, dx \le \frac{1}{a} \int_a^b (\phi'[x])^2 x \, dx.$$

Our claim follows from the last two inequalities.

The following Theorem 1 is the main mathematical result of this paper. Its statement, perhaps a little obscure at a first reading, becomes much clearer after looking at Fig. 1. Point **c** of Fig. 1, although not explicitly mentioned in Theorem 1, corresponds to point **c** of Lemma 1 and Lemma 2. Obviously, any bounded convex domain D satisfies the assumptions of Theorem 1.

THEOREM 1 *Let* $D \subset \mathbb{R}^2$ *be a bounded set whose boundary* ∂D *is a Jordan curve. Assume there is a point O external to* ∂D *such that the intersection between D and any line through O is either empty or a segment. Assume further that there is a continuous arc* $AB \subset D$ *such that the angle* $\widehat{AOB}$ *contains D. Then there is a constant C such that*

$$\int_D \zeta^2 \, dx \, dy \le C \int_D |\nabla \zeta|^2 \, dx \, dy \tag{3}$$

for any smooth function ζ *equal to zero on arc AB.*

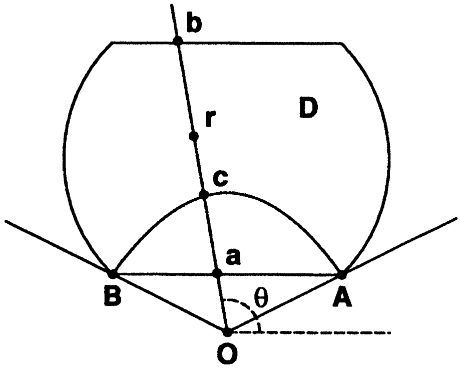

FIGURE 1 Heuristic illustration of the symbols appearing in the statement of Theorem 1. In the application to quasi-geostrophic equations, the curved arcs (straight-line segments) in the boundary of D represent the coastlines (zonal boundaries where the wind forcing vanishes) of the subtropical North Atlantic ocean.

Proof Let (r, θ) be polar coordinates with origin O, and let $[a, b]$ denote the intersection (when nonempty) of D with a straight line through O, where $a = a[\theta]$ and $b = b[\theta]$. Defining $\varphi[r, \theta] = \zeta[r\cos[\theta], r\sin[\theta]]$, we may apply Lemma 2 to the function $r \mapsto \varphi[r, \theta]$ obtaining

$$\int_{a[\theta]}^{b[\theta]} \varphi^2[r, \theta] r \, \mathrm{d}r \leq K[\theta] \int_{a[\theta]}^{b[\theta]} \left(\frac{\partial \varphi}{\partial r}\right)^2 r \, \mathrm{d}r,$$

where

$$K[\theta] = 4\frac{b[\theta]}{a[\theta]}(b[\theta] - a[\theta])^2.$$

Hence, a fortiori,

$$\int_{\theta_A}^{\theta_B} \int_{a[\theta]}^{b[\theta]} \varphi^2[r, \theta] r \, \mathrm{d}r \, \mathrm{d}\theta \leq K_{\max} \int_{\theta_A}^{\theta_B} \int_{a[\theta]}^{b[\theta]} \left(\left(\frac{\partial \varphi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)^2\right) r \, \mathrm{d}r \, \mathrm{d}\theta,$$

where $K_{\max}$ is the constant defined by

$$K_{\max} = \max\{K[\theta]: \ \theta_A \leq \theta \leq \theta_B\},$$

which is well defined because O is assumed to be external to ∂D. Passing from polar to Cartesian coordinates in the last inequality, our claim follows with $C = K_{\max}$.

3 AN APPLICATION TO OCEAN CIRCULATION

The large-scale circulation in the upper oceanic layer is described by the quasi-geostrophic equation [2, p. 32]

$$\frac{\partial}{\partial t}\nabla^2\psi + R\,J[\psi,\nabla^2\psi] + \frac{\partial\psi}{\partial x} = (\mathrm{curl}\,\boldsymbol{\tau})_z + \epsilon\nabla^2\nabla^2\psi, \tag{4}$$

where $\psi = \psi[x,y]$ is the stream function, J is the Jacobian (or Poisson bracket) operator defined by $J[a,b] = \mathrm{div}[a\nabla b \times \mathbf{k}]$, $\boldsymbol{\tau}$ is wind stress, R and ϵ are positive constants, t is time, and x, y, z are Cartesian coordinates with $\mathbf{k}$ the unit vector along z-direction. Since Eq. (4) is of order four in space, one more boundary condition is needed besides the obvious no-mass flux condition:

$$\psi[x,y,t] = 0, \quad \forall\{x,y\} \in \partial D,\ \forall t. \tag{5}$$

The problem of finding a physically appropriate auxiliary (so-called "dynamic") boundary condition is far from being trivial, because boundary conditions affect also the qualitative behavior of the flow, which must fulfill general energy-related constraints. In the following, we shall use Theorem 1 to show that, if the "ocean" D satisfies the assumptions of Theorem 1 (see also Fig. 1), then the solution arising from the mixed boundary condition

$$\frac{\partial}{\partial \mathbf{n}}\nabla^2\psi = 0 \quad \text{on the coastline,} \tag{6}$$

$$\nabla^2\psi = 0 \quad \text{on the sea boundary} \tag{7}$$

successfully passes the following tests:

(1) The kinetic energy of any flow is bounded if the forcing term is in L^2.
(2) The kinetic energy of any flow with zero forcing tends to zero for $t \to \infty$.

Multiplying Eq. (4) by the relative vorticity $\zeta = \nabla^2\psi$, and integrating over the domain D, we have

$$\int_D \zeta\,\frac{\partial}{\partial t}\zeta + R\int_D \zeta\,J[\psi,\zeta] + \int_D \zeta\,\frac{\partial\psi}{\partial x} = \int_D \zeta\,(\mathrm{curl}\,\boldsymbol{\tau})_z + \epsilon\int_D \zeta\nabla^2\zeta. \tag{8}$$

We see immediately that

$$\int_D \zeta \frac{\partial}{\partial t}\zeta = \frac{1}{2}\frac{\partial}{\partial t}\int_D \zeta^2. \tag{9}$$

Straightforward computations (using the identity $bJ[a,b]=J[a,b^2]/2$ and the 2-D divergence theorem $\int_D \mathrm{div} = \int_{\partial D} \mathbf{n}\cdot$ with boundary condition (5)) show that

$$\int_D \zeta\, J[\psi,\zeta] = 0. \tag{10}$$

Integrating over D the identity $\zeta\nabla^2\zeta = \mathrm{div}(\zeta\nabla\zeta) - |\nabla\zeta|^2$, applying the 2-D divergence theorem, and using our mixed boundary conditions yields

$$\int_D \zeta\nabla^2\zeta = -\int_D |\nabla\zeta|^2. \tag{11}$$

Substituting Eqs. (9)–(11) into Eq. (8) gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int_D \zeta^2 + \int_D \zeta\frac{\partial\psi}{\partial x} = \int_D \zeta\,(\mathrm{curl}\,\boldsymbol{\tau})_z - \epsilon\int_D |\nabla\zeta|^2. \tag{12}$$

Integrating over D the identity

$$\zeta\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}(\psi\zeta) - \mathrm{div}\left(\psi\nabla\frac{\partial\psi}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial x}|\nabla\psi|^2,$$

and applying Green's formula $\int_D \partial_x = \int_{\partial D} \mathrm{d}y$, together with boundary condition (5), we get

$$\int_D \zeta\frac{\partial\psi}{\partial x} = \frac{1}{2}\int_{\partial D} \mathrm{d}y|\nabla\psi|^2 \geq 0. \tag{13}$$

From (12) and (13), using Theorem 1, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_D \zeta^2 \leq \int_D \zeta\,(\mathrm{curl}\,\boldsymbol{\tau})_z - \frac{\epsilon}{C}\int_D \zeta^2.$$

We point out that in this case the arc AB of Theorem 1 coincides with segment AB of Fig. 1, where $\zeta = 0$ because of (7). Denoting the L^2-norm by $||\cdot||$, using the Schwarz inequality, and simplifying, the previous relation may be rearranged as

$$\frac{\partial}{\partial t}||\zeta|| \leq ||(\text{curl}\,\boldsymbol{\tau})_z|| - \frac{\epsilon}{C}||\zeta||,$$

which implies

$$||\zeta[t]|| \leq \exp\left[-\frac{\epsilon}{C}t\right]\left(||\zeta[0]|| - \frac{C}{\epsilon}||(\text{curl}\,\boldsymbol{\tau})_z||\right) + \frac{C}{\epsilon}||(\text{curl}\,\boldsymbol{\tau})_z||.$$

On the other hand Crisciani and Purini [3] show that

$$E[t] \leq \frac{1}{2\lambda^2}||\zeta[t]||^2,$$

where $E[t] = (1/2)||\nabla\psi[t]||^2$ represents the kinetic energy, and λ is a constant. From the last two inequalities we get

$$E[t] \leq \frac{1}{2\lambda^2}\left(\exp\left[-\frac{\epsilon}{C}t\right]\left(||\zeta[0]|| - \frac{C}{\epsilon}||(\text{curl}\,\boldsymbol{\tau})_z||\right) + \frac{C}{\epsilon}||(\text{curl}\,\boldsymbol{\tau})_z||\right)^2$$

and hence

$$\lim_{t\to\infty} E[t] = \frac{1}{2}\left(\frac{C}{\lambda\epsilon}||(\text{curl}\,\boldsymbol{\tau})_z||\right)^2,$$

whence we immediately deduce that our mixed boundary condition has successfully passed both the tests previously stated.

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