J. of Inequal. & Appl., 2000, Vol. 5, pp. 433–446
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Some Localization Theorems Using a Majorization Technique

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(Received 3 June 1999; Revised 30 September 1999)

In this note we localize ordered real numbers through their upper and lower bounds solving a class of nonlinear optimization problems. To this aim, a majorization technique, which involves Schur-convex functions, has been applied and maximum and minimum elements of suitable sets are considered. The bounds we develop can be expressed in terms of the mean and higher centered moments of the number distribution. Meaningful results are obtained for real eigenvalues of a matrix of order n. Finally, numerical examples are provided, showing how former results in the literature can be sometimes improved through those methods.

Keywords: Majorization order; Schur-convex (concave) functions; Nonlinear global optimization

AMS 1991 Subject Classifications: Primary 26A51, 62G30; Secondary 34L15, 90C30

1 INTRODUCTION

Given a set of ordered real numbers $x_1 \geq x_2 \geq \cdots \geq x_n$ and their mean μ and standard deviation σ, it is well known (Wolkowicz and Styan [16]) that, by Chebychev and Cauchy–Schwarz inequalities, for any real number x_k the following inequalities hold:

$$\mu - \sigma\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq x_k \leq \mu + \sigma\left(\frac{n-k}{k}\right)^{1/2}, \quad 1 \leq k \leq n. \tag{1.1}$$

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A meaningful application of these inequalities concerns the localization of eigenvalues of a real spectrum matrix, for example a symmetric matrix or, more generally, a symmetrizable matrix (Engel and Schneider [4], Stefani and Torriero [12], [13]). Indeed, in this case, the mean and the standard deviation can be directly computed using the matrix and its second power traces respectively. In this framework, better lower and upper bound of $\{x_1, x_2, \ldots, x_n\}$ have been found by Tarazaga [14,15] and Merikoski *et al.* [8]. More recently Bianchi and Torriero [3] have shown that by using higher moments of the distributions of x_i (where x_i are not necessarily eigenvalues of a matrix of order n), tighter bounds for x_1 are provided which include the ones previously found in the literature as a particular case. Those bounds are obtained by means of nonlinear global optimization problems solved through majorization techniques, which involve Schur-convex functions [9]. Obviously, if x_i are the eigenvalues of a real spectrum matrix A, then the rth moments $\mu^{(r)}$ of their distribution [5] can be expressed as functions of the traces of the matrix itself and of its powers as follows:

$$\mu^{(r)} = \frac{\sum_{i=1}^{n} x_i^r}{n} = \frac{\operatorname{tr}(A^r)}{n}, \quad r = 1, 2, 3, 4, \ldots \tag{1.2}$$

Furthermore

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n} = \frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2, \tag{1.3}$$

where $\mu = \mu^{(1)}$. In this paper we extend some results due to Bianchi and Torriero [3] and upper and lower bounds for all x_i $(i = 1, 2, \ldots, n)$ throughout the solution of a class of suitable nonlinear optimization problems are obtained. In addition, some numerical examples are presented in order to compare our bounds with the known ones.

Finally, even though our results have been obtained by requiring the nonnegativity of x_i, we will show later on that this assumption can be relaxed.

2 NOTATIONS AND PRELIMINARY RESULTS

Let $\mathbf{e}^j, j = 1, \ldots, n$, be the fundamental vectors of $\mathbf{R}^n$ and set

$$\mathbf{s}^j = \sum_{i=1}^{j} \mathbf{e}^i, \quad j = 1, \ldots, n,$$
$$\mathbf{v}^j = \mathbf{s}^n - \mathbf{s}^j.$$

Assuming that the components of the vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are arranged in a nonincreasing order, the majorization order $\mathbf{x} \leq_m \mathbf{y}$ means:

$$\langle \mathbf{x}, \mathbf{s}^k \rangle \leq \langle \mathbf{y}, \mathbf{s}^k \rangle, \quad k = 1, \ldots, (n-1),$$
$$\langle \mathbf{x}, \mathbf{s}^n \rangle = \langle \mathbf{y}, \mathbf{s}^n \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$ [1], [2], [11].

We recall that for the set $\Sigma = \{\mathbf{x} \in \mathbf{R}^n\colon\ x_1 \geq x_2 \geq \cdots \geq x_n \geq 0, \langle \mathbf{x}, \mathbf{s}^n \rangle = a\}$ as is well known (Marshall and Olkin [7]), the maximum and the minimum elements with respect to the majorization order are respectively

$$\mathbf{x}^* = a\mathbf{e}^1 \quad \text{and} \quad \mathbf{x}_* = (a/n)\mathbf{s}^n.$$

Thus $\mathbf{x} \in \Sigma$ implies $\mathbf{x}_* \leq_m \mathbf{x} \leq_m \mathbf{x}^*$.

DEFINITION 2.1 *A function $\phi: A \to \mathbf{R}, A \subseteq \mathbf{R}^n$, is called Schur-convex, on A if $\mathbf{x} \leq_m \mathbf{y}$ implies $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$. ϕ is said to be strictly Schur-convex if $\phi(\mathbf{x}) < \phi(\mathbf{y})$ and $\mathbf{x}$ is not a permutation of $\mathbf{y}$. ϕ is called Schur-concave on A if $\mathbf{x} \leq_m \mathbf{y}$ implies $\phi(\mathbf{x}) \geq \phi(\mathbf{y})$. ϕ is said to be strictly Schur-concave if $\phi(\mathbf{x}) > \phi(\mathbf{y})$ and $\mathbf{x}$ is not a permutation of $\mathbf{y}$.*

Thus, the Schur-convex functions preserve the ordering of majorization.

Observing that permutation preserves majorization, it follows that a function ϕ, defined on a symmetric set A, is Schur-convex on A if ϕ is Schur-convex on $D \cap A$, where $D = \{\mathbf{x} \in \mathbf{R}^n\colon\ x_1 \geq x_2 \geq \cdots \geq x_n\}$. As a consequence, we may assume later on, without loss of generality, that the coordinates of vectors are arranged in nonincreasing order.

In the present paper we only deal with Schur-convex functions. Analogous results can be proved for Schur-concave functions.

Let us firstly recall some basic results proved in Bianchi and Torriero [3]. Let g be a continuous function, homogeneous of degree p, p real, and strictly Schur-convex. Let us assume

$$S = \Sigma \cap \{\mathbf{x} \in \mathbf{R}^n\colon\ g(\mathbf{x}) = b\},$$

where $b \in \mathbf{R}$ is fixed such that $S \neq \emptyset$ and

$$\Sigma = \{\mathbf{x} \in \mathbf{R}^n\colon\ x_1 \geq x_2 \geq \cdots \geq x_n \geq 0, \langle \mathbf{x}, \mathbf{s}^n \rangle = a\}.$$

The following fundamental lemma holds:

LEMMA 2.1 *Let* $S = \Sigma \cap \{\mathbf{x} \in \mathbf{R}^n\colon\ g(\mathbf{x}) = b\} \neq \emptyset$, *being* g *a continuous function, homogeneous of degree* p *and strictly Schur-convex and* $b \in \mathbf{R}$ *fixed. Then either* $(a^p/n^p)g(\mathbf{s}^n) = b$ *or there exists a unique integer* $1 \leq h^* < n$ *such that*

$$\frac{a^p}{(h^*+1)^p} g(\mathbf{s}^{h^*+1}) < b \leq \frac{a^p}{(h^*)^p} g(\mathbf{s}^{h^*}). \tag{2.1}$$

Let us now introduce the following sets:

$$S^{\geq}(\alpha) = \Sigma \cap \{\mathbf{x} \in \mathbf{R}^n\colon\ x_i \geq \alpha,\ i = 1, \ldots, h,\ 0 < \alpha \leq a/h\},\ 1 \leq h \leq n;$$

$$S^{\leq}(\alpha) = \Sigma \cap \{\mathbf{x} \in \mathbf{R}^n\colon\ x_i \leq \alpha,\ i = h, \ldots, n\},\ 1 \leq h \leq n.$$

Denoting with $x^*(\alpha)$ and $x_*(\alpha)$ the maximum and the minimum elements of the previous sets with respect to the majorization order, we get:

for $S^{\geq}(\alpha)$:
$\mathbf{x}^*(\alpha) = (a - h\alpha)\mathbf{e}^1 + \alpha\mathbf{s}^h$,
$\mathbf{x}_*(\alpha) = (a/n)\mathbf{s}^n$ if $\alpha \leq (a/n)$,
$\mathbf{x}_*(\alpha) = \alpha\mathbf{s}^h + \rho\mathbf{v}^h$ if $\alpha > (a/n)$, being $\rho = (a - h\alpha)/(n-h)$.

for $S^{\leq}(\alpha)$ and $h > 1$:
$\mathbf{x}^*(\alpha) = a\mathbf{e}^1$,
$\mathbf{x}_*(\alpha) = (a/n)\mathbf{s}^n$ if $\alpha \geq (a/n)$,
$\mathbf{x}_*(\alpha) = \rho\mathbf{s}^{h-1} + \alpha\mathbf{v}^{h-1}$ if $\alpha < (a/n)$, being $\rho = (a - (n-h+1)\alpha)/(h-1)$,

and finally

for $S^{\leq}(\alpha)$ and $h = 1$:
$\mathbf{x}^*(\alpha) = \alpha\mathbf{s}^k + \theta\mathbf{e}^{k+1}$, being $k = [a/\alpha]$ and $\theta = a - \alpha k$.
$\mathbf{x}_*(\alpha) = (a/n)\mathbf{s}^n$.

As observed in the previous section, the hypothesis of nonnegativity of $x_i\,(i = 1, \ldots, n)$ is not restrictive and can easily be dropped out by means of the following proposition:

PROPOSITION 2.1 (Bianchi and Torriero [3]) *Let* $S \subseteq \Sigma$ *be a subset with maximum and minimum elements, with respect to the majorization order, given by* $\mathbf{x}^*(S)$ *and* $\mathbf{x}_*(S)$ *respectively. For* $r \in \mathbf{R}$ *let* $S' = S + r\mathbf{s}^n$. *Then* $\mathbf{x}^*(S') = \mathbf{x}^*(S) + r\,\mathbf{s}^n$ *and* $\mathbf{x}_*(S') = \mathbf{x}_*(S) + r\,\mathbf{s}^n$.

Indeed, if S' is the original set, the nonnegativity of the elements of S is assured by taking $r=\mu-\sigma(n-1)^{1/2}$ according to (1.1). Hence, by Proposition 2.1, maximum and minimum elements for S' easily follow.

3 A CLASS OF CONSTRAINED OPTIMIZATION PROBLEMS

In this section we develop some new bounds for x_h $(h=1,\dots,n)$ throughout the solution of two nonlinear optimization problems. To this aim, we shall follow the same methodology applied in Bianchi and Torriero [3] for providing upper and lower bounds for x_1.

More precisely, each bound can be obtained by taking advantage of some results on majorization order applied to the following optimization problems:

$$(P(h)) \quad \max F(x_h) \text{ subject to } \mathbf{x} \in S$$
$$(P^*(h)) \quad \min F(x_h) \text{ subject to } \mathbf{x} \in S$$

where $F:\mathbf{R}\to\mathbf{R}$ is an increasing function of x_h.

It is immediate that from problems $(P(h))$ and $(P^*(h))$ we can deduce upper and lower bounds for x_h respectively.

We observe that $(P(1))$ will provide, as a particular case, the same results as those obtained in Bianchi and Torriero [3].

We start from problem $(P(h))$. The main result follows:

THEOREM 3.1 *The solution of the optimization problem $(P(h))$ is the hth component of vector $\mathbf{x}_*=(a/n)\mathbf{s}^n$ if $(a^p/n^p)g(\mathbf{s}^n)=b$. If $(a^p/n^p)g(\mathbf{s}^n)\neq b$ let h^* the integer satisfying condition (2.1) with $1\leq h^*<n$. The solution of the optimization problem $(P(h))$ is α^*, where*

(1) *for $h>h^*$, α^* is the unique root of the function $f(\alpha)=[g(\mathbf{x}^*(\alpha))-b]$ in $I=(0,a/h]$, that is the hth component of the vector $\mathbf{x}^*(\alpha^*)=(a-h\alpha^*)\mathbf{e}^1+\alpha^*\mathbf{s}^h$.*
(2) *for $h\leq h^*$, α^* is the unique root of the function $f(\alpha)=[g(\mathbf{x}_*(\alpha))-b]$ in $I=(a/n,a/h]$, that is the hth component of the vector $\mathbf{x}_*(\alpha^*)=\alpha^*\mathbf{s}^h+\rho^*\mathbf{v}^h$ and $\rho^*=(a-h\alpha^*)/(n-h)$.*

Proof If $(a^p/n^p)g(\mathbf{s}^n)=b$ the set S is the singleton $\{(a/n)\,\mathbf{s}^n\}$ and the problem $(P(h))$ is trivial. Let $S^{\geq}(\alpha)=\Sigma\cap\{\mathbf{x}\in\mathbf{R}^n\colon\ x_i\geq\alpha,\ i=1,\dots,h,$

$0 < \alpha \leq a/h\}$, $1 \leq h \leq n$ and h^* the integer satisfying condition (2.1). We prove the theorem in two steps. In the first one we use the maximum element of $S^{\geq}(\alpha)$ and in the second one the minimum element of $S^{\geq}(\alpha)$.

1. We know that the maximum element of the set $S^{\geq}(\alpha)$ with respect to the majorization order is $\mathbf{x}^*(\alpha) = (a - h\alpha)\mathbf{e}^1 + \alpha\mathbf{s}^h$.

We note that if $\mathbf{x}^*(\alpha) \in S$, then $S^{\geq}(\alpha) \cap S = \{\mathbf{x}^*(\alpha)\}$. In fact if $\mathbf{x} \in S^{\geq}(\alpha) \cap S$ and $\mathbf{x} \neq \mathbf{x}^*(\alpha)$, we find a contradiction, since $\mathbf{x} \leq_m \mathbf{x}^*(\alpha)$ implies $g(\mathbf{x}) < g(\mathbf{x}^*(\alpha)) = b$.

Now we claim that there exists a unique α^* such that $\mathbf{x}^*(\alpha^*) \in S$ and that $\mathbf{x}^*(\alpha^*)$ is the solution of the optimization problem $(P(h))$ for $h > h^* \geq 1$. The last part of the statement is immediate, since any element of S that improves the objective function has the component h which is bigger than α^* and thus belongs to $S^{\geq}(\alpha^*)$ which is impossible from our previous consideration. To complete the proof we have to prove that function

$$f(\alpha) = g(\mathbf{x}^*(\alpha)) - b$$

has a unique root in the interval $I = (0, a/h]$ for $h > h^*$. Since

$$\begin{aligned} f'(\alpha) &= \langle \nabla g(\mathbf{x}^*(\alpha)),\ \mathrm{d}(\mathbf{x}^*(\alpha))/\mathrm{d}\alpha \rangle \\ &= (1-h)\frac{\partial g}{\partial x_1}(\mathbf{x}^*(\alpha)) + \sum_{i=2}^{h} \frac{\partial g}{\partial x_i}(\mathbf{x}^*(\alpha)) \end{aligned}$$

taking into account that (Hardy *et al.* [6]):

$$\frac{\partial g}{\partial x_1}(\mathbf{x}) > \frac{\partial g}{\partial x_2}(\mathbf{x}) > \cdots > \frac{\partial g}{\partial x_n}(\mathbf{x}), \tag{3.1}$$

for all $\mathbf{x} \in S$, we deduce that $f'(\alpha) < 0$ in I since $h > 1$. Thus f has a unique root in the interval I if $f(0) > 0$ and $f(a/h) \leq 0$. We note that

$$\begin{aligned} f(0) &= g(a\mathbf{e}^1) - b, \\ f(a/h) &= g((a/h)\mathbf{s}^h) - b. \end{aligned}$$

Let $\hat{\mathbf{x}} \in S$. If $\hat{\mathbf{x}} = a\mathbf{e}^1$, that is $g(a\mathbf{e}^1) = b$, the set S is the singleton $\{a\mathbf{e}^1\}$ and the optimization problem is trivial.

If $\hat{\mathbf{x}} \neq a\mathbf{e}^1$, set $\hat{\mathbf{w}} = (\hat{\mathbf{x}}/a) \in \Sigma_1 = \{\mathbf{x} \in \mathbf{R}^n\colon\ x_1 \geq x_2 \geq \cdots \geq x_n \geq 0,$ $\langle \mathbf{x}, \mathbf{s}^n \rangle = 1\}$. Since $\mathbf{e}^1$ is the maximum element of the set Σ_1 with respect to the majorization order, it is easy to verify that $g(\hat{\mathbf{w}}) < g(\mathbf{e}^1)$, and thus $f(0) > 0$. Furthermore the integer h^* satisfies the condition

$$g\left(\frac{a}{h^*+1}\mathbf{s}^{h^*+1}\right) < b$$

and thus $f(a/h) < 0$ for $h > h^*$. The continuity of f guarantees now the existence of a solution α^* of the equation $f(\alpha) = 0$ in I.

2. We know that the minimum element of the set $S^{\geq}(\alpha)$ with respect to the majorization order is

$$\mathbf{x}_* = (a/n)\mathbf{s}^n \qquad \text{if } \alpha \leq (a/n),$$
$$\mathbf{x}_*(\alpha) = \alpha\mathbf{s}^h + \rho\mathbf{v}^h \quad \text{if } \alpha > (a/n) \text{ with } \rho = (a - h\alpha)/(n-h).$$

As in part 1, if $\mathbf{x}_*(\alpha) \in S$, then $S^{\geq}(\alpha) \cap S = \{\mathbf{x}_*(\alpha)\}$. Now we claim that there is a unique α^* such that $\mathbf{x}_*(\alpha^*) \in S$ for $h \leq h^*$ and $\mathbf{x}_*(\alpha^*)$ is the solution of the optimization problem $(P(h))$.

The last part of the statement is similar to part 1.

Because $(a/n)\mathbf{s}^n \notin S$, we have to show that the function $f(\alpha) = g(\mathbf{x}_*(\alpha)) - b$ has a unique root in the interval $I = (a/n, a/h]$. Since

$$\begin{aligned} f'(\alpha) &= \langle \nabla g(\mathbf{x}^*(\alpha)), \mathrm{d}(\mathbf{x}_*(\alpha))/\mathrm{d}\alpha \rangle \\ &= \sum_{i=1}^{h} \frac{\partial g}{\partial x_i}(\mathbf{x}_*(\alpha)) - \frac{h}{(n-h)} \sum_{i=h+1}^{n} \frac{\partial g}{\partial x_i}(\mathbf{x}^*(\alpha)) \end{aligned}$$

we can verify $f'(\alpha) > 0$ in I, by condition (3.1). Furthermore

$$f(a/n) = g((a/n)\mathbf{s}^n) - b,$$
$$f(a/h) = g((a/h)\mathbf{s}^h) - b.$$

Let $\hat{\mathbf{x}} \in S$. We know that $\hat{\mathbf{x}} \neq (a/n)\mathbf{s}^n$ and thus $g((a/n)\mathbf{s}^n) < g(\hat{\mathbf{x}})$, that is $g((a/n)\mathbf{s}^n) < b$. This condition implies $f(a/n) < 0$. From Lemma 2.1 we also know that

$$g(\hat{\mathbf{x}}/a) \leq g\left(\frac{1}{h^*}\mathbf{s}^{h^*}\right)$$

and thus $f(a/h) \geq 0$ for $h \leq h^*$. The continuity of f guarantees the existence of a solution of the equation $f(\alpha) = 0$ in I.

Now we study the problem $(P^*(h))$ for $h > 1$ proving the following result:

THEOREM 3.2 *The solution of the optimization problem* $(P^*(h))$ *is the hth component of the vector* $\mathbf{x}_* = (a/n)s^n$ *if* $(a^p/n^p)g(\mathbf{s}^n) = b$. *If* $(a^p/n^p)g(\mathbf{s}^n) \neq b$ *let* h^* *the integer satisfying condition* (2.1) *with* $1 \leq h^* < n$. *The solution of the optimization problem* $(P^*(h))$ *for* $1 < h \leq h^* + 1$ *is the unique root* α^* *of the function* $f(\alpha) = [g(\mathbf{x}_*(\alpha)) - b]$ *in* $I = (0, a/n]$, *that is the hth component of the vector*

$$\mathbf{x}_*(\alpha^*) = \rho^*\mathbf{s}^{h-1} + \alpha^*\mathbf{v}^{h-1} \text{ and } \rho^* = (a - (n - h + 1)\alpha^*)/(h - 1).$$

The solution of the optimization problem $(P^*(h))$ *for* $h > h^* + 1$ *is zero.*

Proof If $(a^p/n^p)g(\mathbf{s}^n) = b$ the set S is the singleton $\{(a/n)\mathbf{s}^n\}$ and the problem $(P^*(h))$ is trivial. Let $S^4(\alpha) = \Sigma \cap \{\mathbf{x} \in \mathbf{R}^n\colon x_i \leq \alpha, i = h, \ldots, n\}$, $1 < h \leq n$ and h^* the integer satisfying condition (2.1).

We know that the minimum element of the set $S^{\leq}(\alpha)$ with respect to the majorization order is

$$\mathbf{x}_*(\alpha) = (a/n)\mathbf{s}^n \quad \text{if } \alpha \geq (a/n),$$

$$\mathbf{x}_*(\alpha) = \rho\mathbf{s}^{h-1} + \alpha\mathbf{v}^{h-1} \quad \text{if } \alpha < (a/n) \text{ with}$$

$$\rho = (a - (n - h + 1)\alpha)/(h - 1).$$

As in Theorem 3.1, if $\mathbf{x}_*(\alpha) \in S$, then $S^{\leq}(\alpha) \cap S = \{\mathbf{x}_*(\alpha)\}$. Now we claim that there is a unique α^* such that $\mathbf{x}_*(\alpha^*) \in S$ for $1 < h \leq h^* + 1$ which is the solution of the optimization problem $(P^*(h))$. The last part of the statement is similar to Theorem 3.1.

Because $(a/n)\mathbf{s}^n \notin S$, we have to show that the function $f(\alpha) = g(\mathbf{x}_*(\alpha)) - b)$ has a unique root in the interval $I = (0, a/n)$. Since

$$f'(\alpha) = \langle \nabla g(\mathbf{x}_*(\alpha)), \mathrm{d}(\mathbf{x}_*(\alpha))/\mathrm{d}\alpha \rangle$$

$$= \frac{(h - 1 - n)}{(h - 1)} \sum_{i=1}^{h-1} \frac{\partial g}{\partial x_i}(\mathbf{x}^*(\alpha)) + \sum_{i=h}^{n} \frac{\partial g}{\partial x_i}(\mathbf{x}^*(\alpha)),$$

condition (3.1) yields $f'(\alpha) < 0$ in I. Furthermore

$$f(a/n) = g((a/n)\mathbf{s}^n) - b,$$
$$f(0) = g\Big(\frac{a}{h-1}\mathbf{s}^{h-1}\Big) - b^{\cdot}$$

Let $\hat{\mathbf{x}} \in S$. We know that $\hat{\mathbf{x}} \neq (a/n)\mathbf{s}^n$ and thus $g((a/n)\mathbf{s}^n) < g(\hat{\mathbf{x}})$, that is $g((a/n)\mathbf{s}^n) < b$. Hence it follows that $f(a/n) < 0$. By (2.1) we get

$$b \leq g\big((a/h^*)\mathbf{s}^{h^*}\big)$$

and thus $f(0) \geq 0$ for $h \leq h^* + 1$. The continuity of f guarantees the existence of a solution of the equation $f(\alpha) = 0$ in I. Now we consider the case $h > h^* + 1$. It is easily verified that under this assumption there exists a vector $\mathbf{x} \in S$ with the last $(n-h)$ components equal to zero. In fact from the definition of h^*, it follows that $g((a/h^* + 1)\mathbf{s}^{h^*+1}) < b$. Now the vector $\mathbf{x}(\varepsilon) = ((a/h^* + 1) - \varepsilon/h^*)\mathbf{s}^{h^*+1} + (\varepsilon + \varepsilon/h^*)\mathbf{e}^1$ belongs to Σ and $\mathbf{x}(\varepsilon) \geq_m (a/h^* + 1)\mathbf{s}^{h^*+1}$ for any $0 < \varepsilon < (h^*a)/(h^* + 1)$. In particular $\mathbf{x}((h^*a)/(h^* + 1)) = a\mathbf{e}^1$. From the previous consideration we obtain $g(\mathbf{x}(\varepsilon)) > g((a/h^* + 1)\mathbf{s}^{h^*+1})$ and since $g(a\mathbf{e}^1) > g((a/h^*)\mathbf{s}^{h^*}) > b$, from the continuity of g we deduce the existence of an ε^* such that $0 < \varepsilon^* < (h^*a)/(h^* + 1)$ and $g(\mathbf{x}(\varepsilon^*)) = b$.

Finally, as proved in Bianchi and Torriero [3], the solution of problem $P^*(1)$ is (a/n) if $(a^p/n^p)g(\mathbf{s}^n) = b$, otherwise the first component of the vector:

$$\mathbf{x}^*(\alpha^*) = \alpha^*\mathbf{s}^{h^*} + \theta^*\mathbf{e}^{h^*+1},$$

where h^* is determined as in Lemma 1, α^* is the unique root of the equation

$$f(\alpha) = g(\alpha\mathbf{s}^{h^*} + (a - \alpha k)\mathbf{e}^{h^*+1}) - b \text{ in } I_{h^*}$$
$$= (a/(h^* + 1), a/h^*] \text{ and } \theta^* = a - \alpha^* h^*.$$

Example 3.1 Let $g(\mathbf{x}) = \sum_{i=1}^n x_i^2$ and set $S = \Sigma \cap \{\mathbf{x} \in \mathbf{R}^n\colon g(\mathbf{x}) = b\}$. If $a^2/n = b$, the set S is the singleton $\{(a/n)\mathbf{s}^n\}$ and the problems $P(h)$ and $P^*(h)$ are trivial. If $a^2/n \neq b$, the integer h^* such that $a^2/(h^* + 1) < b \leq a^2/h^*$, is $h^* = [a^2/b]$.

From Theorem 3.1, part 1 we have

$$f(\alpha) = h(h-1)\alpha^2 - 2a(h-1)\alpha + (a^2 - b)$$

and thus $f(\alpha) = 0$ in $I = (0, a/h]$ if

$$\alpha = \frac{a}{h} - \sqrt{\frac{1}{h(h-1)}\left(b - \frac{a^2}{h}\right)}. \tag{3.2}$$

We note that the condition of reality of α implies $h \geq a^2/b$ and this condition is obviously satisfied if $h > h^* = [a^2/b]$. Hence $\mathbf{x}_h \leq \alpha$ for $h > h^*$.

From Theorem 3.2, part 2 we find

$$f(\alpha) = \frac{nh\alpha^2 - 2a\alpha h + a^2 - b(n-h)}{n-h}$$

and thus $f(\alpha) = 0$ in $I = (a/n, a/h]$ if

$$\alpha = \frac{a}{n} + \frac{\sqrt{h(n-h)(nb - a^2)}}{nh}. \tag{3.3}$$

We note that the condition of reality of α requires $f(a/n) < 0$ and this condition is satisfied. Furthermore we are sure that for $h \leq h^*$ the root α belongs to I. Hence $\mathbf{x}_h \leq \alpha$ for $h \leq h^*$.

Finally from Theorem 4.1 we get

$$f(\alpha) = \frac{n(n-h+1)\alpha - 2a\alpha^2(n-h+1) + a^2 - b(h-1)}{(h-1)}$$

and thus $f(\alpha) = 0$ in $I = (0, a/n]$ if

$$\alpha = \frac{a}{n} - \frac{\sqrt{(h-1)(n-h+1)(nb - a^2)}}{n(n-h+1)}. \tag{3.4}$$

We note that the condition of reality of α requires $f(a/n) < 0$ and this condition is satisfied. Furthermore we are sure that for $1 < h \leq h^* + 1$ the root α belongs to I. Hence $\mathbf{x}_h \geq \alpha$ for $1 < h \leq h^* + 1$. Observe that for $h > h^* + 1$ the root α becomes negative and thus we have not a significant bound since we treat nonnegative ordered real numbers.

As a particular case, we remind that the previous results can be used to give upper and lower bounds for the hth eigenvalue of a symmetric positive semidefinite matrix $\mathbf{A}$, where $a = \mathrm{tr}(\mathbf{A})$ and $b = \mathrm{tr}(\mathbf{A}^2)$.

In particular (3.2) is derived by Merikoski *et al.* [8], through optimization techniques, while (3.3) and (3.4) can be found in Wolkowicz and Styan [16]. As special cases we find also some known upper bounds for the maximum and the minimum eigenvalue (Wolkowicz and Styan [16]). In fact from (3.3) and $h = 1$ we get

$$x_1 \leq \frac{a}{n} + \sqrt{\frac{n-1}{n}\left(b - \frac{a^2}{n}\right)}$$

while for $h = n$ from (3.2) we have

$$x_n \leq \frac{a}{n} - \sqrt{\frac{1}{n(n-1)}\left(b - \frac{a^2}{n}\right)}.$$

4 BOUNDS FOR NONNEGATIVE ORDERED REAL NUMBERS

By means of the optimization problem studied in Theorems 3.1 and 3.2 we can now find lower and upper bounds of $x_h \in \Sigma$ for $h > 1$, generalizing the results presented in Example 1. We recall that the case $h = 1$ has been already studied in [3].

To this aim let $g(\mathbf{x}) = \sum_{i=1}^{n} x_i^p$, being $p > 1$, and set $S = \Sigma \cap \{\mathbf{x} \in \mathbf{R}^n: g(\mathbf{x}) = b\}$. If $b = a^p/n^{p-1}$ then $S = \{(a/n)\mathbf{s}^n\}$ and the solution is (a/n). Otherwise Lemma 2.1 implies the existence of an integer $h^* < n$ such that

$$\frac{a^p}{(h^*+1)^{p-1}} < b \leq \frac{a^p}{(h^*)^{p-1}}; \quad \text{where} \quad h^* = \left[\sqrt[p-1]{\frac{a^p}{b}}\right]$$

is computed easily. Thus, applying Theorem 3, part 1 we get

$$f(\alpha,p) = (h-1)\alpha^p + (\alpha - h\alpha + \alpha)^p - b$$

and the unique root of the equation $f(\alpha,p)=0$ in $I=(0,a/h]$ is an upper bound for x_h with $h > h^*$.

We recall that the root of the function f can be determined by applying classical numerical methods for solving nonlinear equations.

From Theorem 3, part 2 we find

$$f(\alpha,p) = h\alpha^p + (n-h)\frac{(a-h\alpha)^p}{(n-h)^p} - b$$

and thus the unique root α^* of the equation $f(\alpha,p)=0$ in $I=(a/n,a/h]$ is an upper bound for x_h with $h \leq h^*$. Finally from Theorem 4 we get

$$f(\alpha,p) = (n-h+1)\alpha^p + (h-1)\frac{(a-(n-h+1)\alpha)^p}{(h-1)^p} - b$$

and thus the unique root α^* of the equation $f(\alpha,p)=0$ in $I=(0,a/n]$ is a lower bound for x_h with $1 < h \leq h^*$. We observe that it is easy to prove that if α is an upper bound of x_1, then $(a-\alpha)/(n-1)$ is both a lower bound of x_2 and an upper bound of x_n.

In the following tables, lower and upper bounds for x_h, $h=1,\ldots,n$, corresponding to different value of p and n, are listed. As previously mentioned, these results are obtained by applying Theorems 3.1 and 3.2. It can be seen that lower and upper bounds due to Tarazaga [15] and Wolkowicz and Styan [16] are found as a particular case ($p=2$). Given the set of ordered numbers $\{5, 2.5, 2, 0.5\}$ and $\{2.6, 2.5, 1.5, 1.3, 1, 0.9, 0.2, 0.1\}$, we find the following bounds.

By an inspection of the above tables, as p increases, both upper and lower bounds of x_1 improve monotonically, showing that our bounds are better than known ones ($p=2$) [15,16]. This results can also be proved analytically, by using classical optimization methods. However it appears that the bounds of remaining x_i ($i=2,\ldots,n$) are not necessarily tighter than those obtained for $p=2$. For example, the upper bounds of x_2 do well in Table I, but do not do well in Tables II(a) and (b), which show improvements both for lower bounds of x_1, x_2, x_3 and for upper bounds of x_1, x_2, x_5.

TABLE I

p	$x_1=5$		$x_2=2.5$		$x_3=2$		$x_4=0.5$	
	Lower bound	*Upper bound*	*Lower bound*	*Upper bound*	*Lower bound*	*Upper bound*	*Lower bound*	*Upper bound*
2	3.9343	5.3062	1.5646	4.1202	0.8798	2.7324	0.0	1.5646
3	4.1603	5.1428	1.6191	4.1956	0.8044	2.5699	0.0	1.6191
4	4.2878	5.065	1.645	4.2935	0.7065	2.5268	0.0	1.645
5	4.3876	5.0292	1.6569	4.3882	0.61175	2.5113	0.0	1.6569
6	4.4689	5.013	1.6623	4.469	0.53099	2.505	0.0	1.6623
7	4.5347	5.0058	1.6647	4.5347	0.46529	2.5022	0.0	1.6647
8	4.5876	5.0026	1.6658	4.5876	0.41237	2.501	0.0	1.6658
9	4.6305	5.0011	1.6663	4.6305	0.36949	2.5005	0.0	1.6663
10	4.6657	5.0005	1.6665	4.6657	0.33433	2.5002	0.0	1.6665

TABLE II(a)

p	$x_1=2.6$		$x_2=2.5$		$x_3=1.5$		$x_4=1.3$	
	Lower bound	*Upper bound*	*Lower bound*	*Upper bound*	*Lower bound*	*Upper bound*	*Lower bound*	*Upper bound*
2	1.934	3.6	0.9143	2.7883	0.7372	2.3966	0.5621	2.1381
3	2.0664	3.247	0.9648	2.6606	0.7798	2.3617	0.583	2.1608
4	2.1897	3.0591	0.9916	2.6035	0.7988	2.3645	0.5813	2.2035
5	2.2453	2.9465	1.0076	2.577	0.8077	2.3799	0.5720	2.2475
6	2.2857	2.8736	1.0181	2.5646	0.8118	2.3981	0.5611	2.286
7	2.318	2.8233	1.0252	2.5589	0.8137	2.4152	0.5509	2.318
8	2.3443	2.787	1.0304	2.5564	0.8145	2.4302	0.5419	2.3443
9	2.366	2.7597	1.0343	2.5554	0.8149	2.4429	0.5343	2.366
10	2.3841	2.7384	1.0374	2.5552	0.815	2.4536	0.5279	2.3841

TABLE II(b)

p	$x_5=1$		$x_6=0.9$		$x_7=0.2$		$x_8=0.1$	
	Lower bound	*Upper bound*	*Lower bound*	*Upper bound*	*Lower bound*	*Upper bound*	*Lower bound*	*Upper bound*
2	0.3619	1.9379	0.0	1.3994	0.0	1.1004	0.0	0.9143
3	0.3392	1.8405	0.0	1.402	0.0	1.1396	0.0	0.9647
4	0.2964	1.8193	0.0	1.4126	0.0	1.1631	0.0	0.9915
5	0.2525	1.8193	0.0	1.4228	0.0	1.1784	0.0	1.0076
6	0.2140	1.8136	0.0	1.4314	0.0	1.189	0.0	1.0181
7	0.1819	1.8132	0.0	1.4385	0.0	1.1967	0.0	1.0252
8	0.1556	1.815	0.0	1.4442	0.0	1.2024	0.0	1.0304
9	0.1339	1.8174	0.0	1.4489	0.0	1.2068	0.0	1.0343
10	0.1159	1.8201	0.0	1.4528	0.0	1.2103	0.0	1.0374

References

[1] B.C. Arnold. Majorization and Lorenz order: A brief introduction, *Lectures Notes in Statistics*, **43** (1980), Springer Verlag, Berlin.
[2] A.B. Atkinson. On the measurement of inequality, *Journal of Economic Theory*, **2** (1970), 244-263.
[3] M. Bianchi and A. Torriero. A majorization approach for solving nonlinear optimization problems, in *Convessitá e calcolo parallelo* (G. Giorgi and F.A. Rossi, Eds.), Libreria Universitaria Editrice, Verona, 1997, pp. 39–63.
[4] G.M. Engel and H. Schneider. Matrices diagonally similar to a symmetric matrix, *Linear Algebra and its Applications*, **29** (1980), 131–138.
[5] V.B. Frosini. Lezioni di Statistica, Parte prima, Vita e Pensiero, 1980, Milano.
[6] G.H. Hardy, E. Littlewood and G. Polya. Some simple inequalities satisfied by convex functions, *Messenger Math*, **58** (1929), 145–152.
[7] A.W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and its Applications*, Academic Press, London, 1979.
[8] J.K. Merikoski, H. Sarria and P. Tarazaga. Bounds for singular values using traces, *Linear Algebra and its Applications*, **210** (1994), 227–254.
[9] C.T. Pan. A vector majorization method for solving a nonlinear programming problem, *Linear Algebra and its Applications*, **119** (1989), 129–139.
[10] I. Schur. Über ein Klass von Mittelbildungen mit Anwendungen auf Determinantentheorie, *Sitzber. Berl. Math. Ges*, **22** (1929), 9–20.
[11] A.K. Sen. *On Economic Inequality*, Oxford UP, Oxford, 1973.
[12] S. Stefani and A. Torriero. Limiting intervals for spectrum distribution, Cahiers du Départment d'Économétrie 95–03, 1995, Université de Genève.
[13] S. Stefani and A. Torriero. Spectral properties of matrices and graphs, in *Matrices and Graphs. Theory and Applications to Economics* (S. Camiz and S. Stefani, Eds.), World Scientific, Singapore, pp. 31–49, 1996.
[14] P. Tarazaga. Eigenvalue estimate for symmetric matrices, *Linear Algebra and its Applications*, **135** (1990), 171–179.
[15] P. Tarazaga. More estimate for eigenvalues and singular values, *Linear Algebra and its Applications*, **149** (1991), 97–110.
[16] H. Wolkowicz and G.P.H. Styan. Extensions of Samuelson's inequality, *The American Statistician*, **33**(3) (1979), 143–144.