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On a Further Generalization of Steffensen's Inequality

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Pečarić (*J. Math. Anal. Appl.*, **104** (1984), 432–434) proved two theorems generalizing Steffensen's inequality. We extend Pečarić's results to the case of integrals over a general measure spaces. Some applications are also given.

Keywords: Steffensen's inequality; Jensen's inequality; Continuous measure space; Separating subsets

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1. PRELIMINARIES AND STATEMENT OF RESULTS

The most basic inequality which deals with comparison between integrals over a whole set and integrals over a subset is the following inequality, which was established by Steffensen in 1918 [5].

THEOREM A (Steffensen's inequality) *Let f and g be integrable functions from $[a,b]$ into $\mathbb{R}$ such that f is nonincreasing and for every $x \in [a,b]$, $0 \le g(x) \le 1$. Then*

$$\int_{b-\lambda}^{b} f(x)\,\mathrm{d}x \le \int_a^b f(x)g(x)\,\mathrm{d}x \le \int_a^{a+\lambda} f(x)\,\mathrm{d}x,$$

where $\lambda = \int_a^b g(x)\,\mathrm{d}x$.

In [1], Steffensen's inequality was extended from integrals over compact intervals of the real line to integrals over general measure spaces. To state this result we need the notion of a separating subset that was introduced and studied in [1].

Let $X=(X,\mathcal{A},\mu)$ be a measure space.

DEFINITION 1 *Let $f\in L^0(X)$. Let $(U,c)\in\mathcal{A}\times\mathbb{R}$. We say that the pair (U,c) is* upper-separating *for f iff*

$$\{x\in X\colon f(x)>c\}\overset{a}{\subseteq}U\overset{a}{\subseteq}\{x\in X\colon f(x)\geq c\},$$

where $A\overset{a}{\subseteq}B$ means that A is almost contained in B. In this case, we also say that the subset U of X is upper-separating *for f. We say that the pair (V,c) is* lower-separating *for f iff the pair $(X\setminus V,c)$ is upper-separating for f.*

THEOREM B (Steffensen's inequality over a general measure space) *Let $f,g\in L^1(X)$ be such that $0\leq g\leq 1$ a.e. on X. Let U, $V\in\mathcal{A}$ be respectively upper-, lower-, separating subsets for f such that $\mu(U)=\mu(V)=\int_X g\,d\mu$. Then*

$$\int_V f\,d\mu\leq\int_X f\cdot g\,d\mu\leq\int_U f\,d\mu.$$

The following definition can be found, for example in [2].

DEFINITION 2 *We say that the measure space $(X,\mathcal{A},\mu)$ is* continuous *iff for all $A,B\in\mathcal{A}$ such that $A\subseteq B$ and $\mu(A)<\mu(B)$, there exists an increasing mapping $\phi:[\mu(A),\mu(B)]\to\mathcal{A}$ such that $\phi(\mu(A))=A$, $\phi(\mu(B))=B$, and $\mu(\phi(u))=u$ for all $u\in[\mu(A),\mu(B)]$.*

In the case when $(X,\mathcal{A},\mu)$ is continuous with $\mu(X)<\infty$, the following theorem guarantees the existence of separating subsets U and V in Theorem B.

THEOREM C [1] *If $(X,\mathcal{A},\mu)$ is a continuous measure space with $\mu(X)<\infty$, then, given $f\in L^0(X)$, for any real number λ such that $0\leq\lambda\leq\mu(X)$, there exist an upper-separating subset U and a lower-separating subset V for f such that $\mu(U)=\mu(V)=\lambda$.*

The main results of this paper are the following two theorems generalizing the second Steffensen's inequality over a general measure space. These theorems are extentions of the results by Pečarić obtained in [3].

THEOREM 1 *Let* $(X, \mathcal{A}, \mu)$ *be a continuous measure space with* $\mu(X) < \infty$. *Let* $\alpha \geq 1$ *be a real number and let* f *and* g *be functions on* X *such that* $f, f^\alpha \cdot g \in L^1(X)$ *and* $f, g \geq 0$ *a.e. on* X. *Set* $\lambda = (\int_X g \, \mathrm{d}\mu)^\alpha$. *Let* U *be an upper-separating subset for* f *such that* $\mu(U) = \lambda$. *Assume that* $g \cdot (\int_X g \, \mathrm{d}\mu)^{\alpha-1} \leq 1$ *a.e. on* U. *Then*

$$\left(\int_X gf \, \mathrm{d}\mu\right)^\alpha \leq \int_U f^\alpha \, \mathrm{d}\mu.$$

Remark If we assume a little bit more, namely, that $g \cdot (\int_X g \, \mathrm{d}\mu)^{\alpha-1} \leq 1$ a.e. on X, then integrating this inequality over X, we obtain that $0 \leq \lambda \leq \mu(X)$. It will guarantee the existence of an upper-separating subset U with $\mu(U) = \lambda$. Similar remarks can be done in all following theorems and lemmas.

THEOREM 2 *Let* $(X, \mathcal{A}, \mu)$ *be a continuous measure space with* $\mu(X) < \infty$. *Let* $\alpha \geq 1$ *be a real number and let* $f, g \in L^1(X)$ *be such that* $f \geq 0$ *and* $0 \leq g \leq 1$ *a.e. on* X. *Set* $\lambda = (1/(\mu(X))^{\alpha-1}) (\int_X g \, \mathrm{d}\mu)^\alpha$. *Let* U *be an upper-separating subset for* f *such that* $\mu(U) = \lambda$. *Then*

$$\frac{1}{(\mu(X))^{\alpha-1}} \left(\int_X g \cdot f \, \mathrm{d}\mu\right)^\alpha \leq \int_U f^\alpha \, \mathrm{d}\mu.$$

To obtain Pečarić's inequalities from Theorems 1 and 2 we must take $X = [a, b]$ and assume that f is nonincreasing.

As applications of Theorem 1 we obtain the following two theorems.

THEOREM 3 *Let* α *and* β *be real numbers such that* $\alpha > 1, \beta - \alpha + 1 \geq 0$, $(X, \mathcal{A}, \mu)$ *be a continuous measure space with* $\mu(X) < \infty$, f *be a function on* X *such that* $f, f^\alpha \in L^1(X)$, $\int_X f \, \mathrm{d}\mu = A$, $\int_X f^\alpha \, \mathrm{d}\mu = B^\alpha$, *and* $0 \leq f \leq C$ *a.e. on* X, *where* A, B, *and* C *are positive real numbers. Let* U *be an upper-separating subset for* f *with* $\mu(U) = A/C$. *Then*

$$\int_U f^\beta \, \mathrm{d}\mu \geq \frac{1}{C} A^{-(\beta-\alpha+1)/(\alpha-1)} B^{\alpha\beta/(\alpha-1)}.$$

If $C = B$, we obtain a theorem proved in [1] by a different method.

THEOREM 4 *Let α and β be real numbers such that $\alpha > 1, \beta - \alpha + 1 \geq 0$, and let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers such that $\sum_{i=1}^n x_i \leq A$, $\sum_{i=1}^n x_i^\alpha \geq B^\alpha$, where A and B are positive real numbers. Let $k \in \{1, \ldots, n\}$ be such that $k \geq (A/B)^{\alpha\beta/((\alpha-1)(\beta+1))}$. Then*

$$\sum_{i=1}^{k} x_i^\beta \geq \left(\frac{\beta^{\alpha\beta}}{A^{\beta-\alpha+1}}\right)^{\beta/((\alpha-1)(\beta+1))}.$$

Taking $\beta = 1$ in Theorem 4, we obtain the following.

COROLLARY 1 *Let α be a real number such that $1 < \alpha \leq 2$, and let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers such that $\sum_{i=1}^n x_i \leq A$, $\sum_{i=1}^n x_i^\alpha \geq B^\alpha$, where A and B are positive real numbers. Let $k \in \{1, \ldots, n\}$ be such that $k \geq (A/B)^{\alpha/(2(\alpha-1))}$. Then*

$$\sum_{i=1}^{k} x_i \geq \left(\frac{B^\alpha}{A^{2-\alpha}}\right)^{1/(2(\alpha-1))}.$$

Corollary 1 complements the following result proved in [1]: *Let α be a real number such that $\alpha \geq 2$, and let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers such that $\sum_{i=1}^n x_i \leq A$, $\sum_{i=1}^n x_i^\alpha \geq B^\alpha$. Let $k \in \{1, \ldots, n\}$ be such that $k \geq (A/B)^{1/(\alpha-1)}$. Then $\sum_{i=1}^k x_i \geq B$.*

Taking $\alpha = 2$, $\beta = 1$ in Theorem 4, we obtain the following corollary.

COROLLARY 2 *Let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers such that $\sum_{i=1}^n x_i \leq A$, $\sum_{i=1}^n x_i^2 \geq B^2$. Let $k \in \{1, \ldots, n\}$ be such that $k \geq (A/B)$. Then $\sum_{i=1}^k x_i \geq B$.*

Corollary 2 with $A = 300$ and $B = 100$ was a problem in Moscow Mathematical Olympiad in 1954.

2. PROOFS OF THEOREMS 1–4

Proof of Theorem 1 By Jensen's inequality for convex function x^α with $\alpha \geq 1$,

$$\left(\int_X f \cdot g \, \mathrm{d}\mu\right)^\alpha \leq \left(\int_X g \, \mathrm{d}\mu\right)^{\alpha-1} \int_X f^\alpha \cdot g \, \mathrm{d}\mu,$$

(see Theorem 2.3 in [4]). Hence it is enough to prove that

$$\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_X f^\alpha\cdot g\,\mathrm{d}\mu\le\int_U f^\alpha\,\mathrm{d}\mu.$$

We proceed as follows:

$$\begin{aligned}
&\int_U f^\alpha\,\mathrm{d}\mu-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\int_X f^\alpha\cdot g\,\mathrm{d}\mu\\
&\quad=\int_U f^\alpha\,\mathrm{d}\mu-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\left(\int_U f^\alpha\cdot g\,\mathrm{d}\mu+\int_{X\setminus U} f^\alpha\cdot g\,\mathrm{d}\mu\right)\\
&\quad=\int_U f^\alpha\cdot\left(1-g\cdot\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\right)\mathrm{d}\mu-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\int_{X\setminus U} f^\alpha\cdot g\,\mathrm{d}\mu.
\end{aligned}$$

Let c be a real number such that (U,c) is an upper-separating pair for f. Then $f\ge c$ a.e. on U. Since $1-g\cdot(\int_X g\,\mathrm{d}\mu)^{\alpha-1}\ge 0$ a.e. on U, we obtain that

$$\begin{aligned}
&\int_U f^\alpha\,\mathrm{d}\mu-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_X f^\alpha\cdot g\,\mathrm{d}\mu\\
&\ge c^\alpha\int_U\left(1-g\cdot\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\right)\mathrm{d}\mu-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\int_{X\setminus U} f^\alpha\cdot g\,\mathrm{d}\mu\\
&=c^\alpha\left(\mu(U)-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_U g\,\mathrm{d}\mu\right)-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_{X\setminus U} f^\alpha\cdot g\,\mathrm{d}\mu\\
&=c^\alpha\left(\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha}-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\int_U g\,\mathrm{d}\mu\right)-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_{X\setminus U} f^\alpha\cdot g\,\mathrm{d}\mu\\
&=c^\alpha\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\left(\int_X g\,\mathrm{d}\mu-\int_U g\,\mathrm{d}\mu\right)-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\int_{X\setminus U} f^\alpha\cdot g\,\mathrm{d}\mu\\
&=c^\alpha\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_{X\setminus U} g\,\mathrm{d}\mu-\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_{X\setminus U} f^\alpha\cdot g\,\mathrm{d}\mu\\
&=\left(\int_X g\,\mathrm{d}\mu\right)^{\alpha-1}\cdot\int_{X\setminus U} g\cdot(c^\alpha-f^\alpha)\,\mathrm{d}\mu\ge 0,
\end{aligned}$$

where the last inequality holds since $f\le c$ on $X\setminus U$.

Proof of Theorem 2 This theorem can be proved exactly in the same way as Theorem 1 in [3]. Therefore we omit the proof.

LEMMA 1 *Let $(X, \mathcal{A}, \mu)$ be a continuous measure space with $\mu(X) < \infty$. Let α and β be real numbers such that $\alpha > 1$, $\beta - \alpha + 1 \geq 0$, and let f and g be functions on X such that $f^{\beta}, g, f^{\alpha-1} \cdot g \in L^1(X)$ and $f, g \geq 0$ a.e. on X. Set $\lambda = (\int_X g \, \mathrm{d}\mu)^{\beta/(\alpha-1)}$. Let U be an upper-separating subset for f such that $\mu(U) = \lambda$. Assume that $g \cdot (\int_X g \, \mathrm{d}\mu)^{(\beta-\alpha+1)/(\alpha-1)} \leq 1$ a.e. on U. Then*

$$\left(\int_X f^{\alpha-1} \cdot g \, \mathrm{d}\mu\right)^{\beta/(\alpha-1)} \leq \int_U f^{\beta} \, \mathrm{d}\mu.$$

Proof We obtain Lemma 1 if we take $\beta/(\alpha - 1)$ instead of α and $f^{\alpha-1}$ instead of f in Theorem 1.

Proof of Theorem 3 Set $g = C^{(1-\alpha)/\beta} A^{(\alpha-\beta-1)/\beta} f$. Then $\int_X g \, \mathrm{d}\mu = C^{(1-\alpha)/\beta} A^{(\alpha-\beta-1)/\beta} \int_X f \mathrm{d}\mu = C^{(1-\alpha)/\beta} A^{(\alpha-\beta-1)/\beta} A = (A/C)^{(\alpha-1)/\beta}$. Therefore $\mu(U) = A/C = (\int_X g \, \mathrm{d}\mu)^{\beta/(\alpha-1)}$. Moreover,

$$\begin{aligned} & g \cdot \left(\int_X g \, \mathrm{d}\mu\right)^{(\beta-\alpha+1)/(\alpha-1)} \\ & = C^{(1-\alpha)/\beta} \cdot A^{(\alpha-\beta-1)/\beta} \cdot f \cdot \left(\frac{A}{C}\right)^{((\alpha-1)/\beta)\cdot((\beta-\alpha+1)/(\alpha-1))} \\ & = \frac{f}{C} \leq 1 \quad \text{a.e. on } X. \end{aligned}$$

Therefore f, g and U satisfy the conditions of Lemma 1. Hence we obtain that

$$\begin{aligned} \int_U f^{\beta} \, \mathrm{d}\mu & \geq \left(\int_X f^{\alpha-1} \cdot g \, \mathrm{d}\mu\right)^{\beta/(\alpha-1)} \\ & = \left(C^{(1-\alpha)/\beta} \cdot A^{(\alpha-\beta-1)/\beta}\right)^{\beta/(\alpha-1)} \cdot \left(\int_X f^{\alpha} \, \mathrm{d}\mu\right)^{\beta/(\alpha-1)} \\ & = \frac{1}{C} A^{-(\beta-\alpha+1)/(\alpha-1)} \cdot B^{\alpha\beta/(\alpha-1)}. \end{aligned}$$

LEMMA 2 (Discrete Pečarić's inequality) *Let $\alpha \geq 1$ be a real number and let $(x_i)_{i=1}^{n}$ be a nonincreasing sequence of nonnegative real numbers.*

Let $k \in \{1,\dots,n\}$ *be such that* $k \geq (\sum_{i=1}^{n} y_i)^{\alpha}$. *Assume that* $y_i(\sum_{j=1}^{n} y_j) \leq 1$ *for* $i = 1,\dots,k$. *Then*

$$\left(\sum_{i=1}^{n} x_i y_i\right)^{\alpha} \leq \sum_{i=1}^{k} x_i^{\alpha}.$$

Proof We obtain the conclusion by applying Theorem 1 with $X = (0, n]$ and $f = \sum_{i=1}^{n} x_i \chi_{(i-1,i]}$, $g = \sum_{i=1}^{n} y_i \chi_{(i-1,i]}$, where

$$\chi_{(i-1,i]} = \begin{cases} 1, & \text{if } x \in (i-1, i] \\ 0, & \text{if } x \notin (i-1, i]. \end{cases}$$

LEMMA 3 *Let* α *and* β *be real numbers such that* $\alpha > 1$, $\beta - \alpha + 1 \geq 0$, $(x_i)_{i=1}^{n}$ *be a nonincreasing sequence of nonnegative numbers, and* $(y_i)_{i=1}^{n}$ *be a sequence of nonnegative real numbers. Let* $k \in \{1,\dots,n\}$ *be such that* $k \geq (\sum_{i=1}^{n} y_i)^{\beta/(\alpha-1)}$. *Assume that* $y_i(\sum_{j=1}^{n} y_j)^{(\beta-\alpha+1)/(\alpha-1)} \leq 1$ *for* $i = 1,\dots,k$. *Then*

$$\left(\sum_{i=1}^{n} x_i^{\alpha-1} y_i\right)^{\beta/(\alpha-1)} \leq \sum_{i=1}^{k} x_i^{\beta}.$$

Proof We obtain Lemma 3 if we take $\beta/(\alpha-1)$ instead of α and $x_i^{\alpha-1}$ instead of x_i in Lemma 2.

Proof of Theorem 4 If $x_j \geq (B^{\alpha\beta}/A^{\beta-\alpha+1})^{1/((\alpha-1)(\beta+1))}$ for some j, then

$$\sum_{i=1}^{k} x_i^{\beta} \geq x_1^{\beta} \geq x_j^{\beta} \geq \left(\frac{B^{\alpha\beta}}{A^{\beta-\alpha+1}}\right)^{\beta/((\alpha-1)(\beta+1))},$$

and we are done. Therefore without loss of generality, we can assume that

$$x_i \leq \left(\frac{B^{\alpha\beta}}{A^{\beta-\alpha+1}}\right)^{1/((\alpha-1)(\beta+1))} \quad \text{for all } i \in \{1,\dots,n\}. \qquad (*)$$

Set $y_i = A^{-(\beta-\alpha+1)/(\beta+1)} \cdot B^{-\alpha/(\beta+1)} \cdot x_i$. Then

$$\sum_{i=1}^{n} y_i = A^{-(\beta-\alpha+1)/(\beta+1)} B^{-\alpha/(\beta+1)} \sum_{i=1}^{n} x_i$$
$$\leq A^{-(\beta-\alpha+1)/(\beta+1)} \cdot B^{-\alpha/(\beta+1)} \cdot A = \left(\frac{A}{B}\right)^{\alpha/(\beta+1)}.$$

Therefore $(\sum_{i=1}^{n} y_i)^{\beta/(\alpha-1)} \leq (A/B)^{\alpha\beta/((\alpha-1)/(\beta+1))}$. Hence if $k \geq (A/B)^{\alpha\beta/((\alpha-1)(\beta+1))}$, then $k \geq (\sum_{i=1}^{n} y_i)^{\beta/(\alpha-1)}$. Moreover

$$y_i \left(\sum_{j=1}^{n} y_j\right)^{(\beta-\alpha+1)/(\alpha-1)}$$
$$\leq A^{-(\beta-\alpha+1)/(\beta+1)} \cdot B^{-\alpha/(\beta+1)} \cdot x_i \cdot \left(\frac{A}{B}\right)^{(\alpha/(\beta+1))\cdot((\beta-\alpha+1)/(\alpha-1))}$$
$$= x_i \left(\frac{A^{\beta-\alpha+1}}{B^{\alpha\beta}}\right)^{1/((\alpha-1)(\beta+1))}.$$

By $(*)$,

$$y_i \left(\sum_{j=1}^{n} y_j\right)^{(\beta-\alpha+1)/(\alpha-1)}$$
$$\leq \left(\frac{B^{\alpha\beta}}{A^{\beta-\alpha+1}}\right)^{1/((\alpha-1)(\beta+1))} \cdot \left(\frac{A^{\beta-\alpha+1}}{B^{\alpha\beta}}\right)^{1/((\alpha-1)(\beta+1))} = 1.$$

Therefore $(x_i)_{i=1}^{n}$, $(y_i)_{i=1}^{n}$, and k satisfy the conditions of Lemma 3. Hence

$$\sum_{i=1}^{k} x_i^{\beta} \geq \left(\sum_{i=1}^{n} x_i^{\alpha-1} y_i\right)^{\beta/(\alpha-1)}$$
$$= \left(\sum_{i=1}^{n} A^{-(\beta-\alpha+1)/(\beta+1)} \cdot B^{-\alpha/(\beta-1)} \cdot x_i^{\alpha}\right)^{\beta/(\alpha-1)}$$
$$= A^{-(\beta(\beta-\alpha+1))/((\alpha-1)(\beta+1))} \cdot B^{-\alpha\beta/((\alpha-1)(\beta+1))} \cdot \left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\beta/(\alpha-1)}$$
$$\geq A^{-(\beta(\beta-\alpha+1))/((\alpha-1)(\beta+1))} \cdot B^{-\alpha\beta/((\alpha-1)(\beta+1))} \cdot (B^{\alpha})^{\beta/(\alpha-1)}$$
$$= \left(\frac{B^{\alpha\beta}}{A^{\beta-\alpha+1}}\right)^{\beta/((\alpha-1)(\beta+1))}.$$

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