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Positive Decreasing Solutions of Systems of Second Order Singular Differential Equations

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Singular differential systems of the type

$$(p(t)|y'|^{\alpha-1}y')' = \varphi(t)z^{-\lambda}, \qquad (q(t)|z'|^{\beta-1}z')' = \psi(t)y^{-\mu} \tag{$*$}$$

are considered in an interval $[a,\infty)$, where $\alpha, \beta, \lambda, \mu$ are positive constants and p, q, φ, ψ are positive continuous functions on $[a,\infty)$. A positive decreasing solution of $(*)$ is called proper or singular according to whether it exists on $[a,\infty)$ or it ceases to exist at a finite point of (a,∞). First, conditions are given under which there does exist a singular solution of $(*)$. Then, conditions are established for the existence of proper solutions of $(*)$ which are classified into moderately decreasing solutions and strongly decreasing solutions according to the rate of their decay as $t \to \infty$.

Keywords: Nonlinear differential equation; Singular nonlinearity; Positive solution; Singular solution; Asymptotic behavior

AMS Subject Classification: 34C11

1. INTRODUCTION

We consider systems of second order singular differential equations of the type

$$\begin{aligned}(p(t)|y'|^{\alpha-1}y')' &= \varphi(t)z^{-\lambda},\\ (q(t)|z'|^{\beta-1}z')' &= \psi(t)y^{-\mu},\end{aligned} \tag{A}$$

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where α, β, λ and μ are positive constants, and $p(t)$, $q(t)$, $\varphi(t)$ and $\psi(t)$ are positive continuous functions defined on $[a, \infty)$, $a \geq 0$.

By a solution of (A) on $J \subset [a, \infty)$ we mean a vector function (y, z) which has the property that y, z, $p|y'|^{\alpha-1}y'$ and $q|z'|^{\beta-1}z'$ are continuously differentiable on J and satisfies the system (A) at all points of J. Obviously, both components of a solution must be positive on J.

In this paper we are concerned exclusively with positive decreasing solutions of (A), that is, those solutions of (A) whose components are positive and decreasing on intervals of the form $[a, T)$, $T \leq \infty$. Let (y, z) be such a solution and let $[a, T)$ be its maximal interval of existence. There are two possible cases: $T < \infty$ or $T = \infty$. If $T < \infty$, (y, z) has the property $\lim_{t \to T-0} y(t) = \lim_{t \to T-0} z(t) = 0$ and is called an *extinct singular solution*. The question of existence of extinct singular solutions for (A) is discussed in Section 2. If, on the other hand, $T = \infty$, then (y, z) exists on $[a, \infty)$ and both y and z decrease to nonnegative limits as $t \to \infty$. In this case (y, z) is called a *decreasing proper solution*. Motivated by our knowledge [2,5] of positive decreasing proper solutions of single singular differential equations of the form

$$(p(t)|y'|^{\alpha-1}y')' = \varphi(t)y^{-\lambda}, \tag{B}$$

we take up the two cases:

$$\int_a^\infty (p(t))^{-1/\alpha}\,dt = \infty \quad \text{and} \quad \int_a^\infty (q(t))^{-1/\beta}\,dt = \infty \tag{1.1}$$

and

$$\int_a^\infty (p(t))^{-1/\alpha}\,dt < \infty \quad \text{and} \quad \int_a^\infty (q(t))^{-1/\beta}\,dt < \infty, \tag{1.2}$$

and focus our attention on the following two types of decreasing proper solutions (y, z) of (A):

(I-1) $\quad \lim_{t\to\infty} y(t) = \text{const.} > 0, \quad \lim_{t\to\infty} z(t) = \text{const.} > 0,$

and

(II-1) $\quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0$

in case the condition (1.1) holds, and

(I-2) $$\lim_{t\to\infty}\frac{y(t)}{\pi(t)} = \text{const.} > 0, \quad \lim_{t\to\infty}\frac{z(t)}{\rho(t)} = \text{const.} > 0,$$

and

(II-2) $$\lim_{t\to\infty}\frac{y(t)}{\pi(t)} = 0, \quad \lim_{t\to\infty}\frac{z(t)}{\rho(t)} = 0$$

in case the condition (1.2) holds, where $\pi(t)$ and $\rho(t)$ are given by

$$\pi(t) = \int_t^\infty (p(s))^{-1/\alpha}\,\mathrm{d}s, \quad \rho(t) = \int_t^\infty (q(s))^{-1/\beta}\,\mathrm{d}s, \qquad t \geq a. \quad (1.3)$$

A solution of the type (I-1) or (I-2) is called a *moderately decreasing solution* or a *moderately decaying solution*, while that of the type (II-1) or (II-2) is referred to as a *strongly decreasing solution* or a *strongly decaying solution*. The cases (1.1) and (1.2) are examined separately in Sections 3 and 4, where, first, necessary and sufficient conditions are established for the existence of moderately decreasing or moderately decaying solutions of (A), and then sufficient conditions are derived for the existence of strongly decreasing or strongly decaying solutions of (A).

The motivation of our study in Sections 2–4 is to extend some of the results [2,5] obtained for the single singular differential equation (B) to the case of second order singular differential systems, on the one hand, and to generalize the results [3,7] developed for the simplest system $y'' = \varphi(t)z^{-\lambda}$, $z'' = \psi(t)y^{-\mu}$ to as large a class of singular differential systems of the type (A) as possible, on the other.

Because of the generality of the functions $p(t)$ and $q(t)$ in (A), the results for (A) can be applied to provide useful information about the existence of positive spherically symmetric solutions of singular systems of partial differential equations of the type

$$\begin{aligned} \operatorname{div}(|Du|^{m-2}Du) &= f(|x|)v^{-\lambda}, \\ \operatorname{div}(|Dv|^{n-2}Dv) &= g(|x|)u^{-\mu}, \end{aligned} \qquad \text{(C)}$$

in exterior domains E_a in $\mathbb{R}^N$, $N \geq 2$, where $m > 1$, $n > 1$, $\lambda > 0$ and $\mu > 0$ are constants, D stands for the gradient operator $(\partial/\partial x_1, \ldots, \partial/\partial x_N)$, and

$E_a = \{x \in \mathbb{R}^N: |x| \geq a\}$, $a > 0$ and $f(t)$ and $g(t)$ are positive continuous functions on $[a, \infty)$. Some of the results obtained for (C) in this manner are listed in Section 5.

There has been an increasing interest in the study of spherically symmetric solutions of nonsingular differential systems including

$$\operatorname{div}(|Du|^{m-2}Du) = f(|x|)v^{\lambda},$$
$$\operatorname{div}(|Dv|^{n-2}Dv) = g(|x|)u^{\mu},$$

as a special case; see, e.g., [1,4,6]. To the best of our knowledge, however, there is no previous work devoted to the qualitative study of systems of partial differential equations with nonlinear singularities.

2. EXTINCT SINGULAR SOLUTIONS

In our earlier paper [2] we have proved that in case $\lambda < \min\{\alpha, 1\}$ the single singular differential equation (B) possesses a positive singular solution which is extinct at any given point to the right of a. An extension of this result to differential systems of the form (A) is presented below.

THEOREM 2.1 *Suppose that*

$$\lambda < \frac{\beta}{\beta+1} \quad \text{and} \quad \mu < \frac{\alpha}{\alpha+1}. \tag{2.1}$$

Then, for any given $T > a$, there exists a singular solution of the system (A) *which is positive on* $[a, T)$ *and extinct at* T.

Proof We first note that $\alpha\beta > \lambda\mu$ by (2.1). Define the positive constants k and l by

$$k = \frac{\beta(\alpha+1) - \lambda(\beta+1)}{\alpha\beta - \lambda\mu}, \qquad l = \frac{\alpha(\beta+1) - \mu(\alpha+1)}{\alpha\beta - \lambda\mu}. \tag{2.2}$$

It is easy to see that $k > 1$, $l > 1$,

$$1 - l\lambda = \alpha(k-1) > 0 \quad \text{and} \quad 1 - k\mu = \beta(l-1) > 0.$$

Let K_1, K_2, L_1 and L_2 denote the positive constants

$$\begin{aligned} K_1 &= \frac{1}{k}\left(\frac{\varphi_*/p^*}{\alpha(k-1)}\right)^{1/\alpha}, \qquad K_2 = \frac{1}{k}\left(\frac{\varphi^*/p_*}{\alpha(k-1)}\right)^{1/\alpha}, \\ L_1 &= \frac{1}{l}\left(\frac{\psi_*/q^*}{\beta(l-1)}\right)^{1/\beta}, \qquad L_2 = \frac{1}{l}\left(\frac{\psi^*/q_*}{\beta(l-1)}\right)^{1/\beta}, \end{aligned} \tag{2.3}$$

where we have used the notation

$$f^* = \max_{t\in[a,T]} f(t), \qquad f_* = \min_{t\in[a,T]} f(t),$$

and put

$$\begin{aligned} c_1 &= \left(K_1 L_2^{-\lambda/\alpha}\right)^{\alpha\beta/(\alpha\beta-\lambda\mu)}, \qquad c_2 = \left(K_2 L_1^{-\lambda/\alpha}\right)^{\alpha\beta/(\alpha\beta-\lambda\mu)}, \\ d_1 &= \left(L_1 K_2^{-\mu/\beta}\right)^{\alpha\beta/(\alpha\beta-\lambda\mu)}, \qquad d_2 = \left(L_2 K_1^{-\mu/\beta}\right)^{\alpha\beta/(\alpha\beta-\lambda\mu)}. \end{aligned} \tag{2.4}$$

It is clear that $K_1 \le K_2$, $L_1 \le L_2$, $c_1 \le c_2$ and $d_1 \le d_2$.

Let us now consider the set $\mathcal{Y} \subset C[a,T] \times C[a,T]$ consisting of vector functions $(y(t), z(t))$ satisfying

$$\begin{aligned} &c_1(T-t)^k \le y(t) \le c_2(T-t)^k, \\ &d_1(T-t)^l \le z(t) \le d_2(T-t)^l, \qquad t \in [a,T], \end{aligned} \tag{2.5}$$

and the mapping $\mathcal{F} : \mathcal{Y} \to C[a,T] \times C[a,T]$ defined by

$$\mathcal{F}(y,z)(t) = (\mathcal{G}z(t), \mathcal{H}y(t)), \quad t \in [a,T], \tag{2.6}$$

where

$$\mathcal{G}z(t) = \int_t^T \left[(p(s))^{-1} \int_s^T \varphi(r)(z(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha} \mathrm{d}s, \tag{2.7}$$

and

$$\mathcal{H}y(t) = \int_t^T \left[(q(s))^{-1} \int_s^T \psi(r)(y(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta} \mathrm{d}s. \tag{2.8}$$

As is easily verified, $(y,z) \in \mathcal{Y}$ implies that

$$K_1 d_2^{-\lambda/\alpha}(T-t)^k \le \mathcal{G}z(t) \le K_2 d_1^{-\lambda/\alpha}(T-t)^k,$$
$$L_1 c_2^{-\mu/\beta}(T-t)^l \le \mathcal{H}y(t) \le L_2 c_1^{-\mu/\beta}(T-t)^l, \quad t \in [a,T].$$

Since from (2.3) and (2.4) we have $K_1 d_2^{-\lambda/\alpha} = c_1$, $K_2 d_1^{-\lambda/\alpha} = c_2$, $L_1 c_2^{-\mu/\beta} = d_1$ and $L_2 c_1^{-\mu/\beta} = d_2$, we see that $\mathcal{F}(y,z) \in \mathcal{Y}$, showing that $\mathcal{F}$ maps $\mathcal{Y}$ into itself. Let $\{(y_n(t), z_n(t))\}_{n=1}^{\infty}$ be a sequence of vector functions of $\mathcal{Y}$ converging to $(y(t), z(t))$ uniformly on $[a,T]$ as $n \to \infty$. Then $(y,z) \in \mathcal{Y}$ and it can be shown with the help of the Lebesgue convergence theorem that $\mathcal{F}(y_n,z_n)(t)$ converges to $\mathcal{F}(y,z)(t)$ uniformly on $[a,T]$ as $n \to \infty$. This shows that $\mathcal{F}$ is a continuous mapping. Finally, using (2.7), (2.8) and (2.5), we find that

$$|(\mathcal{G}z)'(t)| \le \left(\frac{\varphi^*/p_*}{\alpha(k-1)}\right)^{1/\alpha} d_1^{-\lambda/\alpha}(T-t)^{k-1},$$
$$|(\mathcal{H}y)'(t)| \le \left(\frac{\psi^*/q_*}{\beta(l-1)}\right)^{1/\beta} c_1^{-\mu/\beta}(T-t)^{l-1}, \quad t \in [a,T],$$

whence it follows that the set $\mathcal{F}(\mathcal{Y})$ is compact in $C[a,T] \times C[a,T]$.

Thus we are able to apply the Schauder fixed point theorem to conclude that there exists a vector function $(y,z) \in \mathcal{Y}$ such that $(y,z) = \mathcal{F}(y,z)$, that is,

$$\begin{aligned} y(t) &= \int_t^T \left[(p(s))^{-1} \int_s^T \varphi(r)(z(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha} \mathrm{d}s, \\ z(t) &= \int_t^T \left[(q(s))^{-1} \int_s^T \psi(r)(y(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta} \mathrm{d}s, \quad t \in [a,T]. \end{aligned} \tag{2.9}$$

The system of integral equations (2.9) shows that the restriction of vector function $(y(t), z(t))$ on $[a,T)$ gives a desired extinct singular solution of (A). This completes the proof.

Example 2.2 Consider the singular differential system

$$\begin{aligned} (|y'|^{\alpha-1}y')' &= \alpha(k-1)k^\alpha z^{-\lambda}, \\ (|z'|^{\beta-1}z')' &= \beta(l-1)l^\beta y^{-\mu} \end{aligned} \tag{2.10}$$

on $[0, \infty)$, where λ and μ satisfy (2.1) and k and l are given by (2.2). According to Theorem 2.1 the system (2.10) has an extinct singular solution on any interval of the form $[0, T)$, $T > 0$. One such solution is given by $(y(t), z(t)) = ((T-t)^k, (T-t)^l)$.

Remark 2.3 It is known [2] that the single singular equation (B) admits no extinct singular solution provided $\lambda > 1$ and the functions $p(t)$ and $\varphi(t)$ are smooth. It would be of interest to extend this type of nonexistence result to the case of singular systems of the form (A).

3. DECREASING POSITIVE PROPER SOLUTIONS

Let us turn to the study of positive decreasing solutions of (A) existing on the entire interval $[a, \infty)$.

We begin with the case where the functions $p(t)$ and $q(t)$ in (A) satisfy the condition (1.1), and direct our attention to the two types of positive decreasing solutions (y, z) of (A) on $[a, \infty)$ such that

$$\lim_{t\to\infty} y(t) = \text{const.} > 0, \qquad \lim_{t\to\infty} z(t) = \text{const.} > 0 \tag{3.1}$$

and

$$\lim_{t\to\infty} y(t) = 0, \qquad \lim_{t\to\infty} z(t) = 0. \tag{3.2}$$

A solution (y, z) satisfying (3.1) or (3.2) is referred to as a moderately decreasing solution or a strongly decreasing solution of (A), respectively.

Let (y, z) be a positive decreasing solution of (A) on $[a, \infty)$. Note that the system (A) for this (y, z) takes the form

$$\begin{aligned} &(-p(t)(-y'(t))^{\alpha})' = \varphi(t)(z(t))^{-\lambda} \\ &(-q(t)(-z'(t))^{\beta})' = \psi(t)(y(t))^{-\mu}, \quad t \geq a. \end{aligned} \tag{3.3}$$

We claim that

$$\lim_{t\to\infty} p(t)(-y'(t))^{\alpha} = \lim_{t\to\infty} q(t)(-z'(t))^{\beta} = 0. \tag{3.4}$$

In fact, if $\lim_{t\to\infty} p(t)(-y'(t))^{\alpha} = k > 0$, then

$$p(t)(-y'(t))^{\alpha} \geq k \quad \text{or} \quad -y'(t) \geq k^{1/\alpha}(p(t))^{-1/\alpha}, \qquad t \geq a,$$

and integrating the last inequality gives

$$y(t) - y(a) \leq -k^{1/\alpha} \int_a^t (p(s))^{-1/\alpha}\, ds, \quad t \geq a.$$

Letting $t \to \infty$ in the above and using (1.1), we have $\lim_{t\to\infty} y(t) = -\infty$, which contradicts the assumed positivity of $y(t)$. Thus we must have $\lim_{t\to\infty} p(t)(-y'(t))^{\alpha} = 0$. Similarly, it is impossible that $\lim_{t\to\infty} q(t) \times (-z'(t))^{\beta} > 0$.

We now integrate (3.3) from t to ∞ to obtain in view of (3.4)

$$\begin{aligned} -y'(t) &= \left[(p(t))^{-1} \int_t^\infty \varphi(s)(z(s))^{-\lambda}\, ds\right]^{1/\alpha}, \\ -z'(t) &= \left[(q(t))^{-1} \int_t^\infty \psi(s)(y(s))^{-\mu}\, ds\right]^{1/\beta}, \quad t \geq a. \end{aligned} \tag{3.5}$$

One more integration of (3.5) yields the following system of integral equations for a moderately decreasing solution of (A):

$$\begin{aligned} y(t) &= y(\infty) + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z(r))^{-\lambda}\, dr\right]^{1/\alpha} ds, \\ z(t) &= z(\infty) + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y(r))^{-\mu}\, dr\right]^{1/\beta} ds, \quad t \geq a, \end{aligned} \tag{3.6}$$

where $y(\infty) = \lim_{t\to\infty} y(t)$ and $z(\infty) = \lim_{t\to\infty} z(t)$. On the basis of (3.6) one can characterize the existence of a moderately decreasing solutions of (A).

THEOREM 3.1 *Suppose that* (1.1) *holds. The system* (A) *has a positive decreasing proper solution* (y, z) *satisfying* (3.1) *if and only if*

$$\int_a^\infty \left[(p(t))^{-1} \int_t^\infty \varphi(s)\, ds\right]^{1/\alpha} dt < \infty \tag{3.7}$$

and

$$\int_a^\infty \left[(q(t))^{-1} \int_t^\infty \psi(s)\, ds\right]^{1/\beta} dt < \infty. \tag{3.8}$$

Proof (The "only if" part) Suppose that (A) has a moderately decreasing solution (y, z). Then (3.6) holds, which implies that the two repeated integrals therein must converge for all $t \geq a$. This fact combined with (3.1) easily shows that both (3.7) and (3.8) are satisfied.

(The "if" part) Suppose that (3.7) and (3.8) hold. Let $c > 0$ and $d > 0$ be given arbitrarily, and choose $t_0 > a$ large enough so that

$$\int_{t_0}^{\infty}\left[(p(t))^{-1}\int_t^{\infty}\varphi(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}t \leq cd^{\lambda/\alpha} \tag{3.9}$$

and

$$\int_{t_0}^{\infty}\left[(q(t))^{-1}\int_t^{\infty}\psi(s)\,\mathrm{d}s\right]^{1/\beta}\mathrm{d}t \leq c^{\mu/\beta}d. \tag{3.10}$$

Define $\mathcal{Y}$ to be the set of vector functions $(y, z) \in C[t_0, \infty) \times C[t_0, \infty)$ satisfying

$$c \leq y(t) \leq 2c \quad \text{and} \quad d \leq z(t) \leq 2d \qquad \text{for } t \geq t_0.$$

Let the mapping $\mathcal{F} : \mathcal{Y} \to C[t_0, \infty) \times C[t_0, \infty)$ be defined by

$$\mathcal{F}(y, z)(t) = (\mathcal{G}z(t), \mathcal{H}y(t)), \quad (y, z) \leq \mathcal{Y}, \tag{3.11}$$

where $\mathcal{G}$ and $\mathcal{H}$ are the integral operators given by

$$\mathcal{G}z(t) = c + \int_t^{\infty}\left[(p(s))^{-1}\int_s^{\infty}\varphi(r)(z(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha}\mathrm{d}s, \tag{3.12}$$

$$\mathcal{H}y(t) = d + \int_t^{\infty}\left[(q(s))^{-1}\int_s^{\infty}\psi(r)(y(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta}\mathrm{d}s, \quad t \geq t_0. \tag{3.13}$$

That $\mathcal{F}(\mathcal{Y}) \subset \mathcal{Y}$ is an immediate consequence of (3.9) and (3.10). It is verified in a routine manner that $\mathcal{F}$ is a continuous mapping and that $\mathcal{F}(\mathcal{Y})$ is a relatively compact subset of the Fréchet space $C[t_0, \infty) \times C[t_0, \infty)$ with the usual product metric topology. The Schauder–Tychonoff fixed point theorem then ensures the existence of a fixed element $(y, z) \in \mathcal{Y}$ of $\mathcal{F}$, which, by the definition of $\mathcal{F}$, satisfies the integral equations

$$\begin{aligned} y(t) &= c + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha} \mathrm{d}s, \\ z(t) &= d + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta} \mathrm{d}s, \quad t \geq t_0. \end{aligned} \tag{3.14}$$

Differentiating (3.14) twice, we conclude that the vector function (y, z) is a moderately decreasing solution of (A) on the subinterval $[t_0, \infty)$ of $[a, \infty)$ such that $y(\infty) = c$ and $z(\infty) = d$. To complete the proof it suffices to continue this (y, z) to the left of t_0 as the solution of the differential system (A) and to observe that the continuation provides a positive decreasing solution of (A) over $[a, \infty)$. It should be noted that in the process of continuation up to the point a no blow-up of (y, z) takes place because of the presence of negative exponents in (A).

Our next task is to study the question of existence of strongly decreasing solutions of (A). To this end we need integral conditions, stronger than (3.7) and (3.8), which are formulated in terms of the functions

$$\Phi(t) = \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)\,\mathrm{d}r\right]^{1/\alpha} \mathrm{d}s, \tag{3.15}$$

$$\Psi(t) = \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)\,\mathrm{d}r\right]^{1/\beta} \mathrm{d}s. \tag{3.16}$$

THEOREM 3.2 *Suppose that* (1.1) *holds and* $\alpha\beta > \lambda\mu$. *In addition to* (3.7) *and* (3.8) *suppose that*

$$\int_a^\infty \left[(p(t))^{-1} \int_t^\infty \varphi(s)(\Psi(s))^{-\lambda}\,\mathrm{d}s\right]^{1/\alpha} \mathrm{d}t < \infty \tag{3.17}$$

and

$$\int_a^\infty \left[(q(t))^{-1} \int_t^\infty \psi(s)(\Phi(s))^{-\mu}\,\mathrm{d}s\right]^{1/\beta} \mathrm{d}t < \infty. \tag{3.18}$$

Then the system (A) *possesses a positive decreasing proper solution* (y, z) *satisfying* (3.2).

Proof Our proof is an adaptation of the method used by Usami [7] for the special case where $\alpha = \beta = 1$ and $p(t) = q(t) \equiv 1$.

According to the proof of Theorem 2.1, there exists for every $n \in \mathbb{N}$ a moderately decreasing solution (y_n, z_n) on $[a, \infty)$ such that $\lim_{t\to\infty} y_n(t) = \lim_{t\to\infty} z_n(t) = 1/n$. Note that (y_n, z_n) satisfies

$$
\begin{aligned}
y_n(t) &= \frac{1}{n} + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z_n(r))^{-\lambda}\, dr \right]^{1/\alpha} ds, \\
z_n(t) &= \frac{1}{n} + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y_n(r))^{-\mu}\, dr \right]^{1/\beta} ds, \quad t \geq a.
\end{aligned}
\tag{3.19}
$$

Using the decreasing nature of $y_n(t)$ and $z_n(t)$, we have the following system of inequalities from (3.19):

$$
y_n(t) \geq (z_n(t))^{-\lambda/\alpha}\Phi(t), \quad z_n(t) \geq (y_n(t))^{-\mu/\beta}\Psi(t), \qquad t \geq a. \tag{3.20}
$$

Combining (3.20) with the equations for the derivatives $y_n'(t)$ and $z_n'(t)$:

$$
\begin{aligned}
-y_n'(t) &= \left[(p(t))^{-1} \int_t^\infty \varphi(s)(z_n(s))^{-\lambda}\, ds \right]^{1/\alpha}, \\
-z_n'(t) &= \left[(q(t))^{-1} \int_t^\infty \psi(s)(y_n(s))^{-\mu}\, ds \right]^{1/\beta},
\end{aligned}
\tag{3.21}
$$

we see that

$$
\begin{aligned}
-y_n'(t) &\leq (y_n(t))^{\lambda\mu/\alpha\beta} \left[(p(t))^{-1} \int_t^\infty \varphi(s)(\Psi(s))^{-\lambda}\, ds \right]^{1/\alpha}, \\
-z_n'(t) &\leq (z_n(t))^{\lambda\mu/\alpha\beta} \left[(q(t))^{-1} \int_t^\infty \psi(s)(\Phi(s))^{-\mu}\, ds \right]^{1/\beta}, \quad t \geq a,
\end{aligned}
\tag{3.22}
$$

from which, after integration over $[t, \infty)$, it follows that

$$\begin{aligned}
&\frac{\alpha\beta(y_n(t))^{(\alpha\beta-\lambda\mu)/\alpha\beta}}{\alpha\beta-\lambda\mu} \\
&\quad\le \frac{\alpha\beta(1/n)^{(\alpha\beta-\lambda\mu)/\alpha\beta}}{\alpha\beta-\lambda\mu} + \int_t^\infty \left[(p(s))^{-1}\int_s^\infty \varphi(r)(\Psi(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha}\mathrm{d}s, \\
&\frac{\alpha\beta(z_n(t))^{(\alpha\beta-\lambda\mu)/\alpha\beta}}{\alpha\beta-\lambda\mu} \\
&\quad\le \frac{\alpha\beta(1/n)^{(\alpha\beta-\lambda\mu)/\alpha\beta}}{\alpha\beta-\lambda\mu} + \int_t^\infty \left[(q(s))^{-1}\int_s^\infty \psi(r)(\Phi(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta}\mathrm{d}s,
\end{aligned} \tag{3.23}$$

for $t \ge a$. The inequalities (3.23) imply that the sequence $\{(y_n(t), z_n(t))\}_{n=1}^{\infty}$ is uniformly bounded on $[a, \infty)$, and the inequalities (3.22) show that the sequence is locally equicontinuous on $[a, \infty)$. Consequently, there is a subsequence of $\{(y_n(t), z_n(t))\}_{n=1}^{\infty}$ which converges to a continuous vector function $(y_*(t), z_*(t))$ as $n \to \infty$, the convergence being uniform on compact subintervals of $[a, \infty)$. The limit function $(y_*(t), z_*(t))$ is positive because of (3.20). The Lebesgue convergence theorem guarantees that $(y_*(t), z_*(t))$ satisfies the integral equations

$$\begin{aligned}
y_*(t) &= \int_t^\infty \left[(p(s))^{-1}\int_s^\infty \varphi(r)(z_*(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha}\mathrm{d}s, \\
z_*(t) &= \int_t^\infty \left[(q(s))^{-1}\int_s^\infty \psi(r)(y_*(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta}\mathrm{d}s, \quad t \ge a,
\end{aligned} \tag{3.24}$$

whence we conclude that $(y_*(t), z_*(t))$ is a strongly decreasing solution of (A) existing on $[a, \infty)$. This completes the proof.

Remark 3.3 Let (y, z) be a strongly decreasing solution of (A) on $[a, \infty)$. Then it satisfies the integral equations (3.6) with $y(\infty) = z(\infty) = 0$, from which, using the decreasing property of $y(t)$ and $z(t)$, we deduce that

$$y(t) \ge (z(t))^{-\lambda/\alpha}\Phi(t), \quad z(t) \ge (y(t))^{-\mu/\beta}\Psi(t), \qquad t \ge a, \tag{3.25}$$

in particular

$$y(t) \ge (z(a))^{-\lambda/\alpha}\Phi(t), \quad z(t) \ge (y(a))^{-\mu/\beta}\Psi(t), \qquad t \ge a. \tag{3.26}$$

Using (3.26) in (3.6), we obtain

$$y(t) \leq (y(a))^{\lambda\mu/\alpha\beta} \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(\Psi(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha} \mathrm{d}s,$$

$$z(t) \leq (z(a))^{\lambda\mu/\alpha\beta} \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(\Psi(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta} \mathrm{d}s, \quad t \geq a. \tag{3.27}$$

The inequalities (3.26) and (3.27) provide estimates for the rate of decay of all positive strongly decreasing solutions of (A).

Example 3.4 Consider the system

$$\begin{aligned} (|y'|^{\alpha-1}y')' &= \alpha \mathrm{e}^{-(\alpha+\lambda)t} z^{-\lambda} \\ (|z'|^{\beta-1}z')' &= \beta \mathrm{e}^{-(\beta+\mu)t} y^{-\mu}, \quad t \geq 0, \end{aligned} \tag{3.28}$$

which is a special case of (A) with $p(t) = q(t) \equiv 1$, $\varphi(t) = \alpha \mathrm{e}^{-(\alpha+\lambda)t}$ and $\psi(t) = \beta \mathrm{e}^{-(\beta+\mu)t}$. Since $\varphi(t)$ and $\psi(t)$ satisfy the conditions (3.7), (3.8), (3.17) and (3.18), Theorem 3.1 implies that (3.28) has a moderately decreasing positive solution (y_m, z_m) such that $\lim_{t\to\infty} y_m(t) = c$ and $\lim_{t\to\infty} z_m(t) = d$ for any given constants $c > 0$ and $d > 0$.

Suppose in addition that $\alpha\beta > \lambda\mu$. We then have

$$\Phi(t) = c_1 \mathrm{e}^{-((\alpha+\lambda)/\alpha)t}, \qquad \Psi(t) = c_2 \mathrm{e}^{-((\beta+\mu)/\beta)t},$$

$$\int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(\Psi(r))^{-\lambda}\,\mathrm{d}r\right]^{1/\alpha} \mathrm{d}s = c_3 \mathrm{e}^{-((\alpha\beta-\lambda\mu)/\alpha\beta)t},$$

$$\int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(\Phi(r))^{-\mu}\,\mathrm{d}r\right]^{1/\beta} \mathrm{d}s = c_4 \mathrm{e}^{-((\alpha\beta-\lambda\mu)/\alpha\beta)t},$$

where the constants $c_1, \ldots, c_4$ depend only on α, β, λ and μ, and so, applying Theorem 3.2, we conclude that there exists a strongly decreasing positive proper solution (y_s, z_s), which, by Remark 3.3, satisfies

$$\begin{aligned} c\mathrm{e}^{-((\alpha+\lambda)/\alpha)t} &\leq y(t) \leq C\mathrm{e}^{-((\alpha\beta-\lambda\mu)/\alpha\beta)t}, \\ d\mathrm{e}^{-((\beta+\mu)/\beta)t} &\leq z(t) \leq D\mathrm{e}^{-((\alpha\beta-\lambda\mu)/\alpha\beta)t}, \quad t \geq 0, \end{aligned} \tag{3.29}$$

where c, C, d and D are constants depending on y, z, α, β, λ and μ. One such solution is $(y_0, z_0) = (e^{-t}, e^{-t})$.

We notice that the assumption $\alpha\beta > \lambda\mu$ in Theorem 3.2 is not absolutely necessary, because (e^{-t}, e^{-t}) is a solution of (3.28) for any positive values of α, β, λ and μ.

4. DECAYING POSITIVE PROPER SOLUTIONS

Our attention is now directed to positive decreasing proper solutions of the system (A) in which $p(t)$ and $q(t)$ are subject to the condition (1.2). Here extensive use is made of the decaying functions $\pi(t)$ and $\rho(t)$ defined by

$$\pi(t) = \int_t^\infty (p(s))^{-1/\alpha}\,ds, \quad \rho(t) = \int_t^\infty (q(s))^{-1/\beta}\,ds, \qquad t \geq a. \tag{4.1}$$

Our purpose here is to examine the existence of a positive decreasing proper solution (y, z) of (A) such that either

$$\lim_{t\to\infty} \frac{y(t)}{\pi(t)} = \text{const.} > 0, \qquad \lim_{t\to\infty} \frac{z(t)}{\rho(t)} = \text{const.} > 0 \tag{4.2}$$

or

$$\lim_{t\to\infty} \frac{y(t)}{\pi(t)} = 0, \qquad \lim_{t\to\infty} \frac{z(t)}{\rho(t)} = 0. \tag{4.3}$$

A solution of (A) satisfying (4.2) or (4.3) is called a moderately decaying solution or strongly decaying solution, respectively.

Let (y, z) be a moderately decaying solution of (A) on $[a, \infty)$. Integrating (3.3) from t to ∞, we have

$$\begin{aligned} -y'(t) &= \left[(p(t))^{-1}\left(\eta^\alpha + \int_t^\infty \varphi(s)(z(s))^{-\lambda}\,ds\right)\right]^{1/\alpha}, \\ -z'(t) &= \left[(q(t))^{-1}\left(\zeta^\beta + \int_t^\infty \psi(s)(y(s))^{-\mu}\,ds\right)\right]^{1/\beta}, \quad t \geq a. \end{aligned} \tag{4.4}$$

where η and ζ are positive constants given by

$$\eta = -\lim_{t\to\infty}(p(t))^{1/\alpha}y'(t) = \lim_{t\to\infty}\frac{y(t)}{\pi(t)},$$

$$\zeta = -\lim_{t\to\infty}(q(t))^{1/\beta}z'(t) = \lim_{t\to\infty}\frac{z(t)}{\rho(t)}.$$

Integration of (4.4) from t to ∞ then yields

$$\begin{aligned} y(t) &= \int_t^\infty \left[(p(s))^{-1}\left(\eta^\alpha + \int_s^\infty \varphi(r)(z(r))^{-\lambda}\,\mathrm{d}r\right)\right]^{1/\alpha} \mathrm{d}s, \\ z(t) &= \int_t^\infty \left[(q(s))^{-1}\left(\zeta^\beta + \int_s^\infty \psi(r)(y(r))^{-\mu}\,\mathrm{d}r\right)\right]^{1/\beta} \mathrm{d}s, \quad t \geq a. \end{aligned} \tag{4.5}$$

Naturally, letting $\eta = \zeta = 0$ in (4.5), we obtain the integral equations for a strongly decaying solution of (A).

The existence of a moderately decreasing solution for (A) can be characterized as the following theorem indicates.

THEOREM 4.1 *Suppose that* (1.2) *holds. The system* (A) *has a positive decreasing proper solution* (y, z) *satisfying* (4.2) *if and only if*

$$\int_a^\infty \varphi(t)(\rho(t))^{-\lambda}\,\mathrm{d}t < \infty \quad \textit{and} \quad \int_a^\infty \psi(t)(\pi(t))^{-\mu}\,\mathrm{d}t < \infty. \tag{4.6}$$

Proof (The "only if" part) In deriving (4.4) we found the convergence of the integrals

$$\int_t^\infty \varphi(s)(z(s))^{-\lambda}\,\mathrm{d}s \quad \text{and} \quad \int_t^\infty \psi(s)(y(s))^{-\mu}\,\mathrm{d}s$$

for all $t \geq a$. This fact combined with (4.2) implies the truth of (4.6).

(The "if" part) Suppose that (4.6) holds. Let η and ζ be arbitrary but fixed positive constants and take $t_0 > a$ so that

$$\int_{t_0}^\infty \varphi(t)(\rho(t))^{-\lambda}\,\mathrm{d}t \leq (2^\alpha - 1)\eta^\alpha\zeta^\lambda \quad \text{and}$$

$$\int_{t_0}^\infty \psi(t)(\pi(t))^{-\mu}\,\mathrm{d}t \leq (2^\beta - 1)\eta^\mu\zeta^\beta. \tag{4.7}$$

Denote by $\mathcal{Y}$ the set of all vector functions $(y,z) \in C[t_0,\infty) \times C[t_0,\infty)$ such that

$$\eta\pi(t) \le y(t) \le 2\eta\pi(t), \quad \zeta\rho(t) \le z(t) \le 2\zeta\rho(t), \qquad t \ge t_0, \tag{4.8}$$

and let $\mathcal{F}$ denote the mapping

$$\mathcal{F}(y,z)(t) = (\mathcal{G}z(t), \mathcal{H}y(t)), \quad (y,z) \in \mathcal{Y}, \tag{4.9}$$

where $\mathcal{G}$ and $\mathcal{H}$ are defined by

$$\mathcal{G}z(t) = \int_t^\infty \left[(p(s))^{-1}\left(\eta^\alpha + \int_s^\infty \varphi(r)(z(r))^{-\lambda}\,\mathrm{d}r\right)\right]^{1/\alpha} \mathrm{d}s, \tag{4.10}$$

$$\mathcal{H}y(t) = \int_t^\infty \left[(q(s))^{-1}\left(\zeta^\beta + \int_s^\infty \psi(r)(y(r))^{-\mu}\,\mathrm{d}r\right)\right]^{1/\beta} \mathrm{d}s, \quad t \ge t_0. \tag{4.11}$$

It can be shown that (i) $\mathcal{F}$ map $\mathcal{Y}$ into itself, (ii) $\mathcal{F}$ is a continuous mapping, and (iii) $\mathcal{F}(\mathcal{Y})$ is a relatively compact subset of $C[t_0,\infty) \times C[t_0,\infty)$. Therefore, by the Schauder–Tychonoff theorem, there exists a fixed element $(y,z) \in \mathcal{Y}$ of $\mathcal{F}$, which satisfies the system (4.5) of integral equations for $t \ge t_0$. Hence (y,z) is a moderately decreasing solution of (A) defined on $[t_0,\infty)$ and satisfying $\lim_{t\to\infty} y(t)/\pi(t) = \eta$ and $\lim_{t\to\infty} z(t)/\rho(t) = \zeta$. To conclude the proof it suffices to continue (y,z) over the entire interval $[a,\infty)$ as the positive decreasing solution of the differential equation (A).

Conditions stronger than (4.6) is needed to ensure the existence of a strongly decreasing positive solution for (A).

THEOREM 4.2 *Suppose that* (1.2) *holds and* $\alpha\beta > \lambda\mu$. *In addition to* (4.6) *suppose that*

$$\int_a^\infty \varphi(t)(\Psi(t))^{-\lambda}\,\mathrm{d}t < \infty, \tag{4.12}$$

$$\int_a^\infty \psi(t)(\Phi(t))^{-\mu}\,\mathrm{d}t < \infty, \tag{4.13}$$

where

$$\Phi(t) = \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)\, dr\right]^{1/\alpha} ds, \tag{4.14}$$

$$\Psi(t) = \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)\, dr\right]^{1/\beta} ds. \tag{4.15}$$

Then, the system (A) *possesses a positive decreasing proper solution* (y, z) *satisfying* (4.3).

Proof Let $\{(y_n, z_n)\}_{n=1}^\infty$ be the sequence of moderately decaying solutions of (A) on $[a, \infty)$ such that

$$\lim_{t\to\infty} \frac{y_n(t)}{\pi(t)} = \frac{1}{n} \quad \text{and} \quad \lim_{t\to\infty} \frac{z_n(t)}{\pi(t)} = \frac{1}{n}.$$

The sequence exists by Theorem 3.1 and satisfies for $t \geq a$

$$\begin{aligned} y_n(t) &= \int_t^\infty \left[(p(s))^{-1}\left(\frac{1}{n^\alpha} + \int_s^\infty \varphi(r)(z_n(r))^{-\lambda}\, dr\right)\right]^{1/\alpha} ds, \\ z_n(t) &= \int_t^\infty \left[(q(s))^{-1}\left(\frac{1}{n^\beta} + \int_s^\infty \psi(r)(y_n(r))^{-\mu}\, dr\right)\right]^{1/\beta} ds, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} -y_n'(t) &= \left[(p(t))^{-1}\left(\frac{1}{n^\alpha} + \int_t^\infty \varphi(s)(z_n(s))^{-\lambda}\, ds\right)\right]^{1/\alpha}, \\ -z_n'(t) &= \left[(q(t))^{-1}\left(\frac{1}{n^\beta} + \int_t^\infty \psi(s)(y_n(s))^{-\mu}\, ds\right)\right]^{1/\beta}. \end{aligned} \tag{4.17}$$

From (4.16) we have

$$y_n(t) \geq (z_n(t))^{-\lambda/\alpha}\Phi(t), \quad z_n(t) \geq (y_n(t))^{-\mu/\beta}\Psi(t), \qquad t \geq a. \tag{4.18}$$

Combining (4.17) with (4.18) shows that

$$-y_n'(t) \leq \left[(p(t))^{-1}\left(1 + (y_n(t))^{\lambda\mu/\beta}\int_t^\infty \varphi(s)(\Psi(s))^{-\lambda}\,\mathrm{d}s\right)\right]^{1/\alpha},$$
$$-z_n'(t) \leq \left[(q(t))^{-1}\left(1 + (z_n(t))^{\lambda\mu/\alpha}\int_t^\infty \psi(s)(\Phi(s))^{-\mu}\,\mathrm{d}s\right)\right]^{1/\beta}. \tag{4.19}$$

We have in particular

$$\begin{aligned} -y_n'(t) &\leq \left[(p(t))^{-1}\left(1 + A(y_n(t))^{\lambda\mu/\beta}\right)\right]^{1/\alpha}, \\ -z_n'(t) &\leq \left[(q(t))^{-1}\left(1 + B(z_n(t))^{\lambda\mu/\alpha}\right)\right]^{1/\beta}, \quad t \geq a. \end{aligned} \tag{4.20}$$

where

$$A = \int_a^\infty \varphi(s)(\Phi(s))^{-\lambda}\,\mathrm{d}s \quad \text{and} \quad B = \int_a^\infty \psi(s)(\Psi(s))^{-\mu}\,\mathrm{d}s.$$

Rewriting (4.20) as

$$\begin{aligned} -y_n'(t)\left(1 + A(y_n(t))^{\lambda\mu/\beta}\right)^{-1/\alpha} &\leq (p(t))^{-1/\alpha}, \\ -z_n'(t)\left(1 + B(z_n(t))^{\lambda\mu/\alpha}\right)^{-1/\beta} &\leq (q(t))^{-1/\beta}, \quad t \geq a, \end{aligned} \tag{4.21}$$

and integrating these inequalities from t to ∞, we find that

$$\begin{aligned} \int_0^{y_n(t)} \left(1 + Au^{\lambda\mu/\beta}\right)^{-1/\alpha} \mathrm{d}u &\leq \pi(a), \\ \int_0^{z_n(t)} \left(1 + Bv^{\lambda\mu/\alpha}\right)^{-1/\beta} \mathrm{d}v &\leq \rho(a), \quad t \geq a. \end{aligned} \tag{4.22}$$

Since

$$\int_0^\infty \left(1 + Au^{\lambda\mu/\beta}\right)^{-1/\alpha} \mathrm{d}u = \int_0^\infty \left(1 + Bv^{\lambda\mu/\alpha}\right)^{-1/\beta} \mathrm{d}v = \infty$$

because of the assumption $\alpha\beta > \lambda\mu$, it follows from (4.22) that the sequence $\{(y_n(t), z_n(t))\}_{n=1}^{\infty}$ is uniformly bounded on $[a,\infty)$. Using this fact, we see from (4.20) that the sequence is locally equicontinuous on $[a,\infty)$. Consequently, there exists a subsequence of $\{(y_n(t), z_n(t))\}_{n=1}^{\infty}$ which converges to a continuous vector function $(y_*(t), z_*(t))$ uniformly on compact subintervals of $[a,\infty)$. The limit function $(y_*(t), z_*(t))$ is a desired strongly decaying solution of (A), since from (4.16) it follows that

$$\begin{aligned} y_*(t) &= \int_t^{\infty}\left[(p(s))^{-1}\left(\int_s^{\infty}\varphi(r)(z_*(r))^{-\lambda}\,\mathrm{d}r\right)\right]^{1/\alpha}\mathrm{d}s,\\ z_*(t) &= \int_t^{\infty}\left[(q(s))^{-1}\left(\int_s^{\infty}\psi(r)(y_*(r))^{-\mu}\,\mathrm{d}r\right)\right]^{1/\beta}\mathrm{d}s, \quad t\ge a. \end{aligned} \tag{4.23}$$

This completes the proof.

Remark 4.3 Proceeding as in Remark 3.3 one can give estimates for the rate of decay of strongly decaying solutions (y,z) of (A). The estimates are formally the same as those given in (3.26) and (3.27).

Example 4.4 To illustrate the above results we consider the system

$$\begin{aligned} ((\cosh t)^{\alpha}|y'|^{\alpha-1}y')' &= \alpha 2^{1-\alpha}\mathrm{e}^{-(\lambda+2)t}(1+\mathrm{e}^{-2t})^{\alpha-1}z^{-\lambda},\\ ((\cosh t)^{\beta}|z'|^{\beta-1}z')' &= \beta 2^{1-\beta}\mathrm{e}^{-(\mu+2)t}(1+\mathrm{e}^{-2t})^{\beta-1}y^{-\mu}, \end{aligned} \tag{4.24}$$

for $t\ge 0$. It is easy to see that $p(t)=(\cosh t)^{\alpha}$ and $q(t)=(\cosh t)^{\beta}$ satisfy (1.2) and the corresponding functions $\pi(t)$ and $\rho(t)$ (cf. (4.1)) are given by $\pi(t)=\rho(t)=\pi-2\tan^{-1}\mathrm{e}^{t}$, $t\ge 0$. In calculating the integrals in (4.6), (4.12) and (4.13), we can use $2\mathrm{e}^{-t}$ in place of $\pi(t)$ and $\rho(t)$, since $\lim_{t\to\infty}\mathrm{e}^{t}\pi(t)=\lim_{t\to\infty}\mathrm{e}^{t}\rho(t)=2$, and regard the functions

$$\varphi(t)=\alpha 2^{1-\alpha}\mathrm{e}^{-(\lambda+2)t}(1+\mathrm{e}^{-2t})^{\alpha-1}\quad\text{and}$$
$$\psi(t)=\beta 2^{1-\beta}\mathrm{e}^{-(\mu+2)t}(1+\mathrm{e}^{-2t})^{\beta-1}$$

to be positive constant multiples of $\mathrm{e}^{-(\lambda+2)t}$ and $\mathrm{e}^{-(\mu+2)t}$, respectively. Taking these facts into account, we find that (4.6) certainly holds and that (4.12) and (4.13) are satisfied if

$$\beta>\tfrac{1}{2}\lambda(\mu+2)\quad\text{and}\quad\alpha>\tfrac{1}{2}\mu(\lambda+2), \tag{4.25}$$

respectively.

Theorem 4.1 then implies that, for any given constants $\eta > 0$ and $\zeta > 0$, the system (4.24) has a moderately decaying solution (y_m, z_m) such that

$$\lim_{t\to\infty} e^t y_m(t) = \eta \quad \text{and} \quad \lim_{t\to\infty} e^t z_m(t) = \zeta > 0.$$

A concrete example of such solutions is (e^{-t}, e^{-t}). From Theorem 4.2 it follows that if (4.25) holds, then there exists a strongly decreasing solution (y_s, z_s) of (4.24) such that

$$\lim_{t\to\infty} e^t y_s(t) = \lim_{t\to\infty} e^t z_s(t) = 0.$$

5. APPLICATION

The above results for (A) can be used to derive nontrivial information about spherically symmetric solutions to singular systems of partial differential equations of the form (C) in an exterior domain E_a, $a > 0$.

The applicability is endorsed by the fact that a spherically symmetric function $(u, v) = (y(|x|), z(|x|))$ is a solution of (C) in E_a if and only if the function $(y(t), z(t))$ is a solution of the ordinary differential system

$$\begin{aligned} \left(t^{N-1}|y'|^{m-2}y'\right)' &= t^{N-1}f(t)z^{-\lambda}, \\ \left(t^{N-1}|z'|^{n-2}z'\right)' &= t^{N-1}g(t)y^{-\mu}, \quad t \geq a, \end{aligned} \tag{5.1}$$

which is a special case of (A) with $\alpha = m-1, \beta = n-1, p(t) = q(t) = t^{N-1}$, $\varphi(t) = t^{N-1}f(t)$ and $\psi(t) = t^{N-1}g(t)$. It is assumed that $m > 1, n > 1, \lambda > 0$, $\mu > 0$ and $f(t)$ and $g(t)$ are positive and continuous on $[a, \infty)$.

First, we have the following statement from Theorem 2.1 applied to (5.1).

I. Suppose that

$$\lambda < \frac{n-1}{n} \quad \text{and} \quad \mu < \frac{m-1}{m}. \tag{5.2}$$

Then, for any fixed $b > a$ there exists a spherically symmetric solution (u, v) of (C) which is positive on $E(a, b) = \{x \in \mathbb{R}^N\colon\ a \leq |x| < b\}$ and satisfies

$$\lim_{|x|\to b-0} u(x) = \lim_{|x|\to b-0} v(x) = 0. \tag{5.3}$$

We now confine our attention of the following special case of (C)

$$\begin{aligned} \operatorname{div}(|Du|^{m-2}Du) &= |x|^{-k}v^{-\lambda}, \\ \operatorname{div}(|Dv|^{n-2}Dv) &= |x|^{-l}u^{-\mu}, \end{aligned} \tag{D}$$

k and l being positive constants, and examine the existence of positive spherically solutions of (D) defined on E_a in the case where

$$N > m \quad \text{and} \quad N > n. \tag{5.4}$$

In this case $p(t) = q(t) = t^{N-1}$ satisfy the condition (1.2), and the functions $\pi(t)$ and $\rho(t)$ in (5.1) become

$$\pi(t) = \frac{m-1}{N-m} t^{-(N-m)/(m-1)}, \quad \rho(t) = \frac{n-1}{N-n} t^{-(N-n)/(n-1)}, \qquad t \geq a. \tag{5.5}$$

The condition (4.6) for $\varphi(t) = t^{N-1-k}$ and $\psi(t) = t^{N-1-l}$ reads as follows:

$$\int_a^\infty \varphi(t)(\rho(t))^{-\lambda}\, dt < \infty \Leftrightarrow k > N + \frac{\lambda(N-n)}{n-1}, \tag{5.6}$$

$$\int_a^\infty \psi(t)(\pi(t))^{-\mu}\, dt < \infty \Leftrightarrow l > N + \frac{\mu(N-m)}{m-1}. \tag{5.7}$$

Theorem 4.1 regarding the moderately decreasing solutions of (A) yields the following result for (D).

II. If (5.4), (5.6) and (5.7) hold, then there exists a positive spherically symmetric solution (u, v) of (D) on E_a satisfying

$$\begin{aligned} \lim_{|x|\to\infty} |x|^{(N-m)/(m-1)} u(x) &= \text{const.} > 0, \\ \lim_{|x|\to\infty} |x|^{(N-n)/(n-1)} v(x) &= \text{const.} > 0. \end{aligned} \tag{5.8}$$

We present via Theorem 4.2 a sufficient condition for (D) to have a positive spherically symmetric solution (u, v) such that

$$\lim_{|x|\to\infty} |x|^{(N-m)/(m-1)}u(x) = 0, \qquad \lim_{|x|\to\infty} |x|^{(N-n)/(n-1)}v(x) = 0. \quad (5.9)$$

III. Suppose that (5.4) holds. If $(m-1)(n-1) > \lambda\mu$ and k and l satisfy

$$k > N + \frac{\lambda(l-n)}{n-1}, \qquad l > N + \frac{\mu(k-m)}{m-1}, \quad (5.10)$$

then, (D) has a positive spherically symmetric solution (u, v) on E_a satisfying (5.9).

To prove this proposition we first notice that the inequalities in (5.10) are consistent because of the assumption $(m-1)(n-1) > \lambda\mu$. Next we compute the functions $\Phi(t)$ and $\Psi(t)$ in (5.13), finding that they are positive constant multiples of $t^{-(k-m)/(m-1)}$ and $t^{-(l-n)/(n-1)}$, respectively. We finally check that (5.10) ensures the conditions (5.12) and (5.13) and apply Theorem 4.2 to (5.1) with $f(t) = t^{-k}$ and $g(t) = t^{-l}$.

The results of Section 3 could also be applied to formulate two propositions, analogues of the above II and III, for the elliptic system (D) in the case where

$$N \le m \quad \text{and} \quad N \le n.$$

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