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Generalizations of the Results on Powers of p-Hyponormal Operators

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Recently, as a nice application of Furuta inequality, Aluthge and Wang (*J. Inequal. Appl.*, **3** (1999), 279–284) showed that "*if T is a p-hyponormal operator for $p \in (0, 1]$, then T^n is p/n-hyponormal for any positive integer n*," and Furuta and Yanagida (*Scientiae Mathematicae*, to appear) proved the more precise result on powers of p-hyponormal operators for $p \in (0, 1]$. In this paper, more generally, by using Furuta inequality repeatedly, we shall show that "*if T is a p-hyponormal operator for $p > 0$, then T^n is $\min\{1, p/n\}$-hyponormal for any positive integer n*" and a generalization of the results by Furuta and Yanagida in (*Scientiae Mathematicae*, to appear) on powers of p-hyponormal operators for $p > 0$.

Keywords: p-Hyponormal operator; Furuta inequality

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1. INTRODUCTION

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

An operator T is said to be *p-hyponormal* for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$. p-Hyponormal operators were defined as an extension of hyponormal ones, i.e., $T^*T \geq TT^*$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p \geq q > 0$ by Löwner–Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$*," and it is well known that

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there exists a hyponormal operator T such that T^2 is not hyponormal [13], but paranormal [7], i.e., $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. We remark that every p-hyponormal operator for $p > 0$ is paranormal [3] (see also [1,5,10]).

Recently, Aluthge and Wang [2] showed the following results on powers of p-hyponormal operators.

THEOREM A.1 [2] *Let T be a p-hyponormal operator for $p \in (0, 1]$. The inequalities*

$$(T^{n^*}T^n)^{p/n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n^*})^{p/n}$$

hold for all positive integer n.

COROLLARY A.2 [2] *If T is a p-hyponormal operator for $p \in (0, 1]$, then T^n is p/n-hyponormal for any positive integer n.*

By Corollary A.2, if T is a hyponormal operator, then T^2 belongs to the class of 1/2-hyponormal operators which is smaller than that of paranormal.

As a more precise result than Theorem A.1, Furuta and Yanagida [11] obtained the following result.

THEOREM A.3 [11, Theorem 1] *Let T be a p-hyponormal operator for $p \in (0, 1]$. Then*

$$(T^{n^*}T^n)^{(p+1)/n} \geq (T^*T)^{p+1} \quad \text{and} \quad (TT^*)^{p+1} \geq (T^nT^{n^*})^{(p+1)/n}$$

hold for all positive integer n.

Theorem A.3 asserts that the first and third inequalities of Theorem A.1 hold for the larger exponents $(p+1)/n$ than p/n in Theorem A.1. In fact, Theorem A.3 ensures Theorem A.1 by Löwner–Heinz theorem for $p/(p+1) \in (0, 1)$ and p-hyponormality of T.

On the other hand, Fujii and Nakatsu [6] showed the following result.

THEOREM A.4 [6] *For each positive integer n, if T is an n-hyponormal operator, then T^n is hyponormal.*

We remark that Theorem A.1, Corollary A.2 and Theorem A.3 are results on p-hyponormal operators for $p \in (0, 1]$, and Theorem A.4 is a result on n-hyponormal operators for positive integer n. In this paper, more generally, we shall discuss powers of p-hyponormal operators for positive real number $p > 0$.

2. MAIN RESULTS

THEOREM 1 *Let T be a p-hyponormal operator for $p>0$. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq (T^*T)^n$ *and* $(TT^*)^n \geq T^nT^{n^*}$ *hold for positive integer n such that $n<p+1$.*
(2) $(T^{n^*}T^n)^{(p+1)/n} \geq (T^*T)^{p+1}$ *and* $(TT^*)^{p+1} \geq (T^nT^{n^*})^{(p+1)/n}$ *hold for positive integer n such that $n \geq p+1$.*

COROLLARY 2 *Let T be a p-hyponormal operator for $p>0$. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq T^nT^{n^*}$ *holds for positive integer n such that $n<p$.*
(2) $(T^{n^*}T^n)^{p/n} \geq (T^nT^{n^*})^{p/n}$ *holds for positive integer n such that $n \geq p$.*

In other words, if T is a p-hyponormal operator for $p>0$, then T^n is $\min\{1,p/n\}$-hyponormal for any positive integer n.

In case $p \in (0,1]$, Theorem 1 (resp. Corollary 2) means Theorem A.3 (resp. Corollary A.2). Corollary 2 also yields Theorem A.4 in case $p=n$. Theorem 1 and Corollary 2 can be rewritten into the following Theorem 1′ and Corollary 2′, respectively. We shall prove Theorem 1′ and Corollary 2′.

THEOREM 1′ *For some positive integer m, let T be a p-hyponormal operator for $m-1<p\leq m$. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq (T^*T)^n$ *and* $(TT^*)^n \geq T^nT^{n^*}$ *hold for $n=1,2,\ldots,m$.*
(2) $(T^{n^*}T^n)^{(p+1)/n} \geq (T^*T)^{p+1}$ *and* $(TT^*)^{p+1} \geq (T^nT^{n^*})^{(p+1)/n}$ *hold for $n=m+1,m+2,\ldots$*

COROLLARY 2′ *For some positive integer m, let T be a p-hyponormal operator for $m-1<p\leq m$. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq T^nT^{n^*}$ *holds for $n=1,2,\ldots,m-1$.*
(2) $(T^{n^*}T^n)^{p/n} \geq (T^nT^{n^*})^{p/n}$ *holds for $n=m,m+1,\ldots$*

We need the following theorem in order to give a proof of Theorem 1′.

THEOREM B.1 (Furuta inequality [8]) *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{r/2}A^pB^{r/2})^{1/q} \geq (B^{r/2}B^pB^{r/2})^{1/q}$

and

(ii) $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

We remark that Theorem B.1 yields Löwner–Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem B.1 are given in [4,15] and also an elementary one page proof in [9]. It is shown in [16] that the domain drawn for p, q and r in Fig. 1 is the best possible one for Theorem B.1.

Proof of Theorem 1′ We shall prove Theorem 1′ by induction.

Proof of (1) We shall prove

$$T^{n^*}T^n \geq (T^*T)^n \tag{2.1}$$

and

$$(TT^*)^n \geq T^nT^{n^*} \tag{2.2}$$

for $n = 1, 2, \ldots, m$. (2.1) and (2.2) always hold for $n = 1$. Assume that (2.1) and (2.2) hold for some $n \leq m-1$. Then we have

$$T^{n^*}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{n^*} \tag{2.3}$$

and the second inequality holds by p-hyponormality of T and Löwner–Heinz theorem for $n/p \in (0, 1]$. By (2.3), we have

$$T^{n^*}T^n \geq (TT^*)^n \tag{2.4}$$

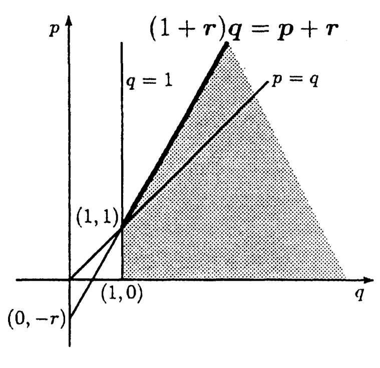

FIGURE 1

and

$$(T^*T)^n \geq T^n T^{n^*}. \tag{2.5}$$

(2.4) ensures

$$T^{n+1^*} T^{n+1} = T^*(T^{n^*} T^n)T \geq T^*(TT^*)^n T = (T^*T)^{n+1},$$

and (2.5) ensures

$$(TT^*)^{n+1} = T(T^*T)^n T^* \geq T(T^n T^{n^*})T^* = T^{n+1} T^{n+1^*}.$$

Hence (2.1) and (2.2) hold for $n+1 \leq m$, so that the proof of (1) is complete.

Proof of (2) We shall prove

$$(T^{n^*} T^n)^{(p+1)/n} \geq (T^*T)^{p+1} \tag{2.6}$$

and

$$(TT^*)^{p+1} \geq (T^n T^{n^*})^{(p+1)/n} \tag{2.7}$$

for $n = m+1, m+2, \ldots$ Let $T = U|T|$ be the polar decomposition of T where $|T| = (T^*T)^{1/2}$ and put $A_n = |T^n|^{2p/n}$ and $B_n = |T^{n^*}|^{2p/n}$. We remark that $T^* = U^*|T^*|$ is also the polar decomposition of T^*.

(a) Case $n = m+1$. (2.1) and (2.2) for $n = m$ ensure

$$(T^{m^*} T^m)^{p/m} \geq (T^*T)^p \geq (TT^*)^p \geq (T^m T^{m^*})^{p/m} \tag{2.8}$$

since the first and third inequalities hold by (2.1), (2.2) and Löwner–Heinz theorem for $p/m \in (0,1]$, and the second inequality holds by p-hyponormality of T. (2.8) ensures the following (2.9) and (2.10).

$$A_m = (T^{m^*} T^m)^{p/m} \geq (TT^*)^p = B_1. \tag{2.9}$$

$$A_1 = (T^*T)^p \geq (T^m T^{m^*})^{p/m} = B_m. \tag{2.10}$$

By using Theorem B.1 for $m/p \geq 1$ and $1/p \geq 0$, we have

$$\begin{aligned}(T^{m+1^*}T^{m+1})^{(p+1)/(m+1)} &= (U^*|T^*|T^{m^*}T^m|T^*|U)^{(p+1)/(m+1)} \\ &= U^*(|T^*|T^{m^*}T^m|T^*|)^{(p+1)/(m+1)}U \\ &= U^*(B_1^{1/2p}A_m^{m/p}B_1^{1/2p})^{(1+1/p)/((m/p)+1/p)}U \\ &\geq U^*B_1^{1+1/p}U \\ &= U^*|T^*|^{2(p+1)}U \\ &= |T|^{2(p+1)} \\ &= (T^*T)^{p+1},\end{aligned}$$

so that (2.6) holds for $n = m+1$.

By using Theorem B.1 again for $m/p \geq 1$ and $1/p \geq 0$, we have

$$\begin{aligned}(T^{m+1}T^{m+1^*})^{(p+1)/(m+1)} &= (U|T|T^mT^{m^*}|T|U^*)^{(p+1)/(m+1)} \\ &= U(|T|T^mT^{m^*}|T|)^{(p+1)/(m+1)}U^* \\ &= U(A_1^{1/2p}B_m^{m/p}A_1^{1/2p})^{(1+1/p)/((m/p)+1/p)}U^* \\ &\leq UA_1^{1+1/p}U^* \\ &= U|T|^{2(p+1)}U^* \\ &= |T^*|^{2(p+1)} \\ &= (TT^*)^{p+1},\end{aligned}$$

so that (2.7) holds for $n = m+1$.

(b) Assume that (2.6) and (2.7) hold for some $n \geq m+1$. Then (2.6) and (2.7) for n ensure

$$(T^{n^*}T^n)^{p/n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n^*})^{p/n} \tag{2.11}$$

since the first and third inequalities hold by (2.6) and (2.7) for n and Löwner–Heinz theorem for $p/(p+1) \in (0,1)$, and the second inequality holds by p-hyponormality of T. (2.11) ensures the following (2.12) and (2.13).

$$A_n = (T^{n^*}T^n)^{p/n} \geq (TT^*)^p = B_1. \tag{2.12}$$

$$A_1 = (T^*T)^p \geq (T^nT^{n^*})^{p/n} = B_n. \tag{2.13}$$

By using Theorem B.1 for $n/p \geq 1$ and $1/p \geq 0$, we have

$$\begin{aligned}
(T^{n+1^*}T^{n+1})^{(p+1)/(n+1)} &= (U^*|T^*|T^{n^*}T^n|T^*|U)^{(p+1)/(n+1)} \\
&= U^*(|T^*|T^{n^*}T^n|T^*|)^{(p+1)/(n+1)}U \\
&= U^*(B_1^{1/2p}A_n^{n/p}B_1^{1/2p})^{(1+1/p)/((n/p)+1/p)}U \\
&\geq U^*B_1^{1+1/p}U \\
&= U^*|T^*|^{2(p+1)}U \\
&= |T|^{2(p+1)} \\
&= (T^*T)^{p+1},
\end{aligned}$$

so that (2.6) holds for $n+1$.

By using Theorem B.1 again for $n/p \geq 1$ and $1/p \geq 0$, we have

$$\begin{aligned}
(T^{n+1}T^{n+1^*})^{(p+1)/(n+1)} &= (U|T|T^nT^{n^*}|T|U^*)^{(p+1)/(n+1)} \\
&= U(|T|T^nT^{n^*}|T|)^{(p+1)/(n+1)}U^* \\
&= U(A_1^{1/2p}B_n^{n/p}A_1^{1/2p})^{(1+1/p)/((n/p)+1/p)}U^* \\
&\leq UA_1^{1+1/p}U^* \\
&= U|T|^{2(p+1)}U^* \\
&= |T^*|^{2(p+1)} \\
&= (TT^*)^{p+1},
\end{aligned}$$

so that (2.7) holds for $n+1$.

By (a) and (b), (2.6) and (2.7) hold for $n = m+1, m+2, \dots$, that is, the proof of (2) is complete.

Consequently the proof of Theorem 1′ is complete.

Proof of Corollary 2′

Proof of (1) By (1) Theorem 1′, for $n = 1, 2, \dots, m-1$,

$$T^{n^*}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{n^*}$$

hold since the second inequality holds by p-hyponormality of T and Löwner–Heinz theorem for $n/p \in (0,1)$. Therefore $T^{n^*}T^n \geq T^nT^{n^*}$ holds for $n = 1, 2, \dots, m-1$.

Proof of (2) By (1) of Theorem 1′ and Löwner–Heinz theorem for $p/m \in (0, 1]$ in case $n = m$, and by (2) of Theorem 1′ and Löwner–Heinz theorem for $p/(p+1) \in (0, 1)$ in case $n = m+1, m+2, \ldots,$

$$(T^{n^*}T^n)^{p/n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n^*})^{p/n}$$

hold since the second inequality holds by p-hyponormality of T. Therefore $(T^{n^*}T^n)^{p/n} \geq (T^nT^{n^*})^{p/n}$ holds for $n = m, m+1, \ldots$

3. BEST POSSIBILITIES OF THEOREM 1 AND COROLLARY 2

Furuta and Yanagida [11] discussed the best possibilities of Theorem A.3 and Corollary A.2 on p-hyponormal operators for $p \in (0, 1]$. In this section, more generally, we shall discuss the best possibilities of Theorem 1 and Corollary 2 on p-hyponormal operators for $p > 0$.

THEOREM 3 *Let n be a positive integer such that* $n \geq 2$, $p > 0$ *and* $\alpha > 1$.

(1) *In case* $n < p+1$, *the following assertions hold:*
  (i) *There exists a p-hyponormal operator T such that* $(T^{n^*}T^n)^\alpha \not\geq (T^*T)^{n\alpha}$.
  (ii) *There exists a p-hyponormal operator T such that* $(TT^*)^{n\alpha} \not\geq (T^nT^{n^*})^\alpha$.

(2) *In case* $n \geq p+1$, *the following assertions hold:*
  (i) *There exists a p-hyponormal operator T such that* $(T^{n^*}T^n)^{((p+1)\alpha)/n} \not\geq (T^*T)^{(p+1)\alpha}$.
  (ii) *There exists a p-hyponormal operator T such that* $(TT^*)^{(p+1)\alpha} \not\geq (T^nT^{n^*})^{((p+1)\alpha)/n}$.

THEOREM 4 *Let n be a positive integer such that* $n \geq 2$, $p > 0$ *and* $\alpha > 1$.

(1) *In case* $n < p$, *there exists a p-hyponormal operator T such that* $(T^{n^*}T^n)^\alpha \not\geq (T^nT^{n^*})^\alpha$.

(2) *In case* $n \geq p$, *there exists a p-hyponormal operator T such that* $(T^{n^*}T^n)^{p\alpha/n} \not\geq (T^nT^{n^*})^{p\alpha/n}$.

Theorem 3 (resp. Theorem 4) asserts the best possibility of Theorem 1 (resp. Corollary 2). We need the following results to give proofs of Theorem 3 and Theorem 4.

THEOREM C.1 [17,19] *Let $p>0$, $q>0$, $r>0$ and $\delta>0$. If $0<q<1$ or $(\delta+r)q<p+r$, then the following assertions hold:*

(i) *There exist positive invertible operators A and B on $\mathbb{R}^2$ such that $A^\delta \geq B^\delta$ and*

$$(B^{r/2}A^pB^{r/2})^{1/q} \not\geq B^{(p+r)/q}.$$

(ii) *There exist positive invertible operators A and B on $\mathbb{R}^2$ such that $A^\delta \geq B^\delta$ and*

$$A^{(p+r)/q} \not\geq (A^{r/2}B^pA^{r/2})^{1/q}.$$

LEMMA C.2 [11] *For positive operators A and B, define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B^{1/2} & 0 & & & & \\ & & B^{1/2} & \boxed{0} & & & \\ & & & A^{1/2} & 0 & & \\ & & & & A^{1/2} & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{3.1}$$

where $\square$ shows the place of the $(0,0)$ matrix element. Then the following assertion holds:

(i) *T is p-hyponormal for $p>0$ if and only if $A^p \geq B^p$.*

Furthermore, the following assertions hold for $\beta>0$ and integers $n\geq 2$:

(ii) $(T^{n^*}T^n)^{\beta/n} \geq (T^*T)^\beta$ *if and only if*

$$(B^{k/2}A^{n-k}B^{k/2})^{\beta/n} \geq B^\beta \quad \textit{holds for } k=1,2,\ldots,n-1. \tag{3.2}$$

(iii) $(TT^*)^\beta \geq (T^nT^{n^*})^{\beta/n}$ *if and only if*

$$A^\beta \geq (A^{k/2}B^{n-k}A^{k/2})^{\beta/n} \quad \textit{holds for } k=1,2,\ldots,n-1. \tag{3.3}$$

(iv) $(T^{n^*}T^n)^{\beta/n} \geq (T^nT^{n^*})^{\beta/n}$ *if and only if*

$$A^\beta \geq B^\beta \quad \textit{holds and}$$

$$(B^{k/2}A^{n-k}B^{k/2})^{\beta/n} \geq B^\beta \quad \textit{and} \quad A^\beta \geq (A^{k/2}B^{n-k}A^{k/2})^{\beta/n}$$

$$\textit{hold for } k = 1, 2, \ldots, n-1. \tag{3.4}$$

Proof of Theorem 3 Let $n \geq 2, p > 0$ and $\alpha > 1$.

Proof of (1) Put $p_1 = n-1 > 0$, $q_1 = 1/\alpha \in (0, 1)$, $r_1 = 1 > 0$ and $\delta = p > 0$.

Proof of (i) By (i) of Theorem C.1, there exist positive operators A and B on H such that $A^\delta \geq B^\delta$ and $(B^{r_1/2}A^{p_1}B^{r_1/2})^{1/q_1} \not\geq B^{(p_1+r_1)/q_1}$, that is,

$$A^p \geq B^p \tag{3.5}$$

and

$$(B^{1/2}A^{n-1}B^{1/2})^\alpha \not\geq B^{n\alpha}. \tag{3.6}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.5) and (i) of Lemma C.2, and $(T^{n^*}T^n)^\alpha \not\geq (T^*T)^{n\alpha}$ by (ii) of Lemma C.2 since the case $k = 1$ of (3.2) does not hold for $\beta = n\alpha$ by (3.6).

Proof of (ii) By (ii) of Theorem C.1, there exist positive operators A and B on H such that $A^\delta \geq B^\delta$ and $A^{(p_1+r_1)/q_1} \not\geq (A^{r_1/2}B^{p_1}A^{r_1/2})^{1/q_1}$, that is,

$$A^p \geq B^p \tag{3.7}$$

and

$$A^{n\alpha} \not\geq (A^{1/2}B^{n-1}A^{1/2})^\alpha. \tag{3.8}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.7) and (i) of Lemma C.2, and $(TT^*)^{n\alpha} \not\geq (T^nT^{n^*})^\alpha$ by (iii) of Lemma C.2 since the case $k = 1$ of (3.3) does not hold for $\beta = n\alpha$ by (3.8).

Proof of (2) Put $p_1 = n-1 > 0$, $q_1 = n/((p+1)\alpha) > 0$, $r_1 = 1 > 0$ and $\delta = p > 0$, then we have $(\delta + r_1)q_1 = n/\alpha < n = p_1 + r_1$.

Proof of (i) By (i) of Theorem C.1, there exist positive operators A and B on H such that $A^\delta \geq B^\delta$ and $(B^{r_1/2}A^{p_1}B^{r_1/2})^{1/q_1} \not\geq B^{(p_1+r_1)/q_1}$, that is,

$$A^p \geq B^p \tag{3.9}$$

and

$$(B^{1/2}A^{n-1}B^{1/2})^{((p+1)\alpha)/n} \not\geq B^{(p+1)\alpha}. \tag{3.10}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.9) and (i) of Lemma C.2, and $(T^{n^*}T^n)^{((p+1)\alpha)/n} \not\geq (T^*T)^{(p+1)\alpha}$ by (ii) of Lemma C.2 since the case $k=1$ of (3.2) does not hold for $\beta=(p+1)\alpha$ by (3.10).

Proof of (ii) By (ii) of Theorem C.1, there exist positive operators A and B on H such that $A^\delta \geq B^\delta$ and $A^{(p_1+r_1)/q_1} \not\geq (A^{r_1/2}B^{p_1}A^{r_1/2})^{1/q_1}$, that is,

$$A^p \geq B^p \tag{3.11}$$

and

$$A^{(p+1)\alpha} \not\geq (A^{1/2}B^{n-1}A^{1/2})^{((p+1)\alpha)/n}. \tag{3.12}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.11) and (i) of Lemma C.2, and $(TT^*)^{(p+1)\alpha} \not\geq (T^nT^{n^*})^{((p+1)\alpha)/n}$ by (iii) of Lemma C.2 since the case $k=1$ of (3.3) does not hold for $\beta=(p+1)\alpha$ by (3.12).

Proof of Theorem 4 Let $n \geq 2, p > 0$ and $\alpha > 1$.

Proof of (1) Put $p_1 = n-1 > 0$, $q_1 = 1/\alpha \in (0,1)$, $r_1 = 1 > 0$ and $\delta = p > 0$. By (i) of Theorem C.1, there exist positive operators A and B on H such that $A^\delta \geq B^\delta$ and $(B^{r_1/2}A^{p_1}B^{r_1/2})^{1/q_1} \not\geq B^{(p_1+r_1)/q_1}$, that is,

$$A^p \geq B^p \tag{3.13}$$

and

$$(B^{1/2}A^{n-1}B^{1/2})^\alpha \not\geq B^{n\alpha}. \tag{3.14}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.13) and (i) of Lemma C.2, and $(T^{n^*}T^n)^\alpha \not\geq (T^nT^{n^*})^\alpha$ by (iv) of Lemma C.2 since the case $k=1$ of the second inequality of (3.4) does not hold for $\beta = n\alpha$ by (3.14).

Proof of (2) It is well known that there exist positive operators A and B on H such that

$$A^p \geq B^p \tag{3.15}$$

and

$$A^{p\alpha} \not\geq B^{p\alpha}. \tag{3.16}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.15) and (i) of Lemma C.2, and $(T^{n^*}T^n)^{p\alpha/n} \not\geq (T^nT^{n^*})^{p\alpha/n}$ by (iv) of Lemma C.2 since the first inequality of (3.4) does not hold for $\beta = p\alpha$ by (3.16).

4. CONCLUDING REMARK

An operator T is said to be *log-hyponormal* if T is invertible and $\log T^*T \geq \log TT^*$. It is easily obtained that every invertible p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function. We remark that log-hyponormal can be regarded as 0-hyponormal since $(T^*T)^p \geq (TT^*)^p$ approaches $\log T^*T \geq \log TT^*$ as $p \to +0$.

As an extension of Theorem A.1, Yamazaki [18] obtained the following Theorem D.1 and Corollary D.2 on log-hyponormal operators.

THEOREM D.1 [18] *Let T be a log-hyponormal operator. Then the following inequalities hold for all positive integer n:*

(1) $T^*T \leq (T^{2^*}T^2)^{1/2} \leq \cdots \leq (T^{n^*}T^n)^{1/n}$.
(2) $TT^* \geq (T^2T^{2^*})^{1/2} \geq \cdots \geq (T^nT^{n^*})^{1/n}$.

COROLLARY D.2 [18] *If T is a log-hyponormal operator, then T^n is also log-hyponormal for any positive integer n.*

The best possibilities of Theorem D.1 and Corollary D.2 are discussed in [12].

As a parallel result to Theorem D.1, Furuta and Yanagida [12] showed the following Theorem D.3 on p-hyponormal operators for $p \in (0, 1]$.

THEOREM D.3 [12] *Let T be a p-hyponormal operator for $p \in (0, 1]$. Then the following inequalities hold for all positive integer n:*

(1) $(T^*T)^{p+1} \leq (T^{2^*}T^2)^{(p+1)/2} \leq \cdots \leq (T^{n^*}T^n)^{(p+1)/n}$.
(2) $(TT^*)^{p+1} \geq (T^2T^{2^*})^{(p+1)/2} \geq \cdots \geq (T^nT^{n^*})^{(p+1)/n}$.

In fact, Theorem D.3 in the case $p \to +0$ corresponds to Theorem D.1.

As a further extension of Theorem D.3, we obtain the following Theorem 5 on p-hyponormal operators for $p > 0$.

THEOREM 5 *For some positive integer m, let T be a p-hyponormal operator for $m-1 < p \leq m$. Then the following inequalities hold for $n = m+1, m+2, \ldots$:*

(1) $(T^*T)^{p+1} \leq (T^{m+1^*}T^{m+1})^{(p+1)/(m+1)} \leq (T^{m+2^*}T^{m+2})^{(p+1)/(m+2)} \leq \cdots \leq (T^{n^*}T^n)^{(p+1)/n}$.

(2) $(TT^*)^{p+1} \geq (T^{m+1}T^{m+1^*})^{(p+1)/(m+1)} \geq (T^{m+2}T^{m+2^*})^{(p+1)/(m+2)} \geq \cdots \geq (T^nT^{n^*})^{(p+1)/n}$.

We remark that Theorem 5 yields Theorem D.3 by putting $m = 1$.

Scrutinizing the proof of Theorem D.1 and Theorem D.3, we recognize that the following result plays an important role.

THEOREM D.4 [12,18] *Let T be a p-hyponormal operator for $p \in (0, 1]$ or a log-hyponormal operator. Then the following inequalities hold for all positive integer n:*

(1) $|T^{n+1}|^{2n/(n+1)} \geq |T^n|^2$, *i.e.*, $(T^{n+1^*}T^{n+1})^{n/(n+1)} \geq T^{n^*}T^n$.

(2) $|T^{n^*}|^2 \geq |T^{n+1^*}|^{2n/(n+1)}$, *i.e.*, $T^nT^{n^*} \geq (T^{n+1}T^{n+1^*})^{n/(n+1)}$.

We remark that it was shown in [14] that Theorem D.1 and Theorem D.4 hold even if an invertible operator T belongs to class A (i.e., $|T^2| \geq |T|^2$) which was introduced in [10] as a class of operators including p-hyponormal and log-hyponormal operators.

Proof of Theorem 5 It is easily obtained by Löwner–Heinz theorem that Theorem D.4 remains valid for p-hyponormal operators for $p > 0$.

Proof of (1) By (1) of Theorem D.4 and Löwner–Heinz theorem for $(p+1)/n \in (0, 1)$,

$$(T^{n+1^*}T^{n+1})^{(p+1)/(n+1)} \geq (T^{n^*}T^n)^{(p+1)/n} \tag{4.1}$$

holds for $n = m+1, m+2, \ldots$ Then

$$(T^*T)^{p+1} \leq (T^{m+1^*}T^{m+1})^{(p+1)/(m+1)} \leq (T^{m+2^*}T^{m+2})^{(p+1)/(m+2)}$$
$$\leq \cdots \leq (T^{n^*}T^n)^{(p+1)/n}$$

holds by (2) of Theorem 1′ and (4.1).

Proof of (2) By (2) of Theorem D.4 and Löwner–Heinz theorem for $(p+1)/n \in (0,1)$,

$$(T^n T^{n^*})^{(p+1)/n} \geq (T^{n+1} T^{n+1^*})^{(p+1)/(n+1)} \tag{4.2}$$

holds for $n = m+1, m+2, \ldots$ Then

$$(TT^*)^{p+1} \geq (T^{m+1} T^{m+1^*})^{(p+1)/(m+1)} \geq (T^{m+2} T^{m+2^*})^{(p+1)/(m+2)}$$
$$\geq \cdots \geq (T^n T^{n^*})^{(p+1)/n}$$

holds by (2) of Theorem 1′ and (4.2).

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