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Global and Blowup Solutions of Quasilinear Parabolic Equation with Critical Sobolev Exponent and Lower Energy Initial Value

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In this paper, by means of the energy method, we first study the existence and asymptotic estimates of global solution of quasilinear parabolic equations involving p-Laplacian ($p > 2$) and critical Sobolev exponent and *lower energy* initial value in a bounded domain in $R^N (N \geq 3)$, and also study the sufficient conditions of finite time blowup of local solution by the classical concave method. Finally, we study the asymptotic behavior of any global solutions $u(x, t; u_0)$ which may possess *high energy* initial value function $u_0(x)$. We can prove that there exists a time subsequence $\{t_n\}$ such that the asymptotic behavior of $u(x, t_n; u_0)$ as $t_n \to \infty$ is similar to the Palais–Smale sequence of stationary equation of the above parabolic problem.

Keywords: Quasilinear parabolic equation; Critical Sobolev exponent; *Lower energy* initial value; Asymptotic estimates; Finite time blow up

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1 INTRODUCTION

In this paper we are concerned with the asymptotic estimates of global solutions, and blow-up of local solutions of quasilinear parabolic

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equations of the following form:

$$\begin{aligned} &u_t - \Delta_p u = |u|^{q-2}u, && (x,t) \in \Omega \times (0,T), \\ &u(x,t) = 0, && (x,t) \in \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x), && u_0(x) \geq 0,\ u_0(x) \not\equiv 0, \end{aligned} \tag{1.1}$$

with *lower energy* initial value, and asymptotic behavior of any global solutions which may possess *high energy* initial value function. Here $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $2 < p < N$, $q = p^* = pN/(N-p)$ is the critical Sobolev exponent. Ω is a bounded domain in $R^N (N \geq 3)$ with smooth boundary $\partial\Omega$.

Equation (1.1) is a class of degenerate parabolic equations and appears in the relevant theory of nonNewtonian fluids [1]. For the case of $p = 2$, various authors have derived sufficient conditions for the existence and asymptotic behavior of global solutions of (1.1) [2,6,10,11]. For the case of $p \neq 2$, Tsutsumi [19], Ishii [9], Otani [16], Nakao [14] have studied the existence and the asymptotic behavior of global solution with $q < p^*$. In [15], Nakao considered the problem with critical or supercritical nonlinear and the condition imposed on the initial data is $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ or $u_0 \in W_0^{1,p}(\Omega) \cap L^{p_0}(\Omega) (p_0 > p^*)$, and obtained precise estimates about the asymptotic behavior as $t \to \infty$. The first object of this paper is to relax this additional condition of $u_0 \in L^\infty(\Omega)$ or $u \in L^{p_0}(\Omega)$ and to study the time-asymptotic estimates and finite time blow up of (1.1) with *lower energy* initial value. The second object of this paper is to consider the asymptotic behavior of any global solution which may possess *high energy* initial value function. We can prove that there exists a subsequence $\{t_n\}$ such that the asymptotic behavior of $u(t_n)$ as $t_n \to \infty$ is similar to the Palais-Smale sequence of stationary equation of (1.1).

To state the main idea, we first give some useful definitions and notations.

Denotes the usual Sobolev space by $W_0^{1,p}(\Omega)$, endowed with the norm $\|\nabla u\|_p = (\int_\Omega |\nabla u|^p \,\mathrm{d}x)^{1/p}$, denote the norm of $L^r(\Omega)$ by $\|\cdot\|_r$. Denote $\Omega \times (0,T)$ by Q_T.

DEFINITION 1.1 *We say that a function u is a solution of* (1.1) *in* Q_T *iff*

$$\begin{aligned} u &\in L^\infty(0,T; W_0^{1,p}(\Omega)), \\ u_t &\in L^2(Q_T) = L^2(0,T; L^2(\Omega)), \end{aligned}$$

and satisfies (1.1) *in the distribution sense. If* $T=\infty$, *u is called global solution. We always denote by* $u(x,t;u_0)$ *the solution with initial value* $u_0(x)$.

Let S be the best constant for the Sobolev embedding $W_0^{1,p}(\Omega)\subset L^{p^*}(\Omega)$ which defined as follows:

$$S=\inf_{\substack{u\in W_0^{1,p}(\Omega)\\ \|u\|_{p^*}=1}}\|\nabla u\|_p^p.$$

Remark 1.1 Let S be the best constant for the Sobolev embedding of $W_0^{1,p}(\Omega)\subset L^{p^*}(\Omega)$. Then

(a) S is independent of Ω and depends only on N.
(b) The infimum S is never achieved when Ω is a bounded domain.

The proof can be found in Talenti [18].
Denote the energy function of (1.1) by

$$J(u)=\frac{1}{p}\int_\Omega|\nabla u|^p\,\mathrm{d}x-\frac{1}{p^*}\int_\Omega|u|^{p^*}\,\mathrm{d}x.$$

DEFINITION 1.2 *We say that a function* $u_0(x)$ *possesses* lower energy *if*

$$J(u_0)<\frac{1}{N}S^{N/p}.$$

where S is the best Sobolev constant.

Remark 1.2 The value $(1/N)S^{N/p}$ is the energy of the unique positive radial solution of the quasilinear elliptic equation

$$\begin{aligned}&-\Delta_p u=|u|^{p^*-2}u,\quad x\in R^N,\\&u(|x|)\to 0,\quad \text{as } |x|\to\infty,\end{aligned}\tag{1.2}$$

to the energy functional

$$J^\infty(u)=\frac{1}{p}\int_{R^N}|\nabla u|^p\,\mathrm{d}x-\frac{1}{p^*}\int_{R^N}|u|^{p^*}\,\mathrm{d}x.$$

A number of authors have studied the perturbation problem of (1.2) (or in $p=2$ and bounded domain) comparing the energy functional of perturbation problem with $(1/N)S^{N/p}$ (e.g. [3,7,8,13,20]).

Now we can state the main results of the first object. First we consider the case: $J(u_0)\leq 0$, we have

THEOREM 1.1 *Let $u_0(x)$ be a* lower energy *initial value and $J(u_0)\leq 0$. Then $u(x,t;u_0)$ blowup in finite time.*

Now we consider the case of positive energy, i.e. $0<J(u_0)<(1/N)S^{N/p}$, we have

THEOREM 1.2 *Let $u_0(x)(\neq 0)$ be a* lower energy *initial value.*

(1) *If $\int_\Omega |u_0|^{p^*}\,\mathrm{d}x < S^{N/p}$, then* (1.1) *has a global solution $u(x,t;u_0)$. Moreover*

$$\|\nabla u(t)\|_p^p = O(t^{-2/(p-2)}), \quad \text{as } t\to\infty. \tag{1.3}$$

and

$$\|u\|_2^2 = O(t^{-2/(p-2)}), \quad \text{as } t\to\infty, \tag{1.4}$$

(2) *If $\int_\Omega |u_0|^{p^*}\,\mathrm{d}x \geq S^{N/p}$, then the local solution blows up in finite time.*

Remark 1.3 Obviously, if $\int_\Omega |u_0|^{p^*}\,\mathrm{d}x < S^{N/p}$, then $J(u_0)>0$. Indeed, if

$$\int_\Omega |u_0|^{p^*}\,\mathrm{d}x < S^{N/p},$$

then

$$\int_\Omega |\nabla u_0|^p\,\mathrm{d}x > \int_\Omega |u_0|^{p^*}\,\mathrm{d}x.$$

Thus, from $u_0(x)\neq 0$, we have

$$J(u_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u_0|^{p^*}\,\mathrm{d}x > \frac{1}{N}\int_\Omega |\nabla u_0|^p\,\mathrm{d}x > 0.$$

Now we state the main results of the second object:

THEOREM 1.3 *If* $u(x,t;u_0)$ *is a global solution of* (1.1), *and uniformly bounded in* $W_0^{1,p}(\Omega)$ *with respect to t, then, for any subsequence* $t_n \to \infty$, *there exists a stationary solution w such that* $u(x,t_n;u_0) \rightharpoonup w$ *in* $W_0^{1,p}(\Omega)$.

THEOREM 1.4 *If* $u(x,t;u_0)$ *is a global solution of* (1), *then the* ω*-limit set of u contains a stationary solution w.*

The rest of this paper is organized as follows: In Section 2, we prove Theorems 1.1 and 1.2. In Section 3 we prove Theorems 1.3 and 1.4.

2 THE PROOF OF THEOREMS 1.1 AND 1.2

In this section we consider the existence and the time-asymptotic estimates of global solutions and finite time blowup of (1.1). We first prove Theorem 1.1.

Proof of Theorem 1.1 In fact, we can prove a more general result:

If there exists some t_0 such that $J(u(t_0)) \le 0$, then $u(x,t;u_0)$ blows up in finite time.

We shall employ the classical concavity method (see [4,5,8,12,17]). Suppose that $t_{\max} = \infty$ and denote $f(t) = \frac{1}{2}\int_{t_0}^t \|u\|_2^2\,\mathrm{d}s$. Performing standard manipulations

$$\int_{t_0}^t \int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x = J(u(t_0)), \tag{2.1}$$

$$\begin{aligned} f'(t) &= \frac{1}{2}\|u_0\|_2^2 + \int_{t_0}^t \int_\Omega (-|\nabla u|^p + |u|^{p^*})\,\mathrm{d}x\,\mathrm{d}s, \\ f''(t) &= -\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega |u|^{p^*}\,\mathrm{d}x. \end{aligned} \tag{2.2}$$

By (2.1), we have

$$\begin{aligned} f''(t) &\ge -\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{p^*}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - p^*J(u(t_0)) + p^*\int_{t_0}^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s \\ &= \left(\frac{p^*}{p} - 1\right)\int_\Omega |\nabla u|^p\,\mathrm{d}x - p^*J(u(t_0)) + p^*\int_{t_0}^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{2.3}$$

From the assumption, $J(u(t_0)) \leq 0$ such that

$$\left(\frac{p^*}{p} - 1\right) \int_\Omega |\nabla u|^p \, dx - p^* J(u(t_0)) > 0,$$

for all $t \geq t_0$. If we had $t_{\max} = \infty$, then this inequality would yield

$$\lim_{t \to \infty} f'(t) = \lim_{t \to \infty} f(t) = \infty,$$

and

$$f''(t) \geq p^* \int_{t_0}^t \int_\Omega u_t^2 \, dx \, ds,$$

and

$$\begin{aligned} f(t) f''(t) &\geq \frac{p^*}{p} \left(\int_{t_0}^t \|u(s)\|_2^2 \, ds \right) \left(\int_{t_0}^t \|u_s(s)\|_2^2 \, ds \right) \\ &\geq \frac{p^*}{p} \left(\int_{t_0}^t \int_\Omega u u_t \, dx \, ds \right)^2 = \frac{p^*}{p} (f'(t) - f'(t_0))^2, \end{aligned}$$

and as $t \to \infty$ we have for some $\alpha > 0$ and $\forall t \geq t_0$ such that

$$f(t) f''(t) \geq (1 + \alpha)(f'(t))^2.$$

Hence $f(t)^{-\alpha}$ is concave on $[t_0, \infty]$. But $f(t)^{-\alpha} > 0$ and $\lim_{t \to \infty} f(t)^{-\alpha} = 0$. This contradiction proves that $t_{\max} < \infty$; which completes the proof of Theorem 1.1.

Proof of Theorem 1.2 We divide the proof into several steps

Step 1 Proof of Existence

(i) *A priori estimates and local existence* From [6] and [10], for each $n > 0$, there is a unique classical solution $u_n \in C(Q_T) \cap C^{2,1}(Q_T)$ of the following equation:

$$\begin{aligned} &u_t = \operatorname{div}\left(\left(|\nabla u|^2 + \frac{1}{n}\right)^{(p-2)/2} \nabla u\right) + \min\{n, u^{p^*-1}\}, && (x, t) \in \Omega \times (0, T), \\ &u(x, t) = 0, && (x, t) \in \partial\Omega \times (0, T), \\ &u(x, 0) = u_{n0}, && \end{aligned} \qquad (2.4)$$

where $u_{n0} \in C_0^\infty(\Omega)$, such that

$$u_{n0} \to u_0, \quad \text{strongly in } W_0^{1,p}(\Omega),$$

and $J(u_{n0}) < (1/N)S^{N/p}$, $\int_\Omega |u_{n0}|^{p^*}\,\mathrm{d}x < S^{N/p}$. On the other hand, multiplying (2.4) by u_{nt} and integrating, we have

$$\iint_{Q_{T_1}} u_{nt}^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u_n|^{p^*}\,\mathrm{d}x \le J(u_{n0}). \tag{2.5}$$

For the sake of convenience, define:

$$\Sigma = \left\{ u \,|\, u \in W_0^{1,p}(\Omega), u \ge 0, u \ne 0, J(u) < \frac{1}{N}S^{N/p}, \int_\Omega |u|^{p^*}\,\mathrm{d}x < S^{N/p} \right\}.$$

Now we show that

$$u_n(t) \in \Sigma, \quad \text{for any } t \ge 0.$$

Suppose that it does not hold and let t^* be the smallest time for which $u_n(t^*) \notin \Sigma$. Then in virtue of the continuity of $u_n(t)$ we see that $u_n(t^*) \in \partial\Sigma$. Hence

$$J(u_n(t^*)) = \frac{1}{N}S^{N/p},$$

or

$$\int_\Omega |\nabla u_n|^p\,\mathrm{d}x = \int_\Omega |u_n|^{p^*}\,\mathrm{d}x,$$

which contradicts to (2.5). Then from (2.5) and note that if $\int_\Omega u^{p^*}\,\mathrm{d}x < S^{N/p}$, then $\int_\Omega |\nabla u|^p\,\mathrm{d}x > \int_\Omega u^{p^*}\,\mathrm{d}x$, we have

$$\int_0^t \|u_n'(s)\|_{L^2(\Omega)}^2\,\mathrm{d}s + \frac{1}{N}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x \le J(u_{n0}) < \frac{1}{N}S^{N/p}. \tag{2.6}$$

Thus, we obtain

$$\iint_{Q_{T_1}} u_{nt}^2\,\mathrm{d}x\,\mathrm{d}t < \frac{1}{N}S^{N/p}, \tag{2.7}$$

$$\int_\Omega |\nabla u_n|^p\,\mathrm{d}x < S^{N/p}. \tag{2.8}$$

From (2.8) we have

$$|\nabla u_n|_{L^p(Q_{T_1})} \le C(T_1), \tag{2.9}$$

where $C(T_1)$ is the constant independent of n. From the prior estimates (2.7), (2.8) and (2.9), we see that there exists a subsequence (not relabeled) and a function u such that

$$\begin{gathered} u_n \to u,\ u_n^{p^*-1} \to u^{p^*-1}, \quad \text{a.e. on } Q_{T_1}, \\ \nabla u_n \rightharpoonup \nabla u, \quad \text{weakly in } L^p(Q_{T_1}), \\ u_{nt} \rightharpoonup u_t, \quad \text{weakly in } L^2(Q_{T_1}), \\ u_n \rightharpoonup u, \quad \text{in } L^\infty(0, T_1; W_0^{1,p}(\Omega)) \text{ weak star}, \\ |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup w, \quad \text{weakly in } L^{p/(p-1)}(Q_{T_1}). \end{gathered}$$

Then well known arguments of the theory of monotone operators yields $w = |\nabla u|^{p-2}\nabla u$; which implies the function u is a desired local solution of the problem (1.1).

(ii) *Global existence* To prove that it is a global solution. Multiplying (1.1) by u_t and integrating, we obtain

$$\int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s + J(u(x,t)) = J(u_0) < \frac{1}{N} S^{N/p}.$$

Note if $\int_\Omega |u|^{p^*} \mathrm{d}x < S^{N/p}$, then $\int_\Omega |\nabla u|^p \mathrm{d}x > \int_\Omega |u|^{p^*} \mathrm{d}x$. Thus $J(u(x,t)) < (1/N)S^{N/p}$ for any $t > 0$. Now we prove $u(x,t) \in \Sigma$, for any $t > 0$. Indeed, if there exists a t^* such that $u(x,t) \in \partial\Sigma$, then we have $J(u(x,t)) \ge (1/N)S^{N/p}$, which is a contradiction. Hence $\int_\Omega |\nabla u(t)|^p \mathrm{d}x > \int_\Omega |u(t)|^{p^*} \mathrm{d}x$ for any $t > 0$. Therefore

$$\int_0^t \|u'(s)\|_2^2 \,\mathrm{d}x + \frac{1}{N}\int_\Omega |\nabla u|^p \,\mathrm{d}x \le J(u_0) < \frac{1}{N} S^{N/p},$$

which implies

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x < S^{N/p}, \tag{2.10}$$

$$\|u'(s)\|_{L^2(0,T;L^2(\Omega))} < \frac{1}{N} S^{N/p}, \tag{2.11}$$

for any $T > 0$. Thus $u(x, t)$ is a global solution of (1.1); which completes Step 1.

Step 2 Proof of (1.3) Note if $\int_\Omega |u|^{p^*}\,\mathrm{d}x < S^{N/p}$, then $\int_\Omega |\nabla u|^p\,\mathrm{d}x > \int_\Omega |u|^{p^*}\,\mathrm{d}x$. Thus $J(u(x,t)) < (1/N)S^{N/p}$ for any $t > 0$. It is easy to prove $u(x,t) \in \Sigma$, for any $t > 0$. Indeed, if there exists a t^* such that $u(x,t) \in \partial\Sigma$, then we have $J(u(x,t)) \geq (1/N)S^{N/p}$, which is a contradiction. Hence $\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x > \int_\Omega |u(t)|^{p^*}\,\mathrm{d}x$ for any $t > 0$. Let

$$h(u(t)) = \int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega u^{p^*}\,\mathrm{d}x,$$

then

$$h(u(t)) > 0, \quad \text{for all } t \geq 0.$$

By Sobolev inequality

$$\int_\Omega |u|^{p^*}\,\mathrm{d}x < (1/S^{p^*/p})\left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)^{p^*/p},$$

and

$$J(u_0) > \frac{1}{N}\int_\Omega |\nabla u|^p\,\mathrm{d}x,$$

implies

$$\int_\Omega |u|^{p^*}\,\mathrm{d}x < (1/S^{p^*/p})(NJ(u_0))^{p^*/p-1}\int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{2.12}$$

For simplicity, denote $(1/S^{p^*/p})(NJ(u_0))^{p^*/p-1}$ by $0 < \delta < 1$. Let $\gamma = 1 - \delta$, we have

$$\int_\Omega |u(t)|^{p^*}\,\mathrm{d}x \leq (1-\gamma)\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x. \tag{2.13}$$

Let $T > t_0$ be a fixed number, then from

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(t)|^2\,\mathrm{d}x = -h(u(t))$$

and Poincare's inequality, we have

$$\begin{aligned}\int_t^T h(u(s))\,\mathrm{d}s &= \frac{1}{2}\int_\Omega |u(t)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u(T)|^2\,\mathrm{d}x\\ &\le \frac{1}{2}\int_\Omega |u(t)|^2\,\mathrm{d}x \le \frac{1}{2\lambda_1}\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x\\ &\le C(\Omega)\left(\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x\right)^{2/p}, \end{aligned} \tag{2.14}$$

where λ_1 is the first eigenvalue of $-\Delta_p u = \lambda |u|^{p-2}u$, $x\in\Omega$, $u=0$, $x\in\partial\Omega$. Furthermore, (2.13) implies

$$\begin{aligned}J(u(t)) &= \frac{1}{p}\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u(t)|^{p^*}\,\mathrm{d}x\\ &= \frac{1}{p}\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x + \frac{1}{p^*}\left[h(u(t)) - \int_\Omega |\nabla u(t)|^p\,\mathrm{d}x\right]\\ &= \frac{1}{N}\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x + \frac{1}{p^*}h(u(t)) \ge \frac{1}{N}\int_\Omega |\nabla u(t)|^p\,\mathrm{d}z, \end{aligned} \tag{2.15}$$

on $[t_0,\infty)$. Therefore, by (2.13) and (2.14) we obtain

$$\int_t^T h(u(s))\,\mathrm{d}s \le C(\Omega)(J(u(t)))^{2/p}, \tag{2.16}$$

on $[t_0, T]$. On the other hand, (2.13) implies

$$\gamma\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x \le h(u(t)), \tag{2.17}$$

on $[t_0,\infty)$. By (2.15) and (2.17), we have

$$J(u(t)) \le \left(\frac{1}{N\gamma} + \frac{1}{p^*}\right)h(u(t)). \tag{2.18}$$

Further (2.16) and (2.18) give

$$C_1\int_t^T J(u(s))\,\mathrm{d}s \le (J(u(t)))^{2/p}$$

on $[t_0, T]$, where

$$C_1 = \left(C(\Omega)\left(\frac{1}{N\gamma} + \frac{1}{p^*}\right)\right)^{-1}.$$

Then, from the arbitrariness of $T > t_0$, we have

$$C_1 \int_t^\infty J(u(s))\, \mathrm{d}s \le (J(u(t)))^{2/p},$$

i.e.

$$C_1^{p/2}\left(\int_t^\infty J(u(s))\, \mathrm{d}s\right)^{p/2} \le -\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty J(u(s))\, \mathrm{d}s. \tag{2.19}$$

Setting $y(t) = \int_t^\infty J(u(s))\, \mathrm{d}s$, it follows from (2.19), we have

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} \le -C_1^{p/2} y(t)^{p/2}.$$

Performing standard manipulations, we have

$$y(t) \le C_2 t^{-2/(p-2)}.$$

Thus, we obtain

$$TJ(u(T+t)) \le \int_t^{T+t} J(u(s))\, \mathrm{d}s \le \int_t^\infty J(u(s))\, \mathrm{d}s \le C_2 t^{-2/(p-2)},$$

By (2.15) we have

$$\frac{1}{N}\int_\Omega |\nabla u(T)|^p\, \mathrm{d}x \le J(u(T)) \le C_3 t^{-2/(p-2)},$$

with some constant $C_3 > 0$ for enough large $t > T$. Hence

$$\int_\Omega |\nabla u(t)|^p\, \mathrm{d}x = O(t^{-2/(p-2)}), \quad \text{as } t \to \infty.$$

Step 3 Proof of (1.4) Obviously

$$\|\nabla u(x, t; u_0)\|_p^p \le r^{N/p} < S^{N/p}, \tag{2.20}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u(t)|^2\,\mathrm{d}x+\int_{\Omega}|\nabla u|^p\,\mathrm{d}x=\int_{\Omega}|u|^{p^*}\,\mathrm{d}x,\quad \text{for all } t>0. \tag{2.21}$$

By the same argument with Step 2, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^2\,\mathrm{d}x &< -(1-\delta)\int_{\Omega}|\nabla u|^p\,\mathrm{d}x\\ &\leq -\frac{(1-\delta)}{\lambda_1}\int_{\Omega}|u|^p\,\mathrm{d}x \leq -C\left(\int_{\Omega}|u|^2\,\mathrm{d}x\right)^{p/2},\end{aligned}$$

where λ_1 is the first eigenvalue of $-\Delta_p u=\lambda|u|^{p-2}u$, $x\in\Omega$, $u=0$, $x\in\partial\Omega$,

$$C=\frac{(1-\delta)}{\lambda_1|\Omega|^{(p-2)/2}}.$$

Let $y=\|u\|_2^2$, we see that the estimate

$$\frac{\mathrm{d}y}{\mathrm{d}t}\leq -Cy^{p/2}$$

Therefore, we have

$$y^{-(p-2)/2}\geq\left(\int_{\Omega}|u_0|^2\,\mathrm{d}x\right)^{(p-2)/2}+C\frac{p-2}{2}t,$$

which shows

$$y(t)=O(t^{-2/(p-2)}),\quad \text{as } t\to\infty,$$

which completes the proof of (1.4).

Step 4 Proof of Theorem 1.2 (2) We divide the proof into two steps.

(i) First of all, we define a set which consists of the functions that satisfy the following conditions:

$$J(u_0)<\frac{1}{N}S^{N/p}, \tag{2.22}$$

$$\int_{\Omega}|u_0|^{p^*}\,\mathrm{d}x=S^{N/p}. \tag{2.23}$$

We claim that the set is an empty set. In fact, let u_0 belong to the set. If u_0 satisfies

$$\int_\Omega |\nabla u_0|^p \,\mathrm{d}x \leq \int_\Omega |u_0|^{p^*} \,\mathrm{d}x,$$

then

$$S^{N/p} = \int_\Omega |u_0|^{p^*} \,\mathrm{d}x \geq \int_\Omega |\nabla u_0|^p \,\mathrm{d}x \geq S\left(\int_\Omega |u_0|^{p^*} \,\mathrm{d}x\right)^{p/p^*} = S^{N/p},$$

and hence

$$\int_\Omega |\nabla u_0|^p \,\mathrm{d}x = \int_\Omega |u_0|^{p^*} \,\mathrm{d}x = S^{N/p},$$

$$J(u_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p \,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u_0|^{p^*} \,\mathrm{d}x = \frac{1}{N}S^{N/p},$$

which is contradictory to (2.22).

If u_0 satisfies:

$$\int_\Omega |\nabla u_0|^p \,\mathrm{d}x > \int_\Omega |u_0|^{p^*} \,\mathrm{d}x,$$

then from (2.22) we see that

$$\frac{1}{N}S^{N/p} > J(u_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p \,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u_0|^{p^*} \,\mathrm{d}x > \frac{1}{N}\int_\Omega |u_0|^{p^*} \,\mathrm{d}x.$$

Implies

$$\int_\Omega |u_0|^{p^*} \,\mathrm{d}x < S^{N/p}$$

which is a contradiction because of (2.23).

(ii) Thus we consider only the case of u_0 satisfies

$$J(u) < \frac{1}{N}S^{N/p}, \quad \int_\Omega |u|^{p^*} \,\mathrm{d}x > S^{N/p}. \tag{2.24}$$

Obviously, in this case we have

$$S^{N/p} < \int_\Omega |\nabla u_0|^p \, \mathrm{d}x < \int_\Omega |u_0|^{p^*} \cdot \mathrm{d}x$$

If $u(x, t)$ is a global solution then we can deduce that $u(x, t)$ does not converge strongly to 0 in $W_0^{1,p}(\Omega)$. Otherwise, $\exists t^*, 0 < t^* < \infty$ such that

$$J(u(t^*)) < \frac{1}{N} S^{N/p}, \qquad \int_\Omega |u(t^*)|^{p^*} \, \mathrm{d}x = S^{N/p},$$

which is a contradiction from the first half (i). Now we prove the following claim:

CLAIM *If u_0 satisfies (2.24) and $u(x, t; u_0)$ is a global solution. Then $\forall t \in [0, T]$ the following inequalities hold:*

$$S^{N/p} < \int_\Omega |\nabla u(x,t)|^p \, \mathrm{d}x < \int_\Omega |u(x,t)|^{p^*} \, \mathrm{d}x. \tag{2.25}$$

Indeed, if there exists a t^* such that $\int_\Omega |\nabla u(x,t^*)|^p \, \mathrm{d}x = \int_\Omega |u(x,t^*)|^{p^*} \, \mathrm{d}x$, then we have $\int_\Omega |\nabla u(x,t^*)|^p \, \mathrm{d}x = \int_\Omega |u(x,t^*)|^{p^*} \, \mathrm{d}x \geq S^{N/p}$. But $(1/N)S^{N/p} > J(u(x,t^*)) = (1/N) \int_\Omega |\nabla u(x,t^*)|^p \, \mathrm{d}x$, with a contradiction. Therefore there exists a constant $\eta > 0$ sufficiently small and independent of t, rely on u_0 such that

$$\int_\Omega |u(x,t)|^{p^*} \, \mathrm{d}x \geq (1+\eta) \int_\Omega |\nabla u(x,t)|^p \, \mathrm{d}x, \tag{2.26}$$

for any $t \in [0, \infty]$, which completes the proof of the claim.

From the claim and $p > 2$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2 \, \mathrm{d}x &= \int_\Omega u^{p^*} \, \mathrm{d}x - \int_\Omega |\nabla u|^p \, \mathrm{d}x \\ &\geq \eta \int_\Omega |\nabla u|^p \, \mathrm{d}x \geq C\left(\int_\Omega |u|^2 \, \mathrm{d}x\right)^{p/2}, \end{aligned} \tag{2.27}$$

implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_2^2) \geq C\|u\|_2^p,$$

i.e.

$$\int_{\|u_0\|_2^2}^{\|u\|_2^2} y^{-(p/2)}\,\mathrm{d}y > Ct,$$

and therefore

$$CT \leq \int_{\|u_0\|_2^2}^{\infty} y^{-(p/2)}\,\mathrm{d}y < +\infty$$

which completes the proof of Theorem 1.2.

3 THE PROOF OF THEOREMS 1.3 AND 1.4

First of all, we prove Theorem 1.3.

Proof of Theorem 1.3 For any $t_n \to \infty$, let $u_n = u(x, t_n; u_0)$, from the boundedness we know that there exists a subsequence (still denote by $\{u_n\}$) and a function w such that

$$\begin{aligned} u_n &\rightharpoonup w && \text{in } W_0^{1,p}(\Omega), \\ u_n^{p^*-1} &\rightharpoonup w^{p^*-1} && \text{in } (L^{p^*}(\Omega))^*, \\ u_n &\to w && \text{a.e. in } \Omega. \end{aligned}$$

In order to pass to the limit in (1.1) we first fix some $T < \infty$ and introduce suitable test functions similar to Fila [4]. Take

$$\psi \in W_0^{1,p}(\Omega), \quad \rho \in C_0^2(0, T), \quad \rho \geq 0, \quad \int_0^T \rho(t)\,\mathrm{d}t = 1.$$

Put

$$\varphi(x, t) = \begin{cases} \rho(t - t_n)\psi(x) & \text{for } t > t_n,\ x \in \bar{\Omega}, \\ 0 & \text{for } 0 \leq t \leq t_n,\ x \in \bar{\Omega}. \end{cases}$$

Further, we obtain from Definition 1.1 that

$$\int_{t_n}^{t_n+T} \int_\Omega [u\rho'(t - t_n)\psi - \rho|\nabla u|^{p-2}\nabla u \nabla\psi + u^{p^*-1}\rho(t - t_n)\psi]\,\mathrm{d}x\,\mathrm{d}t = 0.$$

The transformation $s = t - t_n$, leads to

$$\int_0^T \int_\Omega [u(t_n+s)\rho'(s)\psi - \rho|\nabla u(t_n+s)|^{p-2}\nabla u(t_n+s)\nabla\psi + u(t_n+s)^{p^*-1}\rho(s)\psi]\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{3.1}$$

Note that the uniformly boundedness of $u(t_n+s)$ in $W_0^{1,p}(\Omega)$ for $0 \le s \le T$. Therefore, we can choose the same subsequence of $\{t_n\}$ (not relabeled) and functions w_s and w such that

$$u(t_n+s) \to w_s, \quad \text{strongly in } L^q(\Omega)(p \le q < p^*)$$

and

$$u(t_n) \to w, \quad \text{strongly in } L^q(p \le q < p^*).$$

Now we claim: $w_s = w$. Indeed, by the energy inequality we have

$$\int_\Omega |u(t_n+s) - u(t_n)|^2\,\mathrm{d}x = s\int_{t_n}^{t_n+s}\int_\Omega \left|\frac{\partial u}{\partial \tau}\right|^2 \mathrm{d}x\,\mathrm{d}\tau \to 0, \quad \text{as } t_n \to \infty$$

as $0 \le s \le T$ for any fixed $T < \infty$. Thus, we have

$$u(t_n+s) - u(t_n) \to 0, \quad \text{strongly in } L^2(\Omega), \quad \text{as } t_n \to \infty$$

for $0 \le s \le T$ for any fixed $T < \infty$. Hence

$$w_s = w$$

which prove the claim.

Now we rewrite (3.1) as follows:

$$\begin{aligned}
&\int_0^T \int_\Omega [u(t_n)\rho'(s)\psi - \rho|\nabla u(t_n)|^{p-2}\nabla u(t_n)\nabla\psi + u(t_n)^{p^*-1}\rho(s)\psi]\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega [u(t_n+s) - u(t_n)]\rho'(s)\psi\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^T \int_\Omega [|\nabla u(t_n+s)|^{p-2}\nabla u(t_n+s) - |\nabla u(t_n)|^{p-2}\nabla u(t_n)]\nabla\psi\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega [u(t_n+s)^{p^*-1} - u(t_n)^{p^*-1}]\rho(s)\psi\,\mathrm{d}x\,\mathrm{d}s = 0
\end{aligned} \tag{3.2}$$

By the dominated convergence theorem and the choice of ρ and $u(t_n) \to w$ strongly in $L^q(\Omega)(p \leq q < p^*)$, we have

$$\int_0^T \rho \left[\int_\Omega |\nabla u(t_n)|^{p-2} \nabla u(t_n) \nabla \psi \, \mathrm{d}x - \int_\Omega u(t_n)^{p^*-1} \psi \, \mathrm{d}x \right] \mathrm{d}s = o(1),$$

as $n \to \infty$

Denote $u(t_n)$ by u_n, From the choice of ρ, we obtain

$$\int_\Omega |\nabla u(t_n)|^{p-2} \nabla u(t_n) \nabla \psi \, \mathrm{d}x - \int_\Omega u(t_n)^{p^*-1} \psi \, \mathrm{d}x = o(1), \quad \text{as } n \to \infty.$$

Thus, we have

$$\int_\Omega |\nabla w|^{p-2} \nabla w \nabla \psi \, \mathrm{d}x = \int_\Omega w^{p^*-1} \psi \, \mathrm{d}x, \quad \text{for all } \psi \in W_0^{1,p}(\Omega)$$

which completes the proof of Theorem 1.3.

Proof of Theorem 1.4 From now on, denote $u(x, t; u_0)$ by u, we have

$$\int_0^\infty \int_\Omega u_t^2 \, \mathrm{d}x \, \mathrm{d}s \leq C < \infty.$$

Then there exists a sequence $\{t_n\}$ satisfying $t_n \to \infty$ as $n \to \infty$ such that

$$\int_\Omega |u_t(x, t_n; u_0)|^2 \, \mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

For the sake of convenience, denote $u(x, t_n; u_0)$ by u_n. From Theorem 1.1, $J(u(t)) > 0$ for all $t \geq 0$, and

$$0 < J(u(t)) \leq J(u_0). \tag{3.4}$$

If we consider the time sequence $\{t_n\}$ as

$$0 < J(u(t_n)) \leq J(u_0). \tag{3.5}$$

The statement of (3.3) and (3.5) says that $u_n = u(t_n)$, $t_n \to \infty$ is a Palais–Smale sequence related to the statement problem of (1.1). Such a situation has been well studied in the theory of nonlinear elliptic equations. It is easy to prove that there exists a constant $C < +\infty$ such that

$$\int_{\Omega} |\nabla u_n|^p \, dx \leq C.$$

Thus, there exists a subsequence (not relabeled) and a function w such that

$$\begin{aligned} u_n &\rightharpoonup w, \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n &\to w, \quad \text{strongly in } L^q(\Omega)(p \leq q < p^*). \end{aligned}$$

From the theory of elliptic equation we can obtain that w is a stationary solution, which completes the proof of Theorem 1.4.

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