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Positive Multiplication Preserves Dissipativity in Commutative C^*-Algebras

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We prove that multiplication by a positive element preserves dissipativity (accretivity) in the framework of commutative C^*-algebras. A simple counterexample shows that the result is not valid, in general, in commutative involutory Banach algebras.

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The notion of dissipativity (accretivity) of operators plays an important role in analysis and in numerical analysis, being strictly related to key properties like maximum principles for infinite-dimensional models, and stability of the relevant discretization methods. We recall that an operator $\varphi : (\Omega \subseteq X) \to X$, X being a (real or complex) Banach space, is termed *dissipative*, and $-\varphi$ accretive, if

$$\|u - v - \lambda(\varphi(u) - \varphi(v))\| \geq \|u - v\|, \quad \forall \lambda > 0, \ u, v \in \Omega, \qquad (1)$$

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and *strongly* dissipative if there exists $c > 0$ such that

$$\|u - v - \lambda(\varphi(u) - \varphi(v))\| \geq (1 + \lambda c)\|u - v\|, \quad \forall \lambda > 0, \;\; u, v \in \Omega; \;\; (2)$$

cf., e.g. [1, Chap. 3]. Clearly, if φ is dissipative, or strongly dissipative, then such is $a\varphi$ for every scalar $a \geq 0$, or $a > 0$ respectively.

In this note we face the naturally ensuing question in the framework of involutory Banach algebras: does dissipativity (accretivity) be inherited after *multiplication* by a *positive* element? An affirmative answer sounds familiar in specific contexts, like that of linear elliptic operators in spaces of continuous functions, or that of M-matrices in finite dimension. We show that dissipativity is indeed preserved in commutative C^*-algebras, while the result is not valid, in general, in commutative involutory Banach algebras.

Let $\mathcal{A}$ be a commutative C^*-algebra. Here *positive* means self-adjoint with a real nonnegative spectrum, and "strictly" positive refers to a positive spectrum, cf. [3, Chap. 11]; the cone of positive elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^+$.

THEOREM 0.1 *Let $\varphi : (\Omega \subseteq \mathcal{A}) \to \mathcal{A}$ be a dissipative* [*accretive*] *operator. Then, for every $a \in \mathcal{A}^+$, the operator $a\varphi(\cdot)$ is still dissipative* [*accretive*].

If φ is strongly dissipative [*accretive*] *with constant c, and a is strictly positive, then $a\varphi(\cdot)$ is strongly dissipative* [*accretive*] *itself, with constant* $c \min \sigma(a)$ (*where $\sigma(a)$ denotes the spectrum of a*).

Proof First, we prove the result in $C(\Delta, \|\cdot\|_\infty)$, the commutative C^*-algebra of complex-valued continuous functions on the compact Hausdorff space Δ, endowed with the sup-norm. Assume that φ be dissipative, and fix $u, v \in \Omega \subseteq C(\Delta)$: defining for every $\lambda \geq 0$ the continuous function

$$f_\lambda(x) := u(x) - v(x) - \lambda(\varphi(u)(x) - \varphi(v)(x)), \qquad (3)$$

in view of dissipativity of φ there exists a net $\{x_\lambda\}_{\lambda > 0}$ in Δ, such that $|f_\lambda(x_\lambda)| \geq \|u - v\|_\infty$. Here the (pre)order in the directed set $(0, +\infty)$ is given by: "$\lambda_1 \preceq \lambda_2$" if $\lambda_1 \geq \lambda_2$, i.e. convergence is intended as $\lambda \to 0^+$. Being Δ compact, we can extract from $\{x_\lambda\}$ a subnet $\{x_{\lambda_\mu}\}_{\mu \in M}$, convergent to a certain $\hat{x} \in \Delta$ (cf. [2, Chap. 4]). Now, for every $\lambda > 0$ we have

$$\begin{aligned} \|u - v\|_\infty &\le |f_\lambda(x_\lambda)| \\ &\le |u(x_\lambda) - v(x_\lambda)| + \lambda |\varphi(u)(x_\lambda) - \varphi(v)(x_\lambda)|, \end{aligned} \tag{4}$$

from which the estimate

$$0 \le \|u - v\|_\infty - |u(x_\lambda) - v(x_\lambda)| \le \lambda \|\varphi(u) - \varphi(v)\|_\infty \tag{5}$$

follows, and thus from continuity of $u - v$ we obtain

$$\lim_\mu |u(x_{\lambda_\mu}) - v(x_{\lambda_\mu})| = |u(\hat{x}) - v(\hat{x})| = |f_0(\hat{x})| = \|u - v\|_\infty. \tag{6}$$

We will prove below that $|f_\lambda(\hat{x})| \ge \|u - v\|_\infty$ for every $\lambda \ge 0$. Assume that there exists $\tilde{\lambda} > 0$ such that $|f_{\tilde{\lambda}}(\hat{x})| < \|u - v\|_\infty$. Being $f_{\tilde{\lambda}}$ continuous, for a suitable $\mu_1 \in M$ we have that

$$|f_{\tilde{\lambda}}(x_{\lambda_\mu})| < \|u - v\|_\infty, \quad \text{for all } \mu \succeq \mu_1, \tag{7}$$

where " $\preceq$ " is now the preorder in the directed set M, while at the same time the inequality $|f_{\lambda_\mu}(x_{\lambda_\mu})| \ge \|u - v\|_\infty$ holds for every $\mu \in M$. This leads to a contradiction, as $|f_\lambda(x_{\lambda_\mu})|$ is a *convex* function of λ in $[0, +\infty)$ for μ fixed, and thus, being

$$|f_0(x_{\lambda_\mu})| = |u(x_{\lambda_\mu}) - v(x_{\lambda_\mu})| \le \|u - v\|_\infty, \tag{8}$$

we get by (7)

$$|f_{\lambda_\mu}(x_{\lambda_\mu})| < \|u - v\|_\infty, \quad \text{for all } \mu \succeq \mu_1, \ \mu \succeq \mu_0, \tag{9}$$

where μ_0 is such that $\lambda_\mu \le \tilde{\lambda}$ for $\mu \succeq \mu_0$. It follows that

$$|u(\hat{x}) - v(\hat{x}) - \lambda(\varphi(u)(\hat{x}) - \varphi(v)(\hat{x}))| \ge \|u - v\|_\infty \quad \text{for every } \lambda \ge 0, \tag{10}$$

cf. also (6).

Finally, taking $\lambda = \xi a(\hat{x})$ in (10) for every fixed $a \in C^+(\Delta)$ and for every $\xi > 0$, we obtain

$$\begin{aligned} &\|u - v - \xi(a\varphi(u) - a\varphi(v))\|_\infty \\ &\quad \geq |u(\hat{x}) - v(\hat{x}) - \xi a(\hat{x})(\varphi(u)(\hat{x}) - \varphi(v)(\hat{x}))| \\ &\quad \geq \|u - v\|_\infty \quad \forall \xi > 0, \ u, v \in \Omega, \end{aligned} \tag{11}$$

i.e. the operator $a\varphi(\cdot)$ is dissipative on Ω.

Assume now that φ be strongly dissipative, with constant $c > 0$, cf. (2), and that $a(x) > 0$ in Δ, so that $0 < m := \min_{x \in \Delta} a(x) = \min \sigma(a)$. For every $u, v \in \Omega$, $\lambda > 0$, and $x \in \Delta$, we can write

$$\begin{aligned} &|u(x) - v(x) - \lambda a(x)(\varphi(u)(x) - \varphi(v)(x))| \\ &= \Big|(1 + \lambda c a(x))(u(x) - v(x)) \\ &\quad - \lambda a(x)\Big\{\varphi(u)(x) - \varphi(v)(x) + c(u(x) - v(x))\Big\}\Big| \\ &\geq (1 + \lambda c m)\Big|u(x) - v(x) - \lambda \frac{a(x)}{1 + \lambda c a(x)} \\ &\quad \times \Big\{\varphi(u)(x) - \varphi(v)(x) + c(u(x) - v(x))\Big\}\Big|. \end{aligned} \tag{12}$$

Observe that if φ is strongly dissipative with constant c, then $u \mapsto \varphi(u) + cu$ is dissipative. Taking the $\max_{x \in \Delta}$ on both sides of inequality (12), and using the first part of the theorem with the multiplier $a/(1 + \lambda c a) \in C^+(\Delta)$, we get

$$\begin{aligned} &\|u - v - \lambda a(\varphi(u) - \varphi(v))\|_\infty \\ &\quad \geq (1 + \lambda c m)\left\|u - v - \lambda \frac{a}{1 + \lambda c a}\Big\{\varphi(u) - \varphi(v) + c(u - v)\Big\}\right\|_\infty \\ &\quad \geq (1 + \lambda c m)\|u - v\|_\infty, \end{aligned} \tag{13}$$

i.e. $a\varphi(\cdot)$ is strongly dissipative with constant $c \min \sigma(a)$.

Extension to general $\mathcal{A}$ is immediate, recalling that every commutative C^*-algebra, in virtue of the celebrated Gelfand–Naimark theorem [3, Theorem 11.18], is isometrically $*$-isomorphic to $C(\Delta)$, Δ being the compact Hausdorff space of all complex homomorphisms of $\mathcal{A}$ endowed with the weak topology, and that spectra are invariant under such isomorphism. The statements concerning accretivity are trivially recovered by applying the previous results to $-\varphi$.

To conclude, we discuss a simple counterexample, which shows that the result above is not valid in general commutative and involutory Banach algebras. Consider $(\mathbb{C}^2, \|\cdot\|_2)$, where the product is the componentwise product, and the involution is the componentwise complex conjugation. It is immediate to check that this is a commutative involutory Banach algebra, but is not a C^*-algebra, as $\|uu^*\|_2 < \|u\|_2^2$ for all $u \in \mathbb{C}^2$ with no zero components. Here, denoting by $\langle\cdot,\cdot\rangle$ the euclidean scalar product in $\mathbb{C}^2$, dissipativity is equivalent to

$$\mathrm{Re}\langle\varphi(u) - \varphi(v), u - v\rangle \leq 0, \tag{14}$$

as in all Hilbert spaces (cf. [1, Chapter 3]); thus, all hermitian negative-semidefinite matrices are dissipative on the whole space. If we take

$$\varphi = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \ a_1, a_2 \geq 0, \tag{15}$$

which satisfy the assumptions of the first part of Theorem 0.1, then (14) applied to the operator $a\varphi(\cdot)$ would require that, *for all* non-negative a and *for all* $u = (u_1, u_2)^t \in \mathbb{C}^2$, the inequality $\mathrm{Re}((u_1 - u_2) \times (a_2\bar{u}_2 - a_1\bar{u}_1)) \leq 0$ be verified. This is manifestly false, even when $a_1 a_2 > 0$, i.e. a is strictly positive. A familiar interpretation of this counterexample is that the product of a positive diagonal matrix by a hermitian negative-semidefinite matrix is, in general, no more dissipative in the euclidean norm.

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