J. of Inequal. & Appl., 2001, Vol. 6, pp. 305–323
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Existence Theorems for Nonlinear Elliptic Problems

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(Received 12 September 1999; Revised 9 November 1999)

In this paper we prove two theorems for noncoercive elliptic boundary value problems using the critical point theory of Chang and the subdifferentiable of Clarke. The first result is for a Dirichlet noncoercive problem and the second one is for Neumann elliptic problem with nonlinear multivalued boundary conditions. We use the mountain-pass and the saddle-point theorems to obtain nontrivial solutions for these problems.

Keywords: Variational method; Critical points; Locally Lipschitz functional; p-Laplacian

1991 AMS Subject Classification: 35J15, 35J20

1 INTRODUCTION

In this paper, using the critical point theory of Chang [1] for locally Lipschitz functionals, we study nonlinear noncoercive elliptic boundary value problems with multivalued terms. Let $Z \subseteq R^N$ be a bounded domain with C^1-boundary Γ. The first problem under consideration is

$$\begin{aligned} &-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) + \partial j(z,x(z)) \ni f(z,x(z)) \quad \text{a.e. on } Z, \\ &x|_\Gamma = 0, \ 2 \le p < \infty, \end{aligned} \tag{1}$$

where ∂j denotes the subdifferential in the sense of Clarke of $j(z,\cdot)$.

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Problem (1) is a hemivariational inequality. Such inequalities arise in mechanics when one wants to consider more realistic nonmonotone, multivalued mechanical laws. The lack of monotonicity does not permit the use of the convex superpotential of Moreau. Concrete mechanical and engineering applications can be found in the book of Panagiotopoulos [11]. Problems similar to (1) were studied recently by Goeleven *et al.* [4] (semilinear inclusions, i.e. $p=2$) and Gasinski and Papageorgiou [5,6] (quasilinear inclusions).

The second problem is a Neumann elliptic boundary value problem with multivalued nonlinear boundary conditions. Let $Z \subseteq R^N$ be a bounded domain with a C^1-boundary Γ:

$$\begin{aligned} &-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = f(z, x(z)) \quad \text{a.e. on } Z, \\ &-\frac{\partial x}{\partial n_p} \in \partial j(z, \tau(x)(z)) \quad \text{a.e. on } \Gamma,\ 2 \le p < \infty. \end{aligned} \tag{2}$$

Here the boundary condition is in the sense of Kenmochi [8] and the operator τ is the trace operator in $W^{1,p}(Z)$. Our result here is closely related to the work of Halidias and Papageorgiou [7].

In the next section we recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the subdifferentiable of Clarke.

2 PRELIMINARIES

Let Y be a subset of X. A function $f\colon Y \to R$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, one has

$$|f(y) - f(x)| \le K\|y - x\|$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^0(x; v)$ is defined as follows:

$$f^0(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^0(x; v)$ is finite, postively homogeneous, subadditive and satisfies $|f^0(x;v)| \leq K\|v\|$. In addition f^0 satisfies $f^0(x;-v) = -f^0(x;v)$. Now we are ready to introduce the generalized gradient denoted by $\partial f(x)$ as follows:

$$\partial f(x) = \{w \in X^*\colon\ f^0(x;v) \geq \langle w, v\rangle \text{ for all } v \in X\}.$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leq K$ for every w in $\partial f(x)$.
(b) For every v in X, one has

$$f^0(x;v) = \max\{\langle w, v\rangle\colon\ w \in \partial f(x)\}.$$

If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Let us recall the (PS)-condition introduced by Chang.

DEFINITION *We say that Lipschitz function f satisfies the Palais–Smale condition if any sequence $\{x_n\}$ along which $|f(x_n)|$ is bounded and $\lambda(x_n) = \mathrm{Min}_{w \in \partial f(x_n)} \|w\|_{X^*} \to 0$ possesses a convergent subsequence.*

The (PS)-condition can also be formulated as follows (see Costa and Goncalves [3]):

$(PS)^*_{c,+}$ Whenever $(x_n) \subseteq X$, $(\varepsilon_n), (\delta_n) \subseteq R_+$ are sequences with $\varepsilon_n \to 0$, $\delta_n \to 0$, and such that

$$f(x_n) \to c$$
$$f(x_n) \leq f(x) + \varepsilon_n \|x - x_n\| \quad \text{if } \|x - x_n\| \leq \delta_n,$$

then (x_n) posseses a convergent subsequence: $x_{n'} \to \hat{x}$.

Similarly, we define the $(PS)^*_c$ condition from below, $(PS)^*_-$, by interchanging x and x_n in the above inequality. And finally we say that f satisfies $(PS)^*_c$ provided it satisfies $(PS)^*_{c,+}$ and $(PS)^*_{c,-}$.

Note that these two definitions are equivalent when f is locally Lipschitz functional.

Consider the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(Z))$. From Lindqvist [9] we know that $\lambda_1 > 0$ is isolated and simple, that is any two solutions u,v of

$$\left\{ \begin{array}{l} -\Delta_p u = -\mathrm{div}(\|Du\|^{p-2}Du) = \lambda_1 |u|^{p-2}u \text{ a.e. on } Z, \\ u\,|_\Gamma = 0, \ 2 \le p < \infty \end{array} \right\} \tag{3}$$

satisfy $u = cv$ for some $c \in R$. In addition, the λ_1-eigenfunctions do not change sign in Z. Finally we have the following variational characterization of λ_1 (Rayleigh quotient):

$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p}:\ x \in W_0^{1,p}(Z),\ x \neq 0\right].$$

Let us now recall the two basic theorems that we will use to prove the existence results.

The first is the saddle-point theorem.

THEOREM 1 *Let X be a reflexive Banach space f is a locally Lipschitz functional defined on X satisfies (PS)-condition. Suppose $X = X_1 \oplus X_2$, with a finite-dimensional X_1, and that there exist constants $b_1 < b_2$ and a neighborhood of* 0 *in X_1, such that*

$$f|_{X_2} \ge b_2, \quad f|_{\partial N} \le b_1;$$

then f has a critical point.

The second is the mountain-pass theorem.

THEOREM 2 *If a locally Lipschitz functional $f: X \to \mathrm{R}$ on the reflexive Banach space X satisfies the (PS)-condition and the hypotheses*

(i) *there exist positive constants ρ and a such that*

$$f(u) \ge a \quad \textit{for all } x \in X \textit{ with } \|x\| = \rho;$$

(ii) $f(0) = 0$ *and there exists a point $e \in X$ such that*

$$\|e\| > \rho \ \textit{and}\ f(e) \le 0,$$

then there exists a critical value $c \ge a$ of f determined by

$$c = \inf_{g \in G} \max_{t \in [0,1]} f(g(t)),$$

where

$$G = \{g \in C([0,1], X): g(0) = 0,\ g(1) = e\}.$$

One can find a proof for the generalized mountain pass theorem for locally Lipschitz functionals in the paper of Motreanu and Panagiotopoulos [10, Theorem 1 and Corollary 1].

In what follows we will use the well-known inequality

$$\sum_{j=1}^{N}(a_j(\eta) - a_j(\eta'))(\eta - \eta') \geq C|\eta - \eta'|^p.$$

for $\eta, \eta' \in R^N$, with $a_j(\eta) = |\eta|^{p-2}\eta_j$.

3 DIRICHLET PROBLEMS

In this section we prove an existence result for problem (1) using the mountain-pass theorem of Chang for locally Lipschitz functionals.

Let us state the hypothesis on the data, i.e. on f and β.

$H(f)_1$ $f: Z \times R \to R$ is a Carathéodory function such that

(i) for almost all $z \in Z$ and all $x \in R$, $|f(z,x)| \leq c_1|x|^{p-1} + c|x|^{p^*-1}$, with $p^* = Np/N - p$,
(ii) there exists $\theta > p$ and $r_0 > 0$ such that for almost all $z \in Z$ and all $|x| \geq r_0$, $0 < \theta F(z,x) \leq f(z,x)x$,
(iii) $\limsup_{x \to 0}(pF(z,x))/|x|^p \leq \theta(z) \leq \lambda_1$ for almost all $z \in Z$ with $\theta(z) \in L^\infty(Z)$ and $\theta(z) < \lambda_1$ in a set with positive measure.

Remark 1 It is easy to see that the function $f(z,x) = \theta(z)|x|^{p-2}x + |x|^{p^*-2}x$ with $\theta \in L^\infty$ and $\theta(z) < \lambda_1$ in a set with positive measure, satisfies the above hypotheses.

Remark 2 Note that from Hypothesis $H(f)$(ii) we have that $F(z,x) \geq c|x|^\theta$ for $|x| \geq r_0$. Indeed, we have that $\theta/x \leq f(z,x)/F(z,x)$. Integrating on $[r_0, x]$ we have $\theta[\ln|x| - \ln r_0] \leq \ln F(z,x) - \ln F(z,r_0)$ that is $F(z,x) \geq c|x|^\theta$ for $|x| \geq r_0$.

$H(j)_1$: $z \to j(z,x)$ is measurable and $j(z,\cdot)$ is a locally Lipschitz function, for almost all $z \in Z$, $j(z,0) = 0$, for almost all $z \in Z$, all $x \in R$, for all

$v \in \partial j(z,x)$ we have $vx \leq \theta j(z,x)$ and $|v| \leq a_1(z) + c|x|^{p^*-1}$ and finally we have $\limsup_{\xi \to \infty}(1/\xi^p) \int_Z j(z,\xi)\,\mathrm{d}z < \infty$.

THEOREM 3 *If hypotheses* $H(f)_1, H(j)_1$ *hold, then problem* (1) *has a solution* $x \in W^{1,p}(Z)$.

Proof Let $\Phi, \psi : W^{1,p}(Z) \to R$ be defined as

$$\Phi(x) = -\int_Z \int_0^{x(z)} f(z,r)\,\mathrm{d}r\,\mathrm{d}z = -\int_Z F(z,x(z))\,\mathrm{d}z$$
$$\text{with } F(z,x) = \int_0^x f(z,r)\,\mathrm{d}r$$

and

$$\psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_Z j(z,x(z))\,\mathrm{d}z.$$

Then we set the energy functional $R = \Phi + \psi$.

CLAIM 1 $R(\cdot)$ *satisfies the* (PS)*-condition in the sense of Costa and Goncalves.*

Indeed, let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ such that $R(x_n) \to c$ and

$$R(x_n) \leq R(x) + \varepsilon_n\|x - x_n\| \quad \text{with } \|x - x_n\| \leq \delta_n$$

with $\varepsilon_n, \delta_n \to 0$.

Let $x = x_n + \delta x_n$ with $\delta\|x_n\| \leq \delta_n$. Divide with δ and in the limit when $\delta \to 0$ we have that

$$\frac{\Phi(x_n) - \Phi(x_n + \delta x_n)}{\delta} \overset{\delta \to 0}{\to} -\Phi'(x_n; x_n)$$

with $\Phi'(x_n;x_n) = -\int_Z f(z,x_n(z))x_n(z)\,\mathrm{d}z$. Also we have, $\|Dx_n\|_p^p - \|Dx_n + \delta Dx_n\|_p^p = 1/p\|Dx_n\|_p^p(1-(1+\delta)^p)$. Now divide this with δ, then in the limit we have that is equal to $-\|Dx_n\|_p^p$. Let $V_1(x) = \int_Z j(z,x(z))\,\mathrm{d}z$.

Then from above it follows that

$$-\int_Z f(z, x_n(z))x_n(z)\,\mathrm{d}z + \|Dx_n\|_p^p + V_1^0(z, x_n(z); x_n(z))\,\mathrm{d}z \geq -\varepsilon\|x_n\|$$

(see Clarke [2, p. 25]). From Proposition 2.1.2 of Clarke [2] we have that there exists $u_n \in \partial V_1(x_n)$ such that $V_1^0(z, x_n(z); x_n(z)) = \int_Z u_n(z)x_n(z)\,\mathrm{d}z$. Thus, we have

$$\int_Z f(z, x_n(z))x_n(z)\,\mathrm{d}z - \|Dx_n\|_p^p - \int_Z u_n(z)x_n(z)\,\mathrm{d}z \leq \varepsilon\|x_n\|. \tag{5}$$

From the choice of the sequence $\{x_n\} \subseteq W_0^{1,p}(Z)$, we have that

$$\theta R(x_n) \leq M_1 \quad \text{for some } M_1 > 0. \tag{6}$$

Adding (5) and (6) we have

$$\left(\frac{\theta}{p} - 1\right)\|Dx_n\|_p^p + \int_Z (f(z, x_n(z))x_n(z) - \theta F(z, x_n(z))\,\mathrm{d}z)$$
$$+ \int_Z (\theta j(z, x_n(z)) - u_n(z)x_n(z)\,\mathrm{d}z) \leq \|Dx_n\|_p + M_2$$

for some $M_2 > 0$. Since $u_n \in \partial V_1(x_n)$, we have that $u_n(z) \in \partial j(z, x_n(z)) = \beta(z, x_n(z))$ a.e. on Z. Then using the hypotheses $H(f)_1$(ii) and $H(j)_1$, we have

$$\int_Z (f(z, x_n(z))x_n(z) - \theta F(z, x_n(z)))\,\mathrm{d}z \geq 0$$

and

$$\int_Z (\theta j(z, x_n(z)) - u_n(z)x_n(z))\,\mathrm{d}z \geq 0.$$

So, we can say that

$$\left(\frac{\theta}{p} - 1\right)\|Dx_n\|_p^p \leq \|Dx_n\|_p + M_2.$$

Since $\theta > p$ from the last inequality we have that $\{Dx_n\} \subseteq L^p(T, R^N)$ is bounded, thus $\{x_n\} \subseteq W_0^{1,p}(Z)$ is bounded (Poincare inequality).

From the properties of the subdifferential of Clarke [2, p. 83], we have

$$\partial R(x_n) \subseteq \partial\Phi(x_n) + \partial\psi(x_n)$$
$$\subseteq \partial\Phi(x_n) + \partial\left(\frac{1}{p}\|Dx_n\|_p^p\right) + \int_\Gamma \partial j(z, \tau(x_n(z)))\,\mathrm{d}\sigma.$$

So, we have

$$\langle w_n, y\rangle = \langle Ax_n, y\rangle + \langle r_n, y\rangle - \int_Z f(z, x_n(z))y(z)\,\mathrm{d}z$$

with $r_n(z) \in \partial j(z, x_n(z))$ and w_n the element with minimal norm of the subdifferential of R and $A: W^{1,p}(Z) \to W^{1,p}(Z)^*$ such that $\langle Ax, y\rangle = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), Dy(z))R^N\,\mathrm{d}z)$ for all $y \in W^{1,p}(Z)$. But $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W^{1,p}(Z) \subseteq L^p(Z)$. Thus, r_n is bounded in $L^q(Z)$ (see Chang [1, p. 104, Proposition 2]), i.e $r_n \to^w r$ in $L^q(Z)$. In addition we have that $\int_Z f(z, x_n(z))y(z)\mathrm{d}z \to \int_Z f(z, x(z))y(z)\,\mathrm{d}z$. Choose $y = x_n - x$. Then in the limit we have that $\limsup\langle Ax_n, x_n - x\rangle = 0$. By virtue of the inequality (4) we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W^{1,p}(Z)$. The claim is proved.

Now we shall show that there exists $\rho > 0$ such that $R(x) \geq \eta > 0$ with $\|x\| = \rho$. In fact we will show that for every sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ with $\|x_n\| = \rho_n \to 0$ we have $R(x_n) \downarrow 0$. Suppose that this is wrong. Then there exists a sequence as above such that $R(x_n) \leq 0$. Since $\|x_n\| \to 0$ we have $x_n(z) \to 0$ a.e. on Z. So, since $j(z, \cdot) \geq 0$, we have

$$\frac{1}{p}\|Dx_n\|_p^p \leq \int_Z F(z, x_n(z))\,\mathrm{d}z.$$

Dividing the last inequality with $\|x_n\|_p^p$ and using the variational characterization of the first eigenvalue, we have

$$\frac{\lambda_1}{p} \leq \int_Z \frac{pF(z, x_n(z))}{|x_n(z)|^p}\frac{|x_n(z)|^p}{p\|x_n\|_p^p}\,\mathrm{d}z.$$

In the limit and using Fatou's lemma we have that

$$\frac{1}{p} \leq \int_A \limsup_{n\to\infty} \frac{\theta(z)}{\lambda_1} \frac{|x_n(z)|^p}{p\|x_n\|_p^p}\,dz + \int_{Z\setminus A} \limsup_{n\to\infty} \frac{|x_n(z)|^p}{p\|x_n\|_p^p}\,dz$$

with $A \subseteq Z$ such that $\theta(z) < \lambda_1$ on A and $|A| > 0$. In the last inequality we have used the hypothesis $H(f)_1$(iii). Thus we have that $1/p < 1/p$, a contradiction. So, there exists $\rho > 0$ such that $R(x) \geq \eta > 0$ for all $x \in W_0^{1,p}(Z)$ with $\|x\| = \rho$.

Also, from the hypothesis $H(f)_1$(ii), for almost all $x \in Z$ and all $x \in R$ we have

$$F(z,x) \geq c|x|^\theta - c_1, \quad \text{for some } c, c_1 > 0 \tag{7}$$

(see Remark 2). Then for all $\xi > 0$, we have

$$\begin{aligned} R(\xi u_1) &= \frac{\xi^p}{p}\|Du_1\|_p^p + \int_Z j(z,\xi u_1(z))\,dz - \int_Z F(z,\xi u_1(z))\,dz \\ &\leq \frac{\xi^p}{p}\|Du_1\|_p^p + \int_Z j(z,\xi u_1(z))\,dz - c_2\xi^\theta\|u_1\|_\theta^\theta, \\ &\quad \text{for some } c_2 > 0 \\ &\leq \xi^p(c_1 - c_2\xi^{\theta-p}) + \int_Z j(z,\xi u_1(z))\,dz \end{aligned}$$

By virtue of hypothesis $H(j)_1$, for $\xi > p$ big enough we have that $R(\xi u_1) \leq 0$. So we can apply theorem 1 and have that $R(\cdot)$ has a critical point $x \in W_0^{1,p}(Z)$. So $0 \in \partial(\psi(x) + \Phi(x))$. Let $\psi_1(x) = \|Dx\|^p/p$ and $\psi_2(x) = \int_Z j(z,\tau(x)(z))\,dz$. Then let $\hat{\psi}_1 : L^p(Z) \to R$ the extension of ψ_1 in $L^p(Z)$. Then $\partial\psi_1(x) \subseteq \partial\hat{\psi}_1(x)$ (see Chang [1]). It is easy to prove that the nonlinear operator $\hat{A} : D(A) \subseteq L^p(Z) \to L^q(Z)$ such that

$$\langle \hat{A}x, y\rangle = \int_Z \|Dx(Z)\|^{p-2}(Dx(z), Dy(z))\,dz \quad \text{for all } y \in W^{1,p}(Z)$$

with $D(A) = \{x \in W^{1,p}(Z) : \ \hat{A}x \in L^q(Z)\}$, satisfies $\hat{A} = \partial\hat{\psi}_1$. Indeed, first we show that $\hat{A} \subseteq \partial\hat{\psi}$ and then it suffices to show that $\hat{A}$ is maximal monotone:

$$\begin{aligned}
&\langle \hat{A}x, y - x\rangle \\
&= \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z) - Dx(z))_{R^N}\,dz \\
&= \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N}\,dz - \int_Z \|Dx(z)\|^p\,dz \\
&\le \int_Z \left(\frac{\|Dx(z)\|^{q(p-2)}\|Dx(z)\|^q}{q} + \frac{\|Dy(z)\|^p}{p}\right) dz - \|Dx\|_p^p \\
&= \frac{\|Dx\|_p^p}{q} - \|Dx\|^p + \frac{\|Dy\|_p^p}{p} \\
&= \hat{\psi}_1(y) - \hat{\psi}_1(x)
\end{aligned}$$

The monotonicity part is obvious using inequality (4). The maximality needs more work. Let $J: L^p(Z) \to L^q(Z)$ be defined as $J(x) = |x(\cdot)|^{p-2}x(\cdot)$. We will show that $R(\hat{A} + J) = L^q(Z)$. Assume for the moment that this holds. Then let $v \in L^p(Z)$, $v^* \in L^q(Z)$ such that

$$(\hat{A}(x) - v^*, x - v)_{pq} \ge 0$$

for all $x \in D(\hat{A})$. Therefore there exists $x \in D(\hat{A})$ such that $\hat{A}(x) + J(x) = v^* + J(v)$ (recall that we assumed that $R(\hat{A} + J) = L^q(Z)$). Using this in the above inequality we have that

$$(J(v) - J(x), x - v)_{pq} \ge 0.$$

But J is strongly monotone. Thus we have that $v = x$ and $\hat{A}(x) = v^*$. Therefore $\hat{A}$ is maximal monotone. It remains to show that $R(\hat{A} + J) = L^q(Z)$. But $\hat{J} = J|_{W^{1,p}(Z)} : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is maximal monotone, because is demicontinuous and monotone. So $A + \hat{J}$ is maximal monotone. But it is easy to see that the sum is coercive. So is surjective. Therefore, $R(A + \hat{J}) = W^{1,p}(Z)^*$. Then for every $g \in L^q(Z)$, we can find $x \in W^{1,p}(Z)$ such that $A + \hat{J}(x) = g \Rightarrow A(x) = g - \hat{J}(x) \in L^q(Z) \Rightarrow A(x) = \hat{A}(x)$. Thus, $R(\hat{A} + J) = L^q(Z)$.

So, we can say that

$$\int_Z f(z, x(z))y(z) = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))\,dz + \int_Z v(z)y(z)\,dz \tag{8}$$

with $v(z) \in \partial j(z, x(z))$, for every $y \in W^{1,p}(Z)$. Let $y = \phi \in C_0^\infty(Z)$. Then we have

$$\int_Z f(z, x(z))\phi(z)\,\mathrm{d}z = \int_Z \|Dx(z)\|^{p-2}(Dx(z), D\phi(z))\,\mathrm{d}z + \int_Z v(z)\phi(z)\,\mathrm{d}z.$$

But $\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in W^{-1,q}(Z)$ then we have that $\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in L^q(Z)$ because $f(z, x(z)) \in L^q(Z)$ and $v(z) \in L^q(Z)$. So $x \in W^{1,p}(Z)$ solves (1).

Remark Gasinski and Papageorgiou [6] have an existence result when the nonresonance hypothesis at zero $H(f)$(iii) is to the right of λ_1.

4 NEUMANN PROBLEMS

In this section we consider a quasilinear Neumann problem with multivalued boundary condition. More precisely, we study the following problem:

$$\left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = f(z, x(z)) - h(z) \text{ a.e. on } Z \\ -\dfrac{\partial x}{\partial n_p}(z) \in \beta(z, \tau(x)(z)) \text{ a.e. on } \Gamma,\ 2 \leq p < \infty \end{array} \right\} \tag{9}$$

Here $\partial x/\partial n_p(z) = (\|Dx(z)\|^{p-2}Dx(z), n(z))_{R^N}$ with $n(z)$ denoting the outward normal at $z \in \Gamma$ and τ is the trace operator on $W^{1,p}(Z)$. On Γ we consider the $(N-1)$-dimensional Hausdorff measure.

Our hypotheses on $f(z, x)$ and $\beta(z, x)$ are the following:

$H(f)_2$ $f: Z \times R \to R$ is a Carathéodory function such that

(i) for almost all $z \in Z$ and all $x \in R$, $|f(z, x)| \leq \alpha(z) + c|x|^{\theta-1}$ with $\alpha \in L^\infty(Z)$, $c > 0$, $1 \leq \theta < p$;

(ii) Uniformly for almost all $z \in Z$ we have that $f(z,x)/(|x|^{\theta-2}x) \to f_+(z)$ as $|x| \to +\infty$ where $f_+ \in L^1 Z$, $f_+ \geq 0$ with strict inequality on a set of positive Lebesgue measure.

$H(\beta)_2 : \beta(z, x) = \partial j(z, x)$ where $z \to j(z, x)$ is measurable and $j(z, \cdot)$ is a locally Lipschitz function such that for almost all $z \in Z$ and all $x \in R$,

$|\beta(z,x)| = \sup[|u|: u \in \beta(z,x)] \leq \alpha_1(z) + c_1|x|^{\mu}$, $0 \leq \mu < \theta - 1$ (θ the same as $H(f)_2$(i)) with $\alpha_1 \in L^{\infty}$, $c_1 > 0$ and $j(\cdot\,,0) \in L^{\infty}(Z)$ and finally $j(z,\cdot) \geq 0$ for almost all $z \in Z$.

Remark In Halidias and Papageorgiou [7], $j(z,\cdot)$ was assumed also to be convex.

THEOREM 4 *If hypotheses $H(f)_2$ and $H(\beta)_2$ hold, then problem* (9) *has a nontrivial solution.*

Proof Let $\Phi : W^{1,p}(Z) \to R$ and $\psi : W^{1,p}(Z) \to R_+$ be defined by

$$\Phi(x) = -\int_Z F(z,x(z))\,\mathrm{d}z$$

and

$$\psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_{\Gamma} j(z,\tau(x)(z))\,\mathrm{d}\sigma.$$

In the definition of $\Phi(\cdot)$, $F(z,x) = \int_0^x f(z,r)\,\mathrm{d}r$ (the potential of f), $\tau(\cdot)$ is the trace operator on $W^{1,p}(Z)$ and $\mathrm{d}\sigma$ is the $(N-1)$-dimensional Hausdorff measure. Clearly $\Phi \in C^1(W^{1,p}(Z))$, so is locally Lipschitz, while we can check that ψ is locally Lipschitz too. Set $R = \Phi + \psi$.

CLAIM 1 *$R(\cdot)$ satisfies the (PS)-condition (in the sense of Costa and Goncalves).*

Let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ such that $R(x_n) \to c$ when $n \to \infty$ and

$$R(x_n) \leq R(x) + \varepsilon_n\|x - x_n\| \quad \text{with } \|x - x_n\| \leq \delta_n$$

with $\varepsilon_n, \delta_n \to 0$. Choose $x = x_n - \delta x_n$ with $\delta\|x_n\| \leq \delta_n$. Divide with δ and let $n \to \infty$. Note that $\Phi \in C^1(W^{1,p}(Z))$, so we have

$$\frac{\Phi(x_n) - \Phi(x_n - \delta x_n)}{\delta_n} \overset{\delta \to 0}{\longrightarrow} \Phi'(x_n; x_n)$$

with $\Phi'(x_n;x_n) = -\int_Z f(z,x_n(z))x_n(z)\,dz$. Also, $\|Dx_n\|_p^p - \|Dx_n - \delta Dx_n\| = 1/p\|Dx_n\|_p^p(1-(1-\delta)^p)$. So if we divide this with δ and let $n\to\infty$ we have that is equal with $\|Dx_n\|_p^p$. Finally, there exists $w_n \in \partial\eta(x_n)$, where $\eta(x) = \int_\Gamma j(z,\tau(x)(z))\,d\sigma$ such that $\eta^0(x_n;x_n) = \int_\Gamma w_n(z)x_n(z)\,d\sigma$. Note that $w_n(z)\in\partial j(z,\tau(x_n)(z))$ a.e. on Z. So, it follows that

$$\int_Z f(z,x_n(z))x_n(z)\,dz - \|Dx_n\|_p^p - \int_\Gamma w_n(z)\tau(x_n)(z)\,d\sigma \le -\varepsilon_n\|x_n\|.$$

Suppose that $\{x_n\}\subseteq W^{1,p}(Z)$ was unbounded. Then (at least for a subsequence), we may assume that $\|x_n\|\to\infty$. Let $y = x_n/\|x_n\|$, $n\ge 1$. By passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(Z),\quad y_n\to y \text{ in } L^p(Z),$$
$$y_n(z)\to y(z) \text{ a.e. on } Z \text{ as } n\to\infty$$

and $|y_n(z)|\le k(z)$ a.e. on Z with $k\in L^p(Z)$.

Recall that from the choice of the sequence $\{x_n\}$ we have $|R(x_n)|\le M_1$ for some $M_1>0$ and all $n\ge 1$,

$$\Rightarrow \frac{1}{p}\|Dx_n\|_p^p + \int_\Gamma j(z,\tau(x_n)(z))\,d\sigma - \int_Z F(z,x_n(z))\,dz \le M_1$$
$$\Rightarrow \frac{1}{p}\|Dx_n\|_p^p - \int_Z F(z,x_n(z))\,dz \le M_1 \quad (\text{since } j\ge 0).$$

Divide by $\|x_n\|^p$. We obtain

$$\frac{1}{p}\|Dy_n\|_p^p - \int_Z \frac{F(z,x_n(z))}{\|x_n\|^p}\,dz \le \frac{M_1}{\|x_n\|^p}. \tag{10}$$

We have

$$\left|\int_Z \frac{F(z,x_n(z))}{\|x_n\|^p}\,dz\right| \le \frac{1}{\|x_n\|^p}\int_Z\int_0^{|x_n(z)|}|f(z,r)|\,dr\,dz$$
$$\le \frac{1}{\|x_n\|^p}\left(\|\alpha\|_\infty\|x_n\| + \frac{c}{\theta}\|x_n\|^\theta\right)\to 0$$
$$\text{as } n\to\infty.$$

So by passing to the limit as $n \to \infty$ in (10), we obtain

$$\lim \frac{1}{p} \|Dy_n\|_p^p = 0$$
$$\Rightarrow \|Dy\|_p = 0 \quad (\text{recall that } Dy_n \xrightarrow{w} Dy \text{ in } L^p(Z, R^N) \text{ as } n \to \infty)$$
$$\Rightarrow y = \xi \in R$$

Note that $y_n \to \xi$ in $W_0^{1,p}(Z)$ and since $\|y_n\| = 1$, $n \geq 1$ we infer that $\xi \neq 0$. We deduce that $|x_n(z)| \to +\infty$ a.e. on Z as $n \to \infty$.

From the choice of the sequence $\{x_n\} \subseteq W^{1,p}(Z)$, we have

$$\int_Z f(z, x_n(z))x_n(z)\,\mathrm{d}z - \|Dx_n\|_p^p - \int_Z w_n(z)\tau(x_n)(z)\,\mathrm{d}z \geq -\varepsilon_n \|x_n\| \tag{11}$$

and

$$\|Dx_n\|_p^p + p\int_\Gamma j(z, \tau(x)(z)\,\mathrm{d}\sigma - p\int_Z F(z, x_n(z))\,\mathrm{d}z \geq -pM_1 \tag{12}$$

Adding (11) and (12), we obtain

$$\int_\Gamma (pj(z, \tau(x_n)(z)) - w_n(z)\tau(x_n)(z)\,\mathrm{d}\sigma$$
$$+ \int_Z (f(z, x_n(z))x_n(z) - pF(z, x_n(z)))\,\mathrm{d}z \geq -pM_1 - \varepsilon_n \|x_n\|.$$

Divide this inequality by $\|x_n\|^\theta$. We have

$$\int_Z \frac{f(z, x_n(z))}{\|x_n\|^{\theta-1}} y_n(z)\,\mathrm{d}z - \int_Z \frac{pF(z, x_n(z))}{\|x_n\|^\theta}\,\mathrm{d}z$$
$$+ \int_\Gamma \frac{pj(z, \tau(x_n)(z) - w_n(z)\tau(x_n)(z)}{\|x_n\|^\theta}\,\mathrm{d}\sigma$$
$$\geq -\frac{1}{\|x_n\|^\theta} pM_1 - \frac{\varepsilon_n}{\|x_n\|^{\theta-1}}. \tag{13}$$

Note that

$$\int_Z \frac{f(z,x_n(z))}{\|x_n\|^{\theta-1}} y_n(z)\,dz = \int_Z \frac{f(z,x_n(z))}{|x_n(z)|^{\theta-2}x_n(z)} |y_n(z)|^\theta\,dz \to |\xi|^\theta \int_Z f_+(z)\,dz \text{ as } n\to\infty.$$

Also by virtue of Hypothesis $H(f)_2$(ii), given $z\in Z\backslash N$, $|N|=0$ ($|C|$ denotes the Lebesgue measure of a measurable set $C\subseteq Z$) and $\varepsilon>0$, we can find $M_\varepsilon>0$ such that for all $|r|\geq M_\varepsilon$ we have $|f_+(z)-f(z,r)/|r|^{\theta-2}r|\leq\varepsilon$. Then, if $x_n(z)\to+\infty$, we have

$$\begin{aligned}\frac{1}{|x_n(z)|^\theta}F(z,x_n(z))\,dz &\geq \frac{1}{|x_n(z)|^\theta}F(z,M_\varepsilon)\,dz \\ &\quad+\frac{1}{|x_n(z)|^\theta}\int_{M_\varepsilon}^{x_n(z)}(f_+(z)|r|^{\theta-2}r-\varepsilon|r|^{\theta-2}r)\,dr \\ &= \frac{1}{|x_n(z)|^\theta}\eta(z)+\frac{|x_n(z)|^\theta-M_\varepsilon^\theta}{\theta|x_n(z)|^\theta}(f_+(z)-\varepsilon) \\ &\quad\text{for some } \eta\in L^1(Z)\end{aligned}$$

$$\Rightarrow \liminf_{n\to\infty}\frac{F(z,x_n(z))}{|x_n(z)|^\theta}\geq\frac{1}{\theta}(f_+(z)-\varepsilon). \tag{14}$$

Similarly we obtain that

$$\limsup_{n\to\infty}\frac{F(z,x_n(z))}{|x_n(z)|^\theta}\leq\frac{1}{\theta}(f_+(z)+\varepsilon). \tag{15}$$

From (14) and (15) and since $\varepsilon>0$ and $z\in Z\backslash N$ were arbitrary, we infer that

$$\begin{aligned}&\frac{F(z,x_n(z))}{|x_n(z)|^\theta}\to\frac{1}{\theta}f_+(z) \quad \text{a.e. on } Z \text{ as } n\to\infty \\ &\Rightarrow \int_Z\frac{F(z,x_n(z))}{\|x_n\|^\theta}\,dz=\int_Z\frac{F(z,x_n(z))}{|x_n(z)|^\theta}\frac{|x_n(z)|^\theta}{\|x_n\|^\theta}\,dz \\ &\qquad=\int_Z\frac{F(z,x_n(z))}{|x_n(z)|^\theta}|y_n(z)|^\theta\,dz \qquad (16) \\ &\qquad\to\xi^\theta\int_Z\frac{1}{\theta}f_+(z) \text{ as } n\to\infty.\end{aligned}$$

Note that since for almost all $z \in Z j(z, \cdot)$ is locally Lipschitz. So by Lebourg's mean value theorem, for almost all $z \in Z$ and all $x \in R$, we can find $w \in \beta(z, \eta x)$ $0 < \eta < 1$ such that

$$\begin{aligned}
&|j(z,x) - j(z,0)| = wx \\
&\quad \Rightarrow |j(z,x)| \leq |j(z,\cdot)| + |w||x| \leq \beta + |w||x| \quad (\text{since } j(\cdot,\cdot) \in L^\infty(Z)).
\end{aligned}$$

But by $H(\beta)_2$ we have

$$\begin{aligned}
&|w| \leq a_1(z) + c_1|x|^\mu \\
&\quad \Rightarrow |j(z,x)| \leq a_2 + c_2|x|^{\mu+1} \quad \text{for some } a_2, c_2 > 0.
\end{aligned}$$

So it is easy to see that

$$\int_\Gamma \frac{pj(z,\tau(x_n)(z)) - w_n(z)\tau(x_n)(z)}{\|x_n\|^\theta}\, d\sigma \to 0$$
$$\text{as } n \to \infty \ (\text{recall } \mu + 1 < \theta).$$

Thus by passing to the limit in (13), we obtain

$$\left(1 - \frac{p}{\theta}\right)\xi^\theta \int_Z f_+(z) \geq 0$$

a contradiction to Hypothesis $H(f)_2$(ii) (recall $p > \theta$). If $x_n(z) \to -\infty$, with similar arguments as above we show that

$$\int_Z \frac{F(z, x_n(z))}{\|x_n\|^\theta}\, dz \to \xi^\theta \int_Z \frac{1}{\theta} f_+(z) \quad \text{as } n \to \infty$$

(note that $\int_0^{x_n(z)} f(z,r)\, dr = -\int_{x_n(z)}^0 f(z,r)\, dr$). Therefore it follows that $\{x_n\} \subseteq W^{1,p}(Z)$ is bounded. Hence we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$, $x_n \to x$ in $L^p(Z)$, $x_n(z) \to x(z)$ a.e. on Z as $n \to \infty$ and $|x_n(z)| \leq k(z)$ a.e. on Z with $k \in L^p(Z)$.

From the properties of the subdifferential of Clarke [2, p. 83], we have

$$\begin{aligned}
\partial R(x_n) &\subseteq \partial\Phi(x_n) + \partial\psi(x_n) \\
&\subseteq \partial\Phi(x_n) + \partial\left(\frac{1}{p}\|Dx_n\|_p^p\right) + \int_\Gamma \partial j(z, \tau(x_n(z)))\, d\sigma.
\end{aligned}$$

So we have

$$\langle w_n, y\rangle = \langle Ax_n, y\rangle + \langle r_n, y\rangle - \int_Z f(z, x_n(z))y(z)\,\mathrm{d}z$$

with $r_n(z) \in \partial j(z, x_n(z))$ and w_n the element with minimal norm of the subdifferential of R and $A: W^{1,p}(Z) \to W^{1,p}(Z)^*$ such that $\langle Ax, y\rangle = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), Dy(z)))_{R^N}\,\mathrm{d}z$. But $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W^{1,p}(Z) \subseteq L^p(Z)$. Thus, r_n is bounded in $L^q(Z)$ (see Chang [1, p. 104, Proposition 2]), i.e $r_n \xrightarrow{w} r$ in $L^q(Z)$. In addition we have that $\int_Z f(z, x_n(z))y(z)\,\mathrm{d}z \to \int_Z f(z, x(z))y(z)\,\mathrm{d}z$. Choose $y = x_n - x$. Then in the limit we have that $\limsup\langle Ax_n, x_n - x\rangle = 0$. By virtue of the inequality (4) we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W^{1,p}(Z)$. The claim is proved.

Now let $W^{1,p}(Z) = X_1 \oplus X_2$ with $X_1 = R$ and $X_2 = \{y \in W^{1,p}(Z): \int_Z y(z)\,\mathrm{d}z = 0\}$. For every $\xi \in X_1$ we have

$$\begin{aligned} R(\xi) &= \Phi(\xi) + \psi(\xi) = \int_Z j(z,\xi)\,\mathrm{d}\sigma - \int_Z F(z,\xi)\,\mathrm{d}z \\ &\le \|\alpha_1\|_\infty |\xi||\Gamma| + \frac{c_1}{\mu}|\xi|^\mu|\Gamma| - \int_Z F(z,\xi)\,\mathrm{d}z \\ &\quad \text{(see hypothesis } H(\beta)_2) \\ &\Rightarrow \frac{1}{|\xi|^\mu}R(\xi) \le \frac{1}{|\xi|^{\mu-1}}\|\alpha_1\|_\infty|\Gamma| + \frac{c}{\mu}|\Gamma| - \frac{1}{|\xi|^\mu}\int_Z F(z,\xi)\,\mathrm{d}z. \end{aligned}$$

By virtue of Hypothesis $H(f)_2$(ii) we conclude that $R(\xi) \to -\infty$ as $|\xi| \to \infty$. On the other hand for $y \in X_2$, we have

$$\begin{aligned} R(y) &\ge \frac{1}{p}\|Dy\|_p^p - \int_Z F(z, y(z))\,\mathrm{d}z \quad (\text{since } j \ge 0) \\ &\ge \frac{1}{p}\|Dy\|_p^p - c_2\|y\|_p - c_3\|y\|_p^\theta \quad \text{for some } c_2, c_3 > 0 \\ &\quad (\text{since } \theta < p, \text{ see } H(f)_2(i)). \end{aligned}$$

From the Poincare–Wirtinger inequality we know that $\|Dy\|_p$ is an equivalent norm on X_2. So we have

$$R(y) \ge \frac{1}{p}\|Dy\|_p^p - c_4\|Dy\|_p - c_5\|Dy\|_p^\theta \quad \text{for some } c_4, c_5 > 0,$$

$\Rightarrow R(\cdot)$ is coercive on X_2 (recall $\theta < p$), hence bounded below on X_2.

So by Theorem 1 we have that there exists $x \in W^{1,p}(Z)$ such that $0 \in \partial R(x)$. That is $0 \in \partial \Phi(x) + \partial \psi(x)$. Let $\psi_1(x) = \|Dx\|^p/p$ and $\psi_2(x) = \int_\Gamma j(z, \tau(x)(z))\,d\sigma$. Then let $\hat{\psi}_1 : L^p(Z) \to R$ the extension of ψ_1 in $L^p(Z)$. Then $\partial\psi_1(x) \subseteq \partial\hat{\psi}_1(x)$ (see Chang [1]). Then as before we prove that the nonlinear operator $\hat{A} : D(A) \subseteq L^p(Z) \to L^q(Z)$ such that

$$\langle \hat{A}x, y \rangle = \int_Z \|Dx(Z)\|^{p-2}(Dx(z), Dy(z))\,dz \quad \text{for all } y \in W^{1,p}(Z)$$

with $D(A) = \{x \in W^{1,p}(Z)\colon\ \hat{A}x \in L^q(Z)\}$, satisfies $\hat{A} = \partial\hat{\psi}_1$.

So, we can say that

$$\int_Z f(z, x(z))y(z) = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))\,dz + \int_\Gamma v(z)y(z)\,d\sigma \tag{17}$$

with $v(z) \in \partial j(z, \tau(x(z)))$, for every $y \in W^{1,p}(Z)$. Let $y = \phi \in C_0^\infty(Z)$. Then we have

$$\int_Z f(z, x(z))\phi(z)\,dz = \int_Z \|Dx(z)\|^{p-2}(Dx(z), D\phi(z))\,dz.$$

But $\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in W^{-1,q}(Z)$ then we have that $\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in L^q(Z)$ because $f(z, x(z)) \in L^q(Z)$. Then we have that $-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = f(z, x(z))$ a.e. on Z. Going back to (17) and letting $y = C^\infty(Z)$ and finally using the Green formula 1.6 of Kenmochi [8], we have that $-\partial x/\partial n_p \in \partial j(z, \tau(x)(z))$. So $x \in W^{1,p}(Z)$ solves (9).

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