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On Positive Solutions of Functional–Differential Equations in Banach Spaces

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In this paper, we deal with two point boundary value problem (BVP) for the functional–differential equation of second order

$$x''(t)+kx'(t)+f(t,x(h_1(t)),x(h_2(t)))=0,$$
$$ax(-1)-bx'(-1)=0,$$
$$cx(1)+dx'(1)=0,$$

where the function f takes values in a cone K of a Banach space E. For $h_1(t)=t$ and $h_2(t)=-t$ we obtain the BVP with reflection of the argument. Applying fixed point theorem on strict set-contraction from G. Li, *Proc. Amer. Math. Soc.* **97** (1986), 277–280, we prove the existence of positive solution in the space $C([-1,1],E)$. Some inequalities involving f and the respective Green's function are used. We also give the application of our existence results to the infinite system of functional–differential equations in the case $E=l^{\infty}$.

Keywords: Boundary value problem in a Banach space; Positive solution; Cone; Fixed point theorem

1991 Mathematics Subject Classification: 34G20, 34K10

1 INTRODUCTION

Let K be a cone in a real Banach space E. We will assume that the norm $\|\cdot\|_E$ in E is monotonic with respect to K, that is, if $\theta\prec x\prec y$ then $\|x\|_E\leq\|y\|_E$, where $\prec$ denotes the partial ordering defined by K and

θ stands for the zero element of E. Further, denote by $C(I, E)$ the space of all continuous functions defined on the interval $[-1, 1]$ and taking values in E, equipped with the norm

$$\|x\| = \max_{t\in I} \|x(t)\|_E.$$

Obviously, $C(I, E)$ is a Banach space. Let

$$Q = \{x \in C(I, E) \colon \theta \prec x(t) \text{ for } t \in I\}.$$

It is easy to prove that Q is a cone in $C(I, E)$.

In this paper we will study the following boundary value problem (BVP for short) for functional–differential equation of second order

$$\begin{aligned} &x''(t) + kx'(t) + f(t, x(h_1(t)), x(h_2(t))) = 0,\\ &ax(-1) - bx'(-1) = 0,\\ &cx(1) + dx'(1) = 0, \end{aligned} \tag{1}$$

where $t \in I$, $k \in \mathbb{R}$, $a, b, c, d \geq 0$ and $ad + bc + ac > 0$. Throughout the paper we will assume that

(1°) $f\colon I \times K \times K \to K$ is a continuous function,
(2°) $h_1, h_2 \colon I \to I$ are continuous functions mapping the interval I onto itself.

Notice that for $h_1(t) = t$ and $h_2(t) = -t$ we obtain the BVP involving reflection of the argument

$$\begin{aligned} &x''(t) + kx'(t) + f(t, x(t), x(-t)) = 0,\\ &ax(-1) - bx'(-1) = 0,\\ &cx(1) + dx'(1) = 0. \end{aligned}$$

Such problems (that is BVPs with reflection of the argument) have been considered for example in the papers [8,15] for $k = 0$ and $f\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and in [9,10] for f taking values in a real Hilbert space. For more details concerning the differential equations with reflection of the argument we refer the reader to the papers mentioned above and the references therein.

Our purpose is to discuss the existence of positive solutions of (1). We will use the following fixed point theorem from [12] which is a modification of well-known Krasnoselskii theorem on operators compressing and expanding a cone (see [7,11]).

PROPOSITION 1 [12] *Let P be a cone of a real Banach space X, and let the norm $\|\cdot\|$ in X be monotonic with respect to P. Let $B_r=\{x\in X\colon \|x\|<r\}$, $B_R=\{x\in X\colon \|x\|<R\}$, $0<r<R$. Suppose that $F: P\cap\bar{B}_R\to P$ is a strict set-contraction which satisfies one of the following conditions:*

(i) $x\in P\cap\partial B_r\Rightarrow\|Fx\|\le\|x\|$ *and* $x\in P\cap\partial B_R\Rightarrow\|Fx\|\ge\|x\|$ *or*
(ii) $x\in P\cap\partial B_R\Rightarrow\|Fx\|\le\|x\|$ *and* $x\in P\cap\partial B_r\Rightarrow\|Fx\|\ge\|x\|$.

Then F has a fixed point in $P\cap(\bar{B}_R\backslash B_r)$.

Recall that $F\colon D\to X$, $D\subset X$, is said to be a strict set-contraction if F is continuous and bounded and there exists $0\le L<1$ such that $\alpha(F(S))\le L\alpha(S)$ for all bounded subsets S of D, where α denotes the Kuratowski measure of noncompactness (see for instance [2]).

2 PRELIMINARY RESULTS

First we will study some properties of the functions

$$G(t,s)=\begin{cases}\frac{1}{\rho}e^{-kt}[e^{k(s-1)}\mu_1+c][e^{k(t+1)}\mu_2-a], & -1\le t\le s\le 1,\\ \frac{1}{\rho}e^{-kt}[e^{k(s+1)}\mu_2-a][e^{k(t-1)}\mu_1+c], & -1\le s\le t\le 1,\end{cases}\tag{2}$$

where

$$k\neq 0,\quad \mu_1=dk-c,\quad \mu_2=bk+a\ \text{ and }\ \rho=\frac{1}{k}[ae^{-k}\mu_1+ce^{k}\mu_2],$$

and

$$G^*(t,s)=\begin{cases}\frac{1}{\rho^*}(c+d-cs)(a+b+at), & -1\le t\le s\le 1,\\ \frac{1}{\rho^*}(a+b+as)(c+d-ct), & -1\le s\le t\le 1,\end{cases}\tag{3}$$

where $\rho^*=2ac+bc+ad$. It is easy to show that the function (2) fulfils the following inequalities:

$$\bigwedge_{t,s\in I} G(t,s)\ge 0\tag{4}$$

and

$$\bigwedge_{t,s\in I} G(t,s) \leq G(s,s). \tag{5}$$

Moreover, for any $-1 \leq \gamma < \delta \leq 1$ and $t \in [\gamma, \delta]$ we have

$$G(t,s) \geq mG(s,s), \tag{6}$$

where $s \in I$ and

$$m = \min\left\{\frac{e^k\mu_2 - ae^{-k\gamma}}{e^k\mu_2 - ae^{-k}}, \frac{e^{-k}\mu_1 + ce^{-k\delta}}{e^{-k}\mu_1 + ce^k}\right\}. \tag{7}$$

It is easily seen that $m < 1$.

The function G^* also satisfies the inequalities (4), (5) and (6) with m replaced by

$$m^* = \min\left\{\frac{a+b+a\gamma}{2a+b}, \frac{c+d-c\delta}{2c+d}\right\}. \tag{8}$$

Clearly, $m^* < 1$.

Next, consider the integral-functional operator

$$(Fx)(t) = \int_{-1}^{1} G(t,s)f(s, x(h_1(s)), x(h_2(s)))\,\mathrm{d}s, \tag{9}$$

where $t \in I$, $x \in Q$, the function G is defined by (2) and f, h_1 and h_2 satisfy 1° and 2°. Let

$$M = \max_{t,s\in I} G(t,s),$$

$$\bar{T}_r = \{x \in E\colon \|x\|_E \leq r\},$$

and

$$\bar{B}_r = \{x \in C(I,E)\colon \|x\| \leq r\}.$$

The following lemma is a slight modification of that given in [6].

LEMMA 1 *Assume that for any* $r > 0$:

(3°) *the function* f *is uniformly continuous on* $I \times (K \cap \bar{T}_r) \times (K \cap \bar{T}_r)$,

(4°) *there exists a non-negative constant* L_r, *such that* $4ML_r < 1$ *and*

$$\alpha(f(t, \Omega, \Omega)) \leq L_r \alpha(\Omega)$$

for all $t \in I$ *and* $\Omega \subset K \cap \bar{T}_r$.

Then, for any $r > 0$ *the operator* (9) *is a strict set-contraction on* $Q \cap \bar{B}_r$.

Proof From 3° it follows that f is bounded on $I \times (K \cap \bar{T}_r) \times (K \cap \bar{T}_r)$. By the uniform continuity of f (see [3])

$$\alpha(f(I \times \Omega \times \Omega)) = \max_{t \in I} \alpha(f(t, \Omega, \Omega)),$$

hence, in view of 4°

$$\alpha(f(I \times \Omega \times \Omega)) \leq L_r \alpha(\Omega) \tag{10}$$

for every $\Omega \subset K \cap \bar{T}_r$. The uniform continuity and boundedness of f on $I \times (K \cap \bar{T}_r) \times (K \cap \bar{T}_r)$ implies also continuity and boundedness of operator F on $Q \cap \bar{B}_r$. Let $S \subset Q \cap \bar{B}_r$. Since the functions Fx are equicontinuous and uniformly bounded for $x \in S$, we obtain (see [4])

$$\alpha(F(S)) = \sup_{t \in I} \alpha(F(S)(t)),$$

where $F(S)(t)$ denotes the cross-section of $F(S)$ at the point t, that is

$$F(S)(t) = \{(Fx)(t)\colon x \in S,\ t \text{ is fixed}\}.$$

Furthermore, for every $t \in I$ we get (see [14])

$$\begin{aligned}
&\tfrac{1}{2}(Fx)(t) \\
&\quad = \tfrac{1}{2} \int_{-1}^{1} G(t, s) f(s, x(h_1(s)), x(h_2(s)))\, \mathrm{d}s \\
&\quad \in \overline{\operatorname{conv}}\{G(t, s) f(s, x(h_1(s)), x(h_2(s)))\colon s \in I,\ x \in S\} \\
&\quad \subset \overline{\operatorname{conv}}\{\{M f(s, x(h_1(s)), x(h_2(s)))\colon s \in I,\ x \in S\} \cup \{\theta\}\}.
\end{aligned}$$

Thus, in view of the properties of the Kuratowski measure of non-compactness we obtain for $t \in I$

$$\begin{aligned}&\alpha(\tfrac{1}{2}F(S)(t))\\&\quad\le M\alpha(\{f(s,x(h_1(s)),x(h_2(s))): s\in I,\ x\in S\})\\&\quad\le M\alpha(f(I\times S(I)\times S(I))),\end{aligned}$$

where $S(I)=\{x(s): s\in I, x\in S\}$. Hence, by (10) we have

$$\alpha(F(S)(t))\le 2M\alpha(f(I\times S(I)\times S(I)))\le 2ML_r\alpha(S(I)).$$

Finally, proceeding as in the proof of Lemma 2 [6], we can show that

$$\alpha(S(I))\le 2\alpha(S).$$

Hence for any $S\subset Q\cap \bar{B}_r$

$$\alpha(F(S))=\sup_{t\in I}\alpha(F(S)(t))\le 4ML_r\alpha(S),$$

which means that F is a strict set-contraction on $Q\cap\bar{B}_r$.

Remark Obviously, the above lemma remains valid for the operator (9) with the function G^* given by (3) and the constant

$$M^*=\max_{t,s\in I}G^*(t,s).$$

3 EXISTENCE THEOREMS FOR PROBLEM (1)

Now we state and prove our results on positive solutions of (1). First, consider the case $k\ne 0$.

THEOREM 1 *Let G be given by* (2) *and let $-1\le\gamma<\delta\le 1$ be such that $h_i:[\gamma,\delta]\to[\gamma,\delta]$, $i=1,2$. Suppose that the assumptions* 1°–4° *are satisfied and*

(5°) *there exists $\lambda\in K$, $\lambda\ne\theta$, such that*

$$f(t,x,y)\prec\left[\int_{-1}^{1}G(s,s)\,\mathrm{d}s\right]^{-1}\lambda$$

for all $t\in I$ and $x,y\in K$ such that $\|x\|_E, \|y\|_E\in[0,\|\lambda\|_E]$, (6°) *there exist* $\eta \in K$, $\eta \neq \theta$, $\|\eta\|_E \neq \|\lambda\|_E$ *and* $t_0 \in I$ *such that*

$$\left[\int_\gamma^\delta G(t_0, s)\,\mathrm{d}s\right]^{-1} \eta \prec f(t, x, y)$$

for all $t \in I$ *and* $x, y \in K$ *such that* $\|x\|_E, \|y\|_E \in [m\|\eta\|_E, \|\eta\|_E]$, *where* m *is given by* (7).

Then the problem (1) *has at least one positive solution.*

Proof Notice that each positive solution of the problem (1) (with $k \neq 0$) is a fixed point of the integral-functional operator (9), that is

$$(Fx)(t) = \int_{-1}^{1} G(t, s) f(s, x(h_1(s)), x(h_2(s)))\,\mathrm{d}s$$

where $t \in I$, $x \in C(I, E)$ and the function G is given by (2). On the other hand, if x belonging to Q is a fixed point of F, then x is a solution of (1) (see [6]). Thus, to prove our theorem it is enough to show that F has a fixed point in Q. In the space $C(I, E)$ consider the set

$$P = \left\{ x \in C(I, E)\colon \theta \prec x(t) \text{ on } I \text{ and } \bigwedge_{t \in [\gamma,\delta]} \bigwedge_{s \in I} m x(s) \prec x(t) \right\}.$$

Clearly, P is a cone in $C(I, E)$ and the norm $\|\cdot\|$ in $C(I, E)$ is monotonic with respect to P. Consider the operator (9) for $t \in I$ and $x \in P$. We will show that F satisfies the assumptions of Proposition 1. First, we will prove that $F(P) \subset P$. To this end observe that by 1° and (4)

$$\theta \prec (Fx)(t) \tag{11}$$

for every $x \in P$ and $t \in I$. Moreover, it follows from (6) that for any $t \in [\gamma, \delta]$ and $s \in I$

$$\begin{aligned} m(Fx)(s) &= m \int_{-1}^{1} G(s, s) f(s, x(h_1(s)), x(h_2(s)))\,\mathrm{d}s \\ &\prec \int_{-1}^{1} G(t, s) f(s, x(h_1(s)), x(h_2(s)))\,\mathrm{d}s \\ &= (Fx)(t). \end{aligned}$$

Combining it with (11) we conclude that $F(P) \subset P$. Without loss of generality we may assume that $\|\lambda\|_E < \|\eta\|_E$. Fix $r = \|\lambda\|_E$ and $R = \|\eta\|_E$.

By Lemma 1, F is a strict set-contraction on $P \cap \bar{B}_R$. Moreover, for $x \in P \cap \partial B_r$ we have $\theta \prec x(h_1(t))$ on I and $\|x\| = \|\lambda\|_E$, hence

$$\bigwedge_{t \in I} \|x(h_1(t))\|_E \le \|\lambda\|_E.$$

Analogously

$$\bigwedge_{t \in I} \|x(h_2(t))\|_E \le \|\lambda\|_E.$$

Thus, by 5°, for any $t \in I$ we obtain

$$\begin{aligned}(Fx)(t) &\prec \int_{-1}^{1} G(s,s) f(s, x(h_1(s)), x(h_2(s)))\, \mathrm{d}s \\ &\prec \int_{-1}^{1} G(s,s) \left(\int_{-1}^{1} G(\tau,\tau)\, \mathrm{d}\tau \right)^{-1} \lambda\, \mathrm{d}s = \lambda.\end{aligned}$$

Hence, in view of monotonicity of $\|\cdot\|_E$ we get

$$\bigwedge_{t \in I} \|(Fx)(t)\|_E \le \|\lambda\|_E,$$

and in consequence $\|Fx\| \le \|x\|$ on $P \cap \partial B_r$. Furthermore, for $x \in P \cap \partial B_R$ we have

$$\bigwedge_{t \in [\gamma,\delta]} \bigwedge_{s \in I} \theta \prec m x(h_1(s)) \prec x(h_1(t)).$$

Since the norm $\|\cdot\|_E$ is monotonic we obtain

$$\bigwedge_{t \in [\gamma,\delta]} \bigwedge_{s \in I} \|m x(h_1(s))\|_E \le \|x(h_1(t))\|_E,$$

which gives

$$\bigwedge_{t \in [\gamma,\delta]} m \max_{s \in I} \|x(h_1(s))\|_E \le \|x(h_1(t))\|_E.$$

But h_1 maps I onto itself, hence for $\|x\| = \|\eta\|_E$

$$\bigwedge_{t\in[\gamma,\delta]} m\|\eta\|_E \leq \|x(h_1(t))\|_E \leq \|\eta\|_E.$$

In the same manner we get

$$\bigwedge_{t\in[\gamma,\delta]} m\|\eta\|_E \leq \|x(h_2(t))\|_E \leq \|\eta\|_E.$$

Thus in view of 6°

$$\begin{aligned}\eta &= \int_\gamma^\delta G(t_0,s)\left(\int_\gamma^\delta G(t_0,\tau)\,\mathrm{d}\tau\right)^{-1}\eta\,\mathrm{d}s\\ &\prec \int_\gamma^\delta G(t_0,s)f(s,x(h_1(s)),x(h_2(s)))\,\mathrm{d}s\\ &\prec \int_{-1}^1 G(t_0,s)f(s,x(h_1(s)),x(h_2(s)))\,\mathrm{d}s\\ &= (Fx)(t_0),\end{aligned}$$

so

$$\|(Fx)(t_0)\|_E \geq \|\eta\|_E,$$

which implies $\|Fx\| \geq \|x\|$ on $P\cap\partial B_R$. By Proposition 1 the operator F has a fixed point in the set $P\cap(\bar{B}_R\backslash B_r)$. This means that the problem (1) has at least one positive solution $x\in P$ such that

$$\|\lambda\|_E \leq \|x\| \leq \|\eta\|_E.$$

This ends the proof of Theorem 1.

Next, consider the problem (1) with $k=0$. Using the properties of the function G^* given by (3) we can prove the following theorem in the same way as Theorem 1.

THEOREM 2 *Let G^* be given by* (3) *and let $-1\leq\gamma<\delta\leq 1$ be such that $h_i:[\gamma,\delta]\to[\gamma,\delta]$, $i=1,2$. Suppose that* 1°–4° *are satisfied and*

(7°) *there exists* $\lambda \in K$, $\lambda \neq \theta$, *such that*

$$f(t,x,y) \prec \left[\int_{-1}^{1} G^*(s,s)\,\mathrm{d}s\right]^{-1} \lambda$$

for $t \in I$ *and* $x, y \in K$ *such that* $\|x\|_E, \|y\|_E \in [0, \|\lambda\|_E]$,

(8°) *there exist* $\eta \in K$, $\eta \neq \theta$, $\|\eta\|_E \neq \|\lambda\|_E$, *and* $t_0 \in I$ *such that*

$$\left[\int_{\gamma}^{\delta} G^*(t_0,s)\,\mathrm{d}s\right]^{-1} \eta \prec f(t,x,y)$$

for $t \in I$ *and* $x, y \in K$ *such that* $\|x\|_E, \|y\|_E \in [m^*\|\eta\|_E, \|\eta\|_E]$, *where* m^* *is given by* (8).

Then the problem (1) *has at least one positive solution.*

Remark For similar theorems on positive solutions of BVPs in the case $f: I \times [0,\infty) \to [0,\infty)$ we refer the reader to [5,13].

Finally, we will give an example of application of Theorem 2 to the infinite system of functional–differential equations.

Example Let E be the space l^∞ of all bounded sequences $x = \{x_n\}$ with the supremum norm

$$\|x\|_E = \sup_{n\in\mathbb{N}} |x_n|. \tag{12}$$

Then

$$K = \left\{ x \in E \colon \bigwedge_{n\in\mathbb{N}} x_n \geq 0 \right\}$$

is a cone in E and the norm (12) is monotonic with respect to K. Consider the following BVP of an infinite system of functional–differential equations:

$$\begin{gathered} x_n''(t) + A(t)x_n(h_1(t)) + B(t)x_n(h_2(t)) + C(t) \\ + \omega_n\sqrt{x_n(h_1(t)) + x_n(h_2(t))} = 0, \\ x_n(-1) - x_n'(-1) = 0, \quad x_n(1) + x_n'(1) = 0, \end{gathered} \tag{13}$$

where $n = 1, 2, 3, \ldots, t \in I, x = \{x_n\} \in Q \subset C(I, E)$, the functions A, B, C: $[-1, 1] \to [0, \infty)$ are continuous, $\omega = \{\omega_n\} \in K$ and $\lim_{n\to\infty}\omega_n = 0$. In our case

$$M^* = \max_{t,s\in I} G^*(t, s) = 1$$

and

$$\left[\int_{-1}^{1} G^*(s, s)\,\mathrm{d}s\right]^{-1} = \frac{6}{11}.$$

Assume that

$$\max_{t\in I}(A(t) + B(t)) < \frac{1}{4} \text{ and } \min_{t\in I} C(t) > 0.$$

Moreover, suppose that the functions h_i satisfy 2° and $h_i([-\frac{1}{2}, \frac{1}{2}]) \subset [-\frac{1}{2}, \frac{1}{2}]$, $i = 1, 2$. Then for $\gamma = -\frac{1}{2}$, $\delta = \frac{1}{2}$ we have $m^* = \frac{1}{2}$ and for $t_0 = -1$,

$$\left[\int_{-1/2}^{1/2} G^*(-1, s)\,\mathrm{d}s\right]^{-1} = 2.$$

Consider the function

$$f(t, x, y) = A(t)x + B(t)y + C(t) + \omega\sqrt{x + y},$$

where $t \in I$, $f = \{f_n\}$, $x, y \in K$, $x = \{x_n\}$, $y = \{y_n\}$. Obviously, f is uniformly continuous on $I \times (K \cap \bar{T}_r) \times (K \cap \bar{T}_r)$ for any $r > 0$. We will show that f satisfies 4°. Notice that f admits a splitting

$$f = \tilde{f} + \bar{f},$$

where

$$\tilde{f}(t, x, y) = A(t)x + B(t)y + C(t)$$

and

$$\bar{f}(t, x, y) = \omega\sqrt{x + y}.$$

Evidently, the function $\tilde{f}$ is lipschitzian, hence

$$\alpha(\tilde{f}(t, \Omega, \Omega)) \leq \max_{t\in I}(A(t) + B(t))\alpha(\Omega) \tag{14}$$

for all $t \in I$ and $\Omega \subset K \cap \bar{T}_r$. To find $\alpha(\bar{f}(t,\Omega,\Omega))$ we will apply the following compactness criterion in the space l^∞ (see [1]):

If $D \subset l^\infty$ is bounded and $\limsup_{n\to\infty}[\sup_{x\in D}|x_n|] = 0$, then D is relatively compact in l^∞.

Denote

$$X(t) = \bar{f}(t,\Omega,\Omega) = \{\bar{f}(t,x,y) \colon x,y \in \Omega,\ t \text{ is fixed}\}.$$

For $n \in \mathbb{N}$ and $x, y \in \Omega \subset K \cap \bar{T}_r$ we have

$$|\bar{f}_n(t,x,y)| = |\omega_n \sqrt{x_n + y_n}| \leq \sqrt{2r}\omega_n.$$

Since $\lim_{n\to\infty}\omega_n = 0$, we obtain

$$\limsup_{n\to\infty}\left[\sup_{\bar{f}(t,x,y)\in X(t)} \left|\bar{f}_n(t,x,y)\right|\right] = 0,$$

and in consequence $X(t)$ is relatively compact. Therefore

$$\alpha(\bar{f}(t,\Omega,\Omega)) = 0. \tag{15}$$

By (14) and (15) and the property of the Kuratowski measure of non-compactness we have

$$\begin{aligned}\alpha(f(t,\Omega,\Omega)) &\leq \alpha(\tilde{f}(t,\Omega,\Omega) + \bar{f}(t,\Omega,\Omega)) \\ &\leq \max_{t\in I}(A(t) + B(t))\alpha(\Omega)\end{aligned}$$

which means that 4° is fulfilled. Finally, we can show by simple calculation that 7° and 8° are also satisfied with $\lambda = \{\lambda_n\}$, $\eta = \{\eta_n\} \in K$, such that

$$\lambda_n = \left(\frac{22}{13}\right)^2 \left[\sqrt{2}\omega_n + \sqrt{2\omega_n^2 + \frac{13}{11}\max_{t\in I} C(t)}\right]^2$$

and

$$\eta_n = \tfrac{1}{2}\min_{t\in I} C(t), \quad n = 1, 2, \ldots$$

By Theorem 2, the problem (13) has a positive solution $x \in P$ such that

$$\|\eta\|_E \leq \|x\| \leq \|\lambda\|_E.$$

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