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Isoperimetric Inequality for Torsional Rigidity in the Complex Plane

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Suppose Ω is a simply connected domain in the complex plane. In (F.G. Avhadiev, *Matem. Sborn.*, **189**(12) (1998), 3–12 (Russian)), Avhadiev introduced new geometrical functionals, which give two-sided estimates for the torsional rigidity of Ω. In this paper we find sharp lower bounds for the ratio of the torsional rigidity to the new functionals. In particular, we prove that

$$3I_c(\partial\Omega) \le 2P(\Omega),$$

where $P(\Omega)$ is the torsional rigidity of Ω,

$$I_c(\partial\Omega) = \iint_\Omega R^2(z,\Omega)\,dx\,dy$$

and $R(z,\Omega)$ is the conformal radius of Ω at a point z.

Keywords: Torsional rigidity; Isoperimetric inequality;
Moment of inertia of domain about its boundary; Conformal map

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1 INTRODUCTION

Let Ω be a simply connected domain in the complex plane **C**. By $P(\Omega)$ we denote the torsional rigidity of Ω. The classical problem stated by St Venant is to find geometrical functionals of Ω approximating $P(\Omega)$.

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A number of isoperimetrical inequalities for the torsional rigidity can be found in the books of Pólya and Szegö [2], Bandle [3], and Osserman [4]. Most of these inequalities are one-sided estimates.

The following result due to Avhadiev gives two-sided inequalities for $P(\Omega)$. Let $\mathrm{dist}(z, \partial\Omega)$ be the distance from $z \in \Omega$ to the boundary $\partial\Omega$ of Ω, and let $R(z, \Omega)$ be the conformal radius of Ω at z. In [1], Avhadiev introduced new functionals

$$\begin{aligned} I(\partial\Omega) &= \iint_\Omega \mathrm{dist}^2(z, \partial\Omega)\,\mathrm{d}x\,\mathrm{d}y \quad \text{and} \\ I_c(\partial\Omega) &= \iint_\Omega R^2(z, \Omega)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{1}$$

The value $I(\partial\Omega)$ is called the *moment of inertia* of Ω about $\partial\Omega$, and $I_c(\partial\Omega)$ is the *conformal moment* of Ω.

THEOREM A [1] *For simply connected domain Ω the torsional rigidity $P(\Omega) < +\infty$ if and only if $I_c(\partial\Omega) < +\infty$, and*

$$I(\partial\Omega) \le I_c(\partial\Omega) \le P(\Omega) \le 4I_c(\partial\Omega) \le 64I(\partial\Omega).$$

Moreover, in [5] it was proved that $P(\Omega)$, $I(\partial\Omega)$ and $I_c(\partial\Omega)$ have similar isoperimetric properties. In particular,

$$I(\partial\Omega) \le \frac{A^2(\Omega)}{6\pi} \quad \text{and} \quad I_c(\partial\Omega) \le \frac{A^2(\Omega)}{3\pi}, \tag{2}$$

where $A(\Omega)$ is the area of Ω. Note that the inequalities (2) are similar to the famous isoperimetric inequality of St Venant.

2 MAIN THEOREM AND COROLLARIES

THEOREM 1 *If $P(\Omega) < +\infty$, then*

$$\frac{\pi}{2}R^4(\Omega) \le \frac{3}{2}I_c(\partial\Omega) \le P(\Omega), \tag{3}$$

where $R(\Omega) = \max_{z\in\Omega} R(z, \Omega)$. The equality $\pi R^4(\Omega) = 3I_c(\partial\Omega)$ holds only for a disk. If Ω is bounded, then the equality $3I_c(\partial\Omega) = 2P(\Omega)$ holds if and only if Ω is a disk.

Theorem 1 strengthens the Pólya and Szegö inequality

$$\pi R^4(\Omega) \leq 2P(\Omega). \tag{4}$$

Note that Payne (see [3]) gives other strengthening of (4)

$$\frac{\pi}{2} R^4(\Omega) \leq 2\pi v^2(\Omega) \leq P(\Omega),$$

where $v(\Omega) = \max_{(x,y)\in\Omega} v(x,y)$ and the warping function $v(x,y)$ of Ω satisfies (see [3])

$$\begin{aligned} \Delta v &= -2 \quad \text{in } D, \\ v &= 0 \quad \text{on } \partial D. \end{aligned}$$

On the other hand, from Theorem A it follows that there exists a constant $k > 0$ such that $v^2(\Omega) \leq kI_c(\partial\Omega)$.

Further, it is clear that (3) and the St Venant inequality $P(\Omega) \leq A^2(\Omega)/2\pi$ imply the second inequality in (2).

As a straightforward consequence of Theorem 1 we obtain the following inequality for $I(\partial\Omega)$:

Corollary 1 *Under the condition of Theorem* 1, *we have*

$$\tfrac{3}{2} I(\partial\Omega) < P(\Omega).$$

3 PROOF OF THEOREM 1

Let $f(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n$ be a conformal map of $U = \{\zeta : |\zeta| < 1\}$ onto Ω. The first step of the proof is to obtain series expansions of $I_c(\partial\Omega)$ and $P(\Omega)$ in terms of Taylor's coefficients of $f(\zeta)$. Taking into account (1), the well-known formula $R(z,\Omega) = |f'(\zeta)|(1 - |\zeta|^2)$, and Taylor's series of $f(\zeta)$, we have

$$\begin{aligned} I_c(\partial\Omega) &= \iint_E |f'(\zeta)|^4(1 - |\zeta|^2)^2 \mathrm{d}\xi\, \mathrm{d}\eta = 2\pi \sum_{n=0}^{\infty} |B_n|^2 \int_0^1 (1 - r^2)^2 r^{2n+1} \mathrm{d}r \\ &= 2\pi \sum_{n=0}^{\infty} \frac{|B_n|^2}{(n+1)(n+2)(n+3)} = 2\pi \sum_{n=2}^{\infty} \frac{\left|\sum_{k=1}^{n-1} k(n-k)a_k a_{n-k}\right|^2}{(n-1)n(n+1)}, \end{aligned} \tag{5}$$

where $B_n = \sum_{k=0}^{n}(k+1)(n+1-k)a_{k+1}a_{n+1-k}$. From (5) it follows that the left-hand side of (3) is true. Indeed, suppose $R(z,\Omega) = \max_{t\in\Omega} R(t,\Omega)$, and $f(0) = z$. We obtain

$$R^4(z,\Omega) = |a_1|^4 \le \frac{3}{\pi}\left(\frac{\pi}{3}|a_1|^4 + \frac{4\pi}{3}|a_1a_2|^2 + \cdots\right) = \frac{3}{\pi}I_c(\partial\Omega).$$

It is clear that the equality holds if and only if $a_i = 0$, $i = 2,3,\ldots$ Consequently, the equality $\pi R^4(\Omega) = 3I_c(\partial\Omega)$ holds if and only if Ω is a disk.

The right-hand side of (3) is more difficult to prove. First we establish (3) for a bounded domain.

It is well known (see [2]) that

$$P(\Omega) = \frac{\pi}{2}\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\delta}\min\{\alpha,\beta,\gamma,\delta\}a_\alpha a_\beta\overline{a_\gamma a_\delta}, \qquad (6)$$

the sum being restricted to the non-negative indices α, β, γ, and δ for which $\alpha+\beta = \gamma+\delta$. In [2] it was shown that (6) is absolutely convergent.

Substituting $\alpha+\beta$ for n in (6), we get

$$P(\Omega) = \frac{\pi}{2}\sum_{n=2}^{\infty}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\min\{j,n-j,k,n-k\}a_ja_{n-j}\overline{a_ka_{n-k}}. \qquad (7)$$

The next step to prove Theorem 1 is to use the following lemma which allows us to compare the coefficients of the series (5) and (7).

LEMMA 1 *Let n be a integer number, $n \ge 2$, and*

$$l = \begin{cases} (n-1)/2 & \text{for odd } n, \\ n/2 - 1 & \text{for even } n. \end{cases}$$

Then the matrix M with elements

$$m_{jk} = \min\{j,k\} - \frac{6j(n-j)k(n-k)}{(n-1)n(n+1)}, \quad j,k = 1,2,\ldots,l$$

is positive semidefinite.

Proof of Lemma 1 We compute the determinant of the main minors of M to use Sylvester's criteria of positive semidefinity.

Denote by $M(k)$ $(k=1,\ldots,l)$ the main minor of order k. Let $M(k)_j$ be the j-string of $M(k)$. We preserve the denotation $M(k)$ at the following transformations

$$\begin{aligned}
&\text{(i)}\ M(k)_j = M(k)_j - M(k)_{j-1}, \quad j=2,\ldots,k.\\
&\text{(ii)}\ M(k)_j = M(k)_j - M(k)_{j+1}, \quad j=1,\ldots,k-1.\\
&\text{(iii)}\ M(k)_j = M(k)_j - M(k)_1, \quad j=2,\ldots,k-1 \text{ and}\\
&\qquad M(k)_k = M(k)_k - (n-2k+1)M(k)_1/2.\\
&\text{(iv)}\ M(k)_1 = M(k)_1 - \sum_{j=2}^{k} m_j M(k)_j,
\end{aligned}$$

where $m_j = -12j(n-j)/(n-1)n(n+1), j=2,\ldots,k$.

Finally, we obtain

$$M(k) = \begin{pmatrix} \sum_{j=1}^{k-1} m_j + (n-2k+1)m_k/2 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ - & - & - & \ldots & - & - \\ -1 & 0 & 0 & \ldots & 1 & 0 \\ -(n-2k+1)/2 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

where $m_1 = 1 - 12/n(n+1)$.

Hence

$$\det(M(k)) = \sum_{j=1}^{k-1} m_j + \frac{n-2k+1}{2} m_k.$$

The induction on j gives easily

$$\det(M(k)) = 1 - \frac{2k((k-1)(3n-2k+1) + 3(n-2k+1)(n-k))}{(n-1)n(n+1)}.$$

Therefore, $\det(M(k))$ is the polynomial of the third degree. The polynomial equals zero at the points $k=(n-1)/2, n/2, (n+1)/2$ and equals one at $k=0$. Thus

$$\det(M(k)) = \left(1 - \frac{2k}{n-1}\right)\left(1 - \frac{2k}{n}\right)\left(1 - \frac{2k}{n+1}\right).$$

This shows that $\det(M(k)) \geq 0, k=1,\dots,l$; therefore, M is positive semidefinite. Lemma 1 is proved.

Lemma 1 (see [6]) implies that the Hermitian form

$$\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\left[\min\{j, n-j, k, n-k\} - \frac{6j(n-j)k(n-k)}{(n-1)n(n+1)}\right]\zeta_j\overline{\zeta_k} \geq 0 \tag{8}$$

for all complex members $\zeta_1, \zeta_2, \dots, \zeta_{n-1}$, where $n=2,3,\dots$ From (5), (7) and (8) we derive the right-hand side of (3) for bounded domains.

In the general case $P(\Omega) < +\infty$, we apply the following property: if $\Omega_1 \subset \Omega_2$, then

$$P(\Omega_1) \leq P(\Omega_2) \quad \text{and} \quad I_c(\partial\Omega_1) \leq I_c(\partial\Omega_2). \tag{9}$$

Consider a sequence of bounded domains Ω_n ($\Omega_n \subset \Omega$), which converges to Ω as to a kernel by Caratheodory. Hence, Riemann's functions $f_n : \Omega_n \to U$ converge to $f : \Omega \to U$. In particular, Taylor's coefficients of $f_n(\zeta)$ converge to Taylor's coefficients of $f(\zeta)$. From the convergency, the inequality (3) for Ω_n, and the property (9), we get the right-hand side of (3) for Ω.

To complete the proof of Theorem 1 we consider the equality

$$P(\Omega) = \tfrac{3}{2} I_c(\partial\Omega) \tag{10}$$

under the restriction that Ω is bounded.

First, using the equalities (see [2])

$$\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\min\{j, n-j, k, n-k\} = \sum_{k=1}^{n-1} k(n-k) = \frac{(n-1)n(n+1)}{6}, \tag{11}$$

we prove the equality

$$\sum_{j=1}^{n}\sum_{k=1}^{n} b(n+1)_{jk} a_j a_{n+1-j}\,\overline{a_k a_{n+1-k}} = \frac{4|a_1|^2(n-1)(n-2)}{(n+1)(n+2)}|q^{n-1}a_1 - a_n|^2 \tag{12}$$

for all $a_j = q^{j-1}a_1, j = 1, \ldots, n-1$ ($|q| < 1$) and $a_n \in \mathbf{C}$, where

$$b(n+1)_{jk} = \min\{j, n+1-j, k, n+1-k\} - \frac{6j(n+1-j)k(n+1-k)}{n(n+1)(n+2)}. \tag{13}$$

It can be shown in the usual way that

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} = \sum_{j=2}^{n-1}\sum_{k=2}^{n-1} c_{jk} + 2\mathrm{Re}\left\{\sum_{k=1}^{n}(c_{1k} + c_{nk})\right\} - c_{11} - c_{nn} - 2\mathrm{Re}\, c_{1n}, \tag{14}$$

where $c_{jk} \in \mathbf{C}$ for which $c_{jk} = \overline{c_{kj}}$.

Decompose the left-hand side of (12) in the form

$$\sum_{j=1}^{n}\sum_{k=1}^{n} b(n+1)_{jk} a_j a_{n+1-j}\,\overline{a_k a_{n+1-k}} = I_1 + I_2,$$

where

$$I_1 = |q|^{2(n-1)}|a_1|^4 \sum_{j=2}^{n-1}\sum_{k=2}^{n-1} b(n+1)_{jk},$$

$$I_2 = 4\mathrm{Re}\left\{a_1 a_n \sum_{k=1}^{n} b(n+1)_{1k}\overline{a_k a_{n+1-k}}\right\} - 4b(n+1)_{11}|a_1 a_n|^2.$$

Using (14), (13) and (11), we obtain

$$\begin{aligned} I_1 &= -2|q|^{2(n-1)}|a_1|^4\left(\sum_{k=1}^{n} b(n+1)_{1k} + \sum_{k=2}^{n-1} b(n+1)_{1k}\right) \\ &= 2|q|^{2(n-1)}|a_1|^4(-b(n+1)_{11} - b(n+1)_{1n}) \\ &= \frac{4|q|^{2(n-1)}|a_1|^4(n-1)(n-2)}{(n+1)(n+2)} \end{aligned}$$

and

$$I_2 = \frac{4(n-1)(n-2)}{(n+1)(n+2)}|a_1 a_n|^2 + 4\mathrm{Re}\left\{a_1 a_n (\bar{q})^{n-1} \bar{a}_1^2 \sum_{k=2}^{n-1} b(n+1)_{1k}\right\}$$
$$= \frac{4|a_1|^2(n-1)(n-2)}{(n+1)(n+2)}\left(|a_n|^2 - 2\mathrm{Re}\left\{a_n \overline{a_1} (\bar{q})^{n-1}\right\}\right).$$

This proves (12).

It follows from (8) that (10) is equivalent to

$$\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} b(n)_{jk} a_j a_{n-j} \overline{a_k a_{n-k}} = 0, \tag{15}$$

where $n = 2, 3, \ldots$ Now we apply induction on n. Note that $b(2)_{11} = \sum_{j=1}^{2}\sum_{k=1}^{2} b(3)_{jk} = 0$ and suppose $a_j = q^{j-1} a_1$, $j = 1, \ldots, n-1$, where $q = a_2/a_1$. From (12) and (15), we obtain $a_n = q^{n-1} a_1$. Therefore, the equality (10) holds if and only if

$$f(\zeta) = a_0 + \sum_{n=1}^{\infty} a_1 q^{n-1} \zeta^n = a_0 + \frac{a_1 \zeta}{1 - q\zeta}.$$

This concludes the proof of Theorem 1.

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