On Weighted Dyadic Carleson's Inequalities

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We give an alternate proof of weighted dyadic Carleson's inequalities which are essentially proved by Sawyer and Wheeden. We use the Bellman function approach of Nazarov and Treil. As an application we give an alternate proof of weighted inequalities for dyadic fractional maximal operators. A result on weighted inequalities for fractional integral operators is given.

Keywords: Carleson's inequality; Bellman function; Fractional maximal operator

AMS 1991 Subject Classifications: 42B25, 26D15

1. MAIN RESULTS

In this paper we study weighted dyadic Carleson's inequalities. The result in this paper is essentially contained in the results by Sawyer and Wheeden [5]. We give an alternate proof of it. In the proof of our theorem we will use the Bellman function approach which was invented by Nazarov and Treil [2]. Our interest is in applications of Nazarov and Treil's methods.

As an application of our weighted norm inequalities we will give an alternate proof of weighted norm inequalities for dyadic fractional maximal functions which is studied by Genebashvili, Gogatishvili, Kokilashvili and Krbec under more general setting [1]. A result on

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weighted norm inequalities for fractional integral operators will be
given.

Let $\mathcal{D}$ be the set of all dyadic cubes in $\mathbf{R}^n$. By a dyadic cube we mean
a cube of the form $[2^j k_1, 2^j(k_1 + 1)) \times \cdots \times [2^j k_n, 2^j(k_n + 1))$ for some
integers $j, k_1, \ldots, k_n$. For $I \in \mathcal{D}$ and a locally integrable function φ on
$\mathbf{R}^n$ we set

$$\langle\varphi\rangle_I = \frac{1}{|I|}\int_I \varphi(x)dx,$$

where $|A|$ denotes the Lebesgue measure of a measurable set A.

Next we introduce the dyadic reverse doubling condition on
weights. We say a nonnegative measurable function w satisfies the
dyadic reverse doubling condition if w is locally integrable and there is
a constant $d > 1$ such that

$$d\int_{I'} w(x)dx \leq \int_I w(x)dx \qquad (1)$$

for all $I, I' \in \mathcal{D}$ where I' is contained in I and has the half side length of
I.

Let p' be the positive number such that $p^{-1} + p'^{-1} = 1$.

THEOREM 1.1   Let $1 < p < q < \infty$ and w be a nonnegative locally
integrable function on $\mathbf{R}^n$. We assume that $w^{-1/(p-1)}$ satisfies the
dyadic reverse doubling condition. Let $\{\mu_I\}_{I \in \mathcal{D}}$ be nonnegative numbers.
Then the following two statements are equivalent.

(i)   There is a positive constant C such that

$$\sum_{I \in \mathcal{D}} \mu_I \langle\varphi\rangle_I^q \leq C\left(\int_{\mathbf{R}^n} \varphi(x)^p w(x)dx\right)^{q/p} \qquad (2)$$

for all nonnegative locally integrable functions φ.

(ii)   There is a positive constant C' such that

$$\mu_I \leq C'|I|^q\left(\int_I w(x)^{-1/(p-1)}dx\right)^{-q/p'}$$

for all $I \in \mathcal{D}$.

Remark 1.1 If $w^{-1/(p-1)}$ satisfies the dyadic reverse doubling condition, then we can prove that there is a positive constant c such that

$$\sum_{Q'\subset Q, Q' \in \mathcal{D}} \left(\int_{Q'} w^{-1/(p-1)} dx \right)^{q/p} \leq c \left(\int_Q w^{-1/(p-1)} dx \right)^{q/p}$$

for all dyadic cubes Q. By this inequality and Lemma 2.10 in [5] we can prove (5) in the proof of Theorem 1.1 in Section 2. Hence Theorem 1.1 is a corollary of Sawyer and Wheeden's result.

Let $1 < p < \infty$. We say a nonnegative measurable function w is a dyadic A_p weight if there is a positive constant C such that

$$\left(\frac{1}{|I|} \int_I w(x) dx \right)^{1/p} \left(\frac{1}{|I|} \int_I w(x)^{-1/(p-1)} dx \right)^{1/p'} \leq C \qquad (3)$$

for all $I \in \mathcal{D}$.

If w is a dyadic A_p weight, then $w^{-1/(p-1)}$ satisfies the dyadic reverse doubling condition. The proof of this fact will be given in the proof of the following corollary.

COROLLARY 1.1 *Let $1 < p < q < \infty$ and w be a dyadic A_p weight. Let $\{\mu_I\}_{I \in \mathcal{D}}$ be nonnegative numbers. Then the following two statements are equivalent.*

  (i) *There is a positive constant C such that*

$$\sum_{I \in \mathcal{D}} \mu_I \langle \varphi \rangle_I^q \leq C \left(\int_{R^n} \varphi(x)^p w(x) dx \right)^{q/p} \qquad (4)$$

     for all nonnegative locally integrable functions φ.
 (ii) *There is a positive constant C' such that*

$$\mu_I \leq C' \left(\int_I w(x) dx \right)^{q/p}$$

     for all $I \in \mathcal{D}$.

2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.1

Proof of Theorem 1.1   First we show that (i) implies (ii). We fix a $I \in \mathcal{D}$. In the inequality in (i) we set $\varphi(x) = w(x)^{-1/(p-1)}\chi_I(x)$. Then we have

$$\mu_I \left(\frac{1}{|I|} \int_I w(x)^{-1/(p-1)}dx \right)^q \leq C \left(\int_I w(x)^{-1/(p-1)}dx \right)^{q/p}.$$

Hence we get

$$\mu_I \leq C|I|^q \left(\int_I w(x)^{-1/(p-1)}dx \right)^{-q/p'}.$$

Next we shall prove that (ii) implies (i). This is a consequence of the inequality

$$\sum_{I \in \mathcal{D}} |I|^q \left(\int_I w(x)^{-1/(p-1)}dx \right)^{-q/p'} \langle \varphi \rangle_I^q \leq C \left(\int_{\mathbf{R}^n} \varphi(x)^p w(x)dx \right)^{q/p}. \quad (5)$$

Hence we shall prove (5). We fix a $I \in \mathcal{D}$. It is suffice to show

$$\sum_{J \subset I, J \in \mathcal{D}} |J|^q \left(\int_J w(x)^{-1/(p-1)}dx \right)^{-q/p'} \langle \varphi \rangle_J^q \leq C \left(\int_I \varphi(x)^p w(x)dx \right)^{q/p}$$

$$(6)$$

for all nonnegative locally integrable functions φ where C is a constant which does not depend on I. In fact, we can prove (5) by the following argument. Let m be a positive integer and $K_{m,1}, K_{m,2}, \ldots, K_{m,2^n}$ be dyadic cubes which are obtained by dividing the cube $[-2^m, 2^m) \times \cdots \times [-2^m, 2^m)$ in $\mathbf{R}^n$ into 2^n equal parts.

If we apply (6) to $I = K_{m,i}$, $i = 1, \ldots, 2^n$, and if we let $m \to \infty$, then we have

$$\sum_{J \subset K_i, J \in \mathcal{D}} |J|^q \left(\int_J w(x)^{-1/(p-1)}dx \right)^{-q/p'} \langle \varphi \rangle_J^q \leq C \left(\int_{K_i} \varphi(x)^p w(x)dx \right)^{q/p}$$

for $i = 1, \ldots, 2^n$ where $K_i = \bigcup_{m \geq 1} K_{m,i}$. (5) is a consequence of these inequalities.

We shall prove (6). Now the following lemma holds.

LEMMA 2.1   *Let n be a positive integer, $1 < p < q < \infty$, and $0 < b < 2^n$. Let*

$$D = \{(F, f, v) : 0 \leq F, 0 < v, 0 \leq f \leq F^{1/p} v^{1/p'}\}.$$

Then there is a positive constant c such that

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} \geq c\frac{f^q}{v^{q/p'}} + \frac{1}{2^{nq/p}} \sum_{i=1}^{2^n} \left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p}$$

for all (F, f, v), $(F_i, f_i, v_i) \in D$, $i = 1, \ldots, 2^n$, such that

$$F = \frac{F_1 + \cdots + F_{2^n}}{2^n}, \quad f = \frac{f_1 + \cdots + f_{2^n}}{2^n}, \quad v = \frac{v_1 + \cdots + v_{2^n}}{2^n}$$

and

$$v_i \leq bv, \quad i = 1, \ldots, 2^n. \tag{7}$$

The proof of Lemma 2.1 will be given in Section 3.

Let D be the domain in Lemma 2.1. For $(F, f, v) \in D$ we set

$$B(F, f, v) = \frac{1}{c}\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p}$$

where c is the constant in Lemma 2.1. Let φ be a nonnegative measurable function such that

$$\int_I \varphi(x)^p w(x)dx < \infty.$$

We use the notation

$$F_A = \frac{1}{|A|}\int_A \varphi(x)^p w(x)dx, \quad f_A = \frac{1}{|A|}\int_A \varphi(x)dx$$

and

$$v_A = \frac{1}{|A|}\int_A w(x)^{-1/(p-1)}dx,$$

for a measurable set A in I such that $|A| \neq 0$.

Then we have $(F_I, f_I, v_I) \in D$. In fact, by Hölder's inequality, we have

$$\frac{1}{|I|} \int_I \varphi(x) dx \leq \left(\frac{1}{|I|} \int_I \varphi(x)^p w(x) dx \right)^{1/p} \left(\frac{1}{|I|} \int_I w(x)^{-p'/p} dx \right)^{1/p'}.$$

Hence we get

$$0 \leq f_I \leq F_I^{1/p} v_I^{1/p'}.$$

Let $I_1, I_2, \ldots, I_{2^n}$ be dyadic cubes which are obtained by dividing I into 2^n equal parts. Then we have

$$(F_{I_i}, f_{I_i}, v_{I_i}) \in D, \quad i = 1, \ldots, 2^n,$$

$$F_I = \frac{F_{I_1} + \cdots + F_{I_{2^n}}}{2^n}, \quad f_I = \frac{f_{I_1} + \cdots + f_{I_{2^n}}}{2^n},$$

$$v_I = \frac{v_{I_1} + \cdots + v_{I_{2^n}}}{2^n},$$

and

$$v_{I_i} \leq b v_I, \quad i = 1, \ldots, 2^n,$$

by the dyadic reverse doubling condition for $w^{-1/(p-1)}$, where $b = 2^n d^{-1}$ and d is the constant in the dyadic reverse doubling condition for $w^{-1/(p-1)}$. Since $d > 1$, we have $b < 2^n$.

Hence, by Lemma 2.1, we have

$$B(F_I, f_I, v_I) \geq \frac{f_I^q}{v_I^{q/p'}} + \frac{1}{2^{nq/p}} \sum_{i=1}^{2^n} B(F_{I_i}, f_{I_i}, v_{I_i}).$$

Therefore the inequality

$$|I|^{q/p} B(F_I, f_I, v_I) \geq |I|^{q/p} v_I^{-q/p'} f_I^q + \sum_{i=1}^{2^n} |I_i|^{q/p} B(F_{I_i}, f_{I_i}, v_{I_i})$$

holds.

We apply this inequality to I_i, $i = 1, \ldots, 2^n$, in place of I. Repeating this argument, we have, for $k \in \mathbf{N}$,

$$|I|^{q/p} B(F_I, f_I, v_I) \geq \sum_{J \subset I, J \in \mathcal{D}, |J| \geq 2^{-nk}|I|} |J|^{q/p} v_J^{-q/p'} f_J^q$$

$$+ \sum_{J \subset I, J \in \mathcal{D}, |J| = 2^{-n(k+1)}|I|} |J|^{q/p} B(F_J, f_J, v_J).$$

Since $B(F_J, f_J, v_J) \geq 0$, we get

$$|I|^{q/p} B(F_I, f_I, v_I) \geq \sum_{J \subset I, J \in \mathcal{D}, |J| \geq 2^{-nk}|I|} |J|^{q/p} v_J^{-q/p'} f_J^q.$$

Letting $k \to \infty$, we have

$$\sum_{J \subset I, J \in \mathcal{D}} |J|^q \left(\int_J w(x)^{-1/(p-1)} dx \right)^{-q/p'} \langle \varphi \rangle_J^q \leq |I|^{q/p} B(F_I, f_I, v_I)$$

$$= |I|^{q/p} \frac{1}{c} \left(F_I - \frac{f_I^p}{2 v_I^{p/p'}} \right)^{q/p} \leq c' |I|^{q/p} F_I^{q/p}$$

$$= c' \left(\int_I \varphi(x)^p w(x) dx \right)^{q/p}.$$

Hence we proved (6).   Q.E.D

Proof of Corollary 1.1   First we shall show that if w is a dyadic A_p weight, then $w^{-1/(p-1)}$ satisfies the dyadic reverse doubling condition.

Let I be any dyadic cube in $\mathbf{R}^n$. Let $I_1, \ldots, I_{2^n}$ be dyadic sub-cubes of I which are obtained by dividing I into 2^n equal parts. We use the notation

$$u_A = \frac{1}{|A|} \int_A w(x) dx$$

and

$$v_A = \frac{1}{|A|} \int_A w(x)^{-1/(p-1)} dx$$

for a measurable set $A \subset I$ such that $|A| \neq 0$. Then we have

$$u_I = \frac{u_{I_1} + \cdots + u_{I_{2^n}}}{2^n}, \quad v_I = \frac{v_{I_1} + \cdots + v_{I_{2^n}}}{2^n}.$$

We remark that $u_{I_i} \leq 2^n u_I$ for all $i = 1, \ldots, 2^n$ by the first equality.
Since w is a dyadic A_p weight, we have

$$1 \leq u_I^{1/p} v_I^{1/p'} \leq K$$

and

$$1 \le u_{I_i}^{1/p} v_{I_i}^{1/p'} \le K, \quad \text{for all } i = 1, \ldots, 2^n,$$

where K is a positive constant which does not depend on I.

Now we have, for $i = 1, \ldots, 2^n$,

$$v_{I_i} \ge \frac{1}{u_{I_i}^{p'/p}} \ge \frac{1}{(2^n u_I)^{p'/p}} \ge \frac{v_I}{2^{np'/p} K^{p'}}.$$

Hence we get

$$v_{I_i} = 2^n v_I - \sum_{j \ne i} v_{I_j} \le \left(2^n - \frac{2^n - 1}{2^{np'/p} K^{p'}} \right) v_I = 2^n \left(1 - \frac{1 - 2^{-n}}{2^{np'/p} K^{p'}} \right) v_I.$$

Since

$$\frac{1 - 2^{-n}}{2^{np'/p} K^{p'}} < 1,$$

we conclude

$$d \int_{I_i} w(x)^{-1/(p-1)} dx \le \int_I w(x)^{-1/(p-1)} dx$$

for some $d > 1$. Hence $w^{-1/(p-1)}$ satisfies the dyadic reverse doubling condition.

Since w is a dyadic A_p weight, we have

$$|I| \left(\int_I w(x)^{-1/(p-1)} dx \right)^{-1/p'} \le \left(\int_I w(x) dx \right)^{1/p}$$

$$\le K |I| \left(\int_I w(x)^{-1/(p-1)} dx \right)^{-1/p'}$$

for all $I \in \mathcal{D}$. The corollary is easily proved by this inequality and Theorem 1.1.   Q.E.D.

3. PROOF OF LEMMA 2.1

We shall prove Lemma 2.1. In the proof we use the following two lemmas.

LEMMA 3.1   *Let $\alpha > 1$. Then there is a $\gamma > 0$ such that*

$$(x + y)^\alpha \geq \gamma \min \{x^\alpha, y^\alpha\} + x^\alpha + y^\alpha$$

for all x, $y \geq 0$.

LEMMA 3.2   *Let $1 < p < \infty$ and $0 \leq \alpha$, $0 \leq \beta$, $\alpha + \beta = 1$. Then*

$$\frac{f^p}{v^{p/p'}} \leq \alpha \frac{f_+^p}{v_+^{p/p'}} + \beta \frac{f_-^p}{v_-^{p/p'}}$$

for all $0 \leq f, f_+, f_-$, $0 < v$, v_+, v_- such that

$$f = \alpha f_+ + \beta f_-, \quad v = \alpha v_+ + \beta v_-.$$

Lemma 3.2 is a consequence of the convexity of the function $f^p/v^{p/p'}$ on the domain $\{(f, v) | 0 \leq f, 0 < v\}$. We can prove Lemmas 3.1 and 3.2 by easy calculations.

Proof of Lemma 2.1   Let δ be a sufficiently small positive number. We may assume that $f_1 \leq f_2 \leq \cdots \leq f_{2^n}$.
   First we consider the case $f_1 \geq \delta f$. Let

$$G = \frac{F_2 + \cdots + F_{2^n}}{2^n - 1}, \quad g = \frac{f_2 + \cdots + f_{2^n}}{2^n - 1},$$

and

$$u = \frac{v_2 + \cdots + v_{2^n}}{2^n - 1}.$$

Since

$$g \leq \frac{F_2^{1/p} v_2^{1/p'} + \cdots + F_{2^n}^{1/p} v_{2^n}^{1/p'}}{2^n - 1}$$

$$\leq \frac{(F_2 + \cdots + F_{2^n})^{1/p} (v_2 + \cdots v_{2^n})^{1/p'}}{2^n - 1} \leq G^{1/p} u^{1/p'},$$

we have $(G, g, u) \in D$.
   For simplicity we set

$$\alpha_1 = \frac{1}{2^n} \quad \text{and} \quad \beta_1 = \frac{2^n - 1}{2^n}.$$

Then we get

$$F = \alpha_1 F_1 + \beta_1 G, \quad f = \alpha_1 f_1 + \beta_1 g,$$
$$v = \alpha_1 v_1 + \beta_1 u, \quad v_1 \leq bv, \quad u \leq bv, \tag{8}$$

and

$$f_1 \geq \delta f, \quad g \geq \delta f, \tag{9}$$

where we used the condition (7) in the estimates of v_1 and u.

Then, by Lemma 3.2, the inequality

$$\left(F - \frac{f^p}{2v^{p/p'}} \right)^{q/p} \geq \left\{ \alpha_1 \left(F_1 - \frac{f_1^p}{2v_1^{p/p'}} \right) + \beta_1 \left(G - \frac{g^p}{2u^{p/p'}} \right) \right\}^{q/p}$$

holds. By Lemma 3.1 we have

$$\left(F - \frac{f^p}{2v^{p/p'}} \right)^{q/p} \geq \gamma \min \left\{ \alpha_1^{q/p} \left(F_1 - \frac{f_1^p}{2v_1^{p/p'}} \right)^{q/p}, \beta_1^{q/p} \left(G - \frac{g^p}{2u^{p/p'}} \right)^{q/p} \right\}$$
$$+ \alpha_1^{q/p} \left(F_1 - \frac{f_1^p}{2v_1^{p/p'}} \right)^{q/p} + \beta_1^{q/p} \left(G - \frac{g^p}{2u^{p/p'}} \right)^{q/p}. \tag{10}$$

Since $(F_1, f_1, v_1), (G, g, u) \in D$, we have

$$F_1 - \frac{f_1^p}{2v_1^{p/p'}} \geq \frac{f_1^p}{2v_1^{p/p'}} \geq 2^{-1} \delta^p b^{-p/p'} \frac{f^p}{v^{p/p'}} \tag{11}$$

and

$$G - \frac{g^p}{2u^{p/p'}} \geq \frac{g^p}{2u^{p/p'}} \geq 2^{-1} \delta^p b^{-p/p'} \frac{f^p}{v^{p/p'}} \tag{12}$$

where we used (8) and (9).

Furthermore, since

$$G - \frac{g^p}{2u^{p/p'}} \geq \frac{1}{2^n - 1} \sum_{i=2}^{2^n} \left(F_i - \frac{f_i^p}{2v_i^{p/p'}} \right)$$

by Lemma 3.2, we have

$$\left(G - \frac{g^p}{2u^{p/p'}}\right)^{q/p} \geq \frac{1}{(2^n - 1)^{q/p}} \sum_{i=2}^{2^n} \left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p}. \qquad (13)$$

Hence, by (10), (11), (12) and (13), we conclude that

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} \geq c\frac{f^q}{v^{q/p'}} + \frac{1}{2^{nq/p}} \sum_{i=1}^{2^n} \left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p}.$$

Next we consider the case $f_N < \delta f$ and $f_{N+1} \geq \delta f$ for some N such that $1 \leq N \leq 2^n - 2$. If $n = 1$, then this case does not occur. Let

$$G_1 = \frac{F_1 + \cdots + F_{N+1}}{N + 1}, \quad G_2 = \frac{F_{N+2} + \cdots + F_{2^n}}{2^n - N - 1},$$

$$g_1 = \frac{f_1 + \cdots + f_{N+1}}{N + 1}, \quad g_2 = \frac{f_{N+2} + \cdots + f_{2^n}}{2^n - N - 1},$$

and

$$u_1 = \frac{v_1 + \cdots + v_{N+1}}{N + 1}, \quad u_2 = \frac{v_{N+2} + \cdots + v_{2^n}}{2^n - N - 1}.$$

Then we have (G_1, g_1, u_1), $(G_2, g_2, u_2) \in D$.

For simplicity we set

$$\alpha_2 = \frac{N + 1}{2^n} \quad \text{and} \quad \beta_2 = \frac{2^n - N - 1}{2^n}.$$

Then we get

$$F = \alpha_2 G_1 + \beta_2 G_2, \quad f = \alpha_2 g_1 + \beta_2 g_2,$$
$$v = \alpha_2 u_1 + \beta_2 u_2, \quad u_1 \leq bv, \quad u_2 \leq bv, \qquad (14)$$

and

$$g_1 \geq \frac{\delta}{N + 1} f, \quad g_2 \geq \delta f. \qquad (15)$$

Then, by Lemma 3.2, the inequality

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} \geq \left\{\alpha_2\left(G_1 - \frac{g_1^p}{2u_1^{p/p'}}\right) + \beta_2\left(G_2 - \frac{g_2^p}{2u_2^{p/p'}}\right)\right\}^{q/p}$$

holds. By Lemma 3.1 we have

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} \geq \gamma \min\left\{\alpha_2^{q/p}\left(G_1 - \frac{g_1^p}{2u_1^{p/p'}}\right)^{q/p}, \beta_2^{q/p}\left(G_2 - \frac{g_2^p}{2u_2^{p/p'}}\right)^{q/p}\right\}$$

$$+ \alpha_2^{q/p}\left(G_1 - \frac{g_1^p}{2u_1^{p/p'}}\right)^{q/p} + \beta_2^{q/p}\left(G_2 - \frac{g_2^p}{2u_2^{p/p'}}\right)^{q/p}. \quad (16)$$

Now we have

$$G_1 - \frac{g_1^p}{2u_1^{p/p'}} \geq \frac{g_1^p}{2u_1^{p/p'}} \geq 2^{-1}\left(\frac{\delta}{N+1}\right)^p b^{-p/p'} \frac{f^p}{v^{p/p'}} \quad (17)$$

and

$$G_2 - \frac{g_2^p}{2u_2^{p/p'}} \geq \frac{g_2^p}{2u_2^{p/p'}} \geq 2^{-1}\delta^p b^{-p/p'} \frac{f^p}{v^{p/p'}}. \quad (18)$$

Furthermore, since

$$G_1 - \frac{g_1^p}{2u_1^{p/p'}} \geq \frac{1}{N+1}\sum_{i=1}^{N+1}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)$$

and

$$G_2 - \frac{g_2^p}{2u_2^{p/p'}} \geq \frac{1}{2^n - N - 1}\sum_{i=N+2}^{2^n}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)$$

by Lemma 3.1, we have

$$\left(G_1 - \frac{g_1^p}{2u_1^{p/p'}}\right)^{q/p} \geq \frac{1}{(N+1)^{q/p}}\sum_{i=1}^{N+1}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p} \quad (19)$$

and

$$\left(G_2 - \frac{g_2^p}{2u_2^{p/p'}}\right)^{q/p} \geq \frac{1}{(2^n - N - 1)^{q/p}}\sum_{i=N+2}^{2^n}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p}. \quad (20)$$

Hence, by (16), (17), (18), (19) and (20), we conclude that

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} \geq c\frac{f^q}{v^{q/p'}} + \frac{1}{2^{nq/p}}\sum_{i=1}^{2^n}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p}.$$

Next we consider the case $f_{2^n-1} < \delta f$ and $f_{2^n} \geq \delta f$. Let

$$G' = \frac{F_1 + \cdots + F_{2^n-1}}{2^n - 1}, \quad g' = \frac{f_1 + \cdots + f_{2^n-1}}{2^n - 1},$$

and

$$u' = \frac{v_1 + \cdots + v_{2^n-1}}{2^n - 1}.$$

Then we have $(G', g', u') \in D$.
For simplicity we set

$$\alpha_3 = \frac{2^n - 1}{2^n} \quad \text{and} \quad \beta_3 = \frac{1}{2^n}.$$

Then we get

$$F = \alpha_3 G' + \beta_3 F_{2^n}, \quad f = \alpha_3 g' + \beta_3 f_{2^n}, \quad v = \alpha_3 u' + \beta_3 v_{2^n},$$

and

$$u' \leq bv, \quad v_{2^n} \leq bv.$$

Since

$$g' < \delta f,$$

the inequality

$$f_{2^n} = \frac{f - \alpha_3 g'}{\beta_3} \geq \frac{1 - \alpha_3\delta}{\beta_3}f \tag{21}$$

holds.

By Lemma 3.1 we have

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} = \left(\alpha_3 G' + \beta_3 \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}} - \frac{f^p}{2v^{p/p'}} + \beta_3 F_{2^n} - \beta_3 \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}}\right)^{q/p}$$

$$\geq \gamma \min \left\{\left(\alpha_3 G' + \beta_3 \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}} - \frac{f^p}{2v^{p/p'}}\right)^{q/p},\right.$$

$$\left.\beta_3^{q/p}\left(F_{2^n} - \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}}\right)^{q/p}\right\}$$

$$+ \left(\alpha_3 G' + \beta_3 \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}} - \frac{f^p}{2v^{p/p'}}\right)^{q/p}$$

$$+ \beta_3^{q/p}\left(F_{2^n} - \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}}\right)^{q/p}.$$

By (21) we have the inequality

$$\beta_3 \frac{f^p_{2^n}}{v^{p/p'}_{2^n}} - \frac{f^p}{v^{p/p'}} \geq \left\{\frac{(1 - \alpha_3\delta)^p}{(\beta_3 b)^{p-1}} - 1\right\}\frac{f^p}{v^{p/p'}}. \qquad (22)$$

Since $b < 2^n$, we get

$$\beta_3 b = \frac{b}{2^n} < 1.$$

Hence, for sufficiently small δ, we have

$$\frac{(1 - \alpha_3\delta)^p}{(\beta_3 b)^{p-1}} - 1 > 0.$$

Hence we have

$$\alpha_3 G' + \beta_3 \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}} - \frac{f^p}{2v^{p/p'}} \geq c\frac{f^p}{v^{p/p'}}$$

by (22) and

$$F_{2^n} - \frac{f^p_{2^n}}{2v^{p/p'}_{2^n}} \geq 2^{-1}\delta b^{-p/p'}\frac{f^p}{v^{p/p'}}.$$

Furthermore, by Lemma 3.2, we have

$$\beta_3 \frac{f_{2^n}^p}{v_{2^n}^{p/p'}} - \frac{f^p}{v^{p/p'}} \geq -\alpha_3 \frac{g'^p}{u'^{p/p'}}.$$

Hence we get

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} \geq c' \frac{f^q}{v^{q/p'}} + \alpha_3^{q/p}\left(G' - \frac{g'^p}{2u'^{p/p'}}\right)^{q/p}$$
$$+ \beta_3^{q/p}\left(F_{2^n} - \frac{f_{2^n}^p}{2v_{2^n}^{p/p'}}\right)^{q/p}.$$

Since

$$G' - \frac{g'^p}{2u'^{p/p'}} \geq \frac{1}{2^n - 1}\sum_{i=1}^{2^n-1}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right),$$

we have

$$\left(G' - \frac{g'^p}{2u'^{p/p'}}\right)^{q/p} \geq \frac{1}{(2^n - 1)^{q/p}}\sum_{i=1}^{2^n-1}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p}.$$

Hence we conclude

$$\left(F - \frac{f^p}{2v^{p/p'}}\right)^{q/p} \geq c' \frac{f^q}{v^{q/p'}} + \frac{1}{2^{nq/p}}\sum_{i=1}^{2^n}\left(F_i - \frac{f_i^p}{2v_i^{p/p'}}\right)^{q/p}.$$

Finally we remark that the case $f_{2^n} < \delta f$ does not occur for sufficiently small δ.   Q.E.D.

4. APPLICATIONS

In this section we shall study the weighted norm inequalities for dyadic fractional maximal operators. The result is a corollary of Theorem 4.2.2. in [1, p. 161]. We give an alternate proof of it.

Let $0 < \alpha < n$. For a locally integrable function φ we define the dyadic fractional maximal function $M_\alpha^d \varphi$ by

$$M_\alpha^d \varphi(x) = \sup_{x \in I, I \in \mathcal{D}} \frac{1}{|I|^{1-\alpha/n}} \int_I |\varphi(y)| dy \quad (x \in \mathbf{R}^n).$$

THEOREM 4.1 *Let $1 < p < q < \infty$ and $0 < \alpha < n$. Let w be a nonnegative locally integrable function on $\mathbf{R}^n$. We assume that $w^{-1/(p-1)}$ satisfies the dyadic reverse doubling condition. Let σ be a nonnegative locally integrable function on $\mathbf{R}^n$. Then the following two statements are equivalent.*

(i) *There is a positive constant C such that*

$$\left(\int_{\mathbf{R}^n} M_\alpha^d \varphi(x)^q \sigma(x) dx \right)^{1/q} \le C \left(\int_{\mathbf{R}^n} \varphi(x)^p w(x) dx \right)^{1/p} \quad (23)$$

  for all nonnegative locally integrable functions φ.

(ii) *There is a positive constant $K > 0$ such that*

$$|I|^{1/q-1/p+\alpha/n} \left(\frac{1}{|I|} \int_I \sigma(x) dx \right)^{1/q} \left(\frac{1}{|I|} \int_I w(x)^{-1/(p-1)} dx \right)^{1/p'} \le K \quad (24)$$

  for all $I \in \mathcal{D}$.

Proof First we shall show that (i) implies (ii). Let I be any dyadic cube in $\mathbf{R}^n$. In the inequality in (i) we set

$$\varphi(x) = w(x)^{-1/(p-1)} \chi_I(x).$$

Then we get

$$\left(\int_I M_\alpha^d \varphi(x)^q \sigma(x) dx \right)^{1/q} \le C \left(\int_I w(x)^{-1/(p-1)} dx \right)^{1/p}.$$

For $x \in I$ we have

$$M_\alpha^d \varphi(x) \ge \frac{1}{|I|^{1-\alpha/n}} \int_I w(y)^{-1/(p-1)} dy.$$

Hence we get

$$\frac{1}{|I|^{1-\alpha/n}} \int_I w(x)^{-1/(p-1)} dx \left(\int_I \sigma(x) dx \right)^{1/q} \le C \left(\int_I w(x)^{-1/(p-1)} dx \right)^{1/p}$$

This inequality is equivalent to (24).

Next we shall show that (ii) implies (i). The proof is similar to the arguments in Nazarov and Treil [2, p. 817]. Let φ be a nonnegative locally integrable function on $\mathbf{R}^n$. For every $x \in \mathbf{R}^n$, we choose a $I(x) \in \mathcal{D}$ such that

$$M_\alpha \varphi(x) \leq \frac{2}{|I(x)|^{1-\alpha/n}} \int_{I(x)} \varphi(y)dy. \tag{25}$$

For each $I \in \mathcal{D}$ set

$$E_I = \{x \in I : I(x) = I\}.$$

Then we have

$$E_I \subset I,$$
$$E_I \cap E_J = \emptyset \quad \text{for all } I, J \in \mathcal{D}, I \neq J,$$

and

$$\bigcup_{I \in \mathcal{D}} E_I = \mathbf{R}^n.$$

By (25) we have

$$\int_{\mathbf{R}^n} M_\alpha^d \varphi(x)^q \sigma(x)dx = \sum_{I \in \mathcal{D}} \int_{E_I} M_\alpha^d \varphi(x)^q \sigma(x)dx$$

$$\leq \sum_{I \in \mathcal{D}} \int_{E_I} \left(\frac{2}{|I|^{1-\alpha/n}} \int_I \varphi(y)dy \right)^q \sigma(x)dx$$

$$\leq 2^q \sum_{I \in \mathcal{D}} |I|^{\alpha q/n} \langle \varphi \rangle_I^q \int_{E_I} \sigma(x)dx. \tag{26}$$

Since

$$|I|^{\alpha q/n - q} \int_{E_I} \sigma(x)dx \left(\int_I w(x)^{-1/(p-1)}dx \right)^{q/p'}$$

$$\leq \left\{ |I|^{1/q-1/p+\alpha/n} \left(\frac{1}{|I|} \int_I \sigma(x)dx \right)^{1/q} \right.$$

$$\left. \left(\frac{1}{|I|} \int_I w(x)^{-1(p-1)}dx \right)^{1/p'} \right\}^q \leq K^q,$$

we get

$$|I|^{\alpha q/n} \int_{E_I} \sigma(x)dx \leq K^q |I|^q \left(\int_I w(x)^{-1/(p-1)}dx \right)^{-q/p'}.$$

Hence (26) is bounded by

$$2^q K^q \sum_{I \in \mathcal{D}} |I|^q \left(\int_I w(x)^{-1/(p-1)} dx \right)^{-q/p'} \langle \varphi \rangle_I^q \le C \left(\int_{\mathbf{R}^n} \varphi(x)^p w(x) dx \right)^{1/p}$$

by Theorem 1.1.   Q.E.D.

Next we shall give a result on fractional integral operators. The result is a corollary of Theorem 4.2.2 in [1]. We give it here because it is not mentioned in [1].

Let $0 < \alpha < n$ and I_α be the fractional integral operator, that is,

$$I_\alpha \varphi(x) = \int_{\mathbf{R}^n} \frac{\varphi(y)}{|x - y|^{n-\alpha}} dy \quad (x \in \mathbf{R}^n).$$

Let σ be an A_∞ weight on $\mathbf{R}^n$, that is, σ satisfies the following property: there are constants $c, \delta > 0$ so that, for each cube Q

$$\frac{\sigma(E)}{\sigma(Q)} \le c \left(\frac{|E|}{|Q|} \right)^\delta$$

for all measurable set E in Q, where $\sigma(E) = \int_E \sigma(x) dx$.

As Pérez pointed out in [3, p. 34], we have

$$\int_{\mathbf{R}^n} |I_\alpha \varphi(x)|^q \sigma(x) dx \approx \int_{\mathbf{R}^n} M_\alpha^d \varphi(x)^q \sigma(x) dx$$

for $0 < q < \infty$. Hence we have the following result.

COROLLARY 4.1   Let $1 < p < q < \infty$ and $0 < \alpha < n$. Let w be a nonnegative locally integrable function on $\mathbf{R}^n$. We assume that $w^{-1/(p-1)}$ satisfies the dyadic reverse doubling condition. Let σ be an A_∞ weight on $\mathbf{R}^n$. Then the following two statements are equivalent.

(i)   There is a positive constant C such that

$$\left(\int_{\mathbf{R}^n} I_\alpha \varphi(x)^q \sigma(x) dx \right)^{1/q} \le C \left(\int_{\mathbf{R}^n} \varphi(x)^p w(x) dx \right)^{1/p} \qquad (27)$$

for all nonnegative locally integrable functions φ.

(ii) *There is a positive constant $K > 0$ such that*

$$|I|^{1/q-1/p+\alpha/n} \left(\frac{1}{|I|} \int_I \sigma(x)dx \right)^{1/q} \left(\frac{1}{|I|} \int_I w(x)^{-1/(p-1)}dx \right)^{1/p'} \le K$$

for all $I \in \mathcal{D}$.

Remark 4.1   In [3, p. 34] Pérez proved (27) assuming that $w^{-1/(p-1)}$ is a dyadic A_∞ weight. If $w^{-1/(p-1)}$ is a dyadic A_∞ weight, then we can prove that $w^{-1/(p-1)}$ satisfies the dyadic reverse doubling condition. Hence this corollary includes Pérez's result.

Remark 4.2   In [4, Theorem 1] Sawyer and Wheeden proved that (27) holds if σ and $w^{-1/(p-1)}$ satisfy the reverse doubling condition and (ii). Our Corollary 4.1 is not a direct consequence of Sawyer and Wheeden's result because we assumed that $w^{-1/(p-1)}$ satisfies the "dyadic" reverse doubling condition.

Remark 4.3   By Theorem 4.1 and the argument in [3, p. 39], we can get a result on weighted norm inequalities for ordinary fractional maximal operators. It is a corollary of Theorem 4.2.2 of [1].

Acknowledgment

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