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Viscosity Solutions of Two Classes of Coupled Hamilton-Jacobi-Bellman Equations*

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This paper studies viscosity solutions of two sets of linearly coupled Hamilton-Jacobi-Bellman (HJB) equations (one for finite horizon and the other one for infinite horizon) which arise in the optimal control of nonlinear piecewise deterministic systems where the controls could be unbounded. The controls enter through the system dynamics as well as the transitions for the underlying Markov chain process, and are allowed to depend on both the continuous state and the current state of the Markov chain. The paper establishes the existence and uniqueness of viscosity solutions for these two sets of HJB equations, whose Hamiltonian structures are different from the standard ones.

Keywords and Phrases: Viscosity solutions; Coupled Hamilton-Jacobi-Bellman equations; Piecewise deterministic systems

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1. INTRODUCTION

This paper studies viscosity solutions of *first order*, *linearly coupled* partial differential equations of the following types, where Ω is a subset

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of $\mathbb{R}^n$ (allowed also to be $\mathbb{R}^n$ itself):

(I) Finite horizon:

$$\beta V(t,x,i) + \sup_{u^1 \in \mathbb{R}^r, u^2 \in U_2(i)} \left[-\mathcal{A}^{(u^1,u^2)} V(t,x,i) - L(x,i,u^1) \right] = 0 \quad \text{in } (0,t_f] \times \mathbb{R}^n$$

$$V(t_f,x,i) = g(x,i) \quad \text{on } \Omega \ i \in \mathcal{S} := \{1,2,\ldots,s\}, \quad \text{where } s \text{ is a positive integer,} \tag{1.1}$$

and $U_2(i)$ is a finite set for each $i \in \mathcal{S}$.

(II) Infinite horizon:

$$\beta V(x,i) + \sup_{u^1 \in \mathbb{R}^r, u^2 \in U_2(i)} \left[-G^{(u^1,u^2)} V(x,i) - L(x,i,u^1) \right] = 0 \quad \text{in } \Omega \text{ for each } i \in \mathcal{S}, \tag{1.2}$$

where $U_2(i)$ is again a finite set for each $i \in \mathcal{S}$. Here, the operators $\mathcal{A}$ and G are defined as follows for each $u^1 \in \mathbb{R}^r$, $a \in U_2(i)$, $i \in \mathcal{S}$:

$$\mathcal{A}^{(u^1,a)} V(t,x,i) := \frac{\partial V(t,x,i)}{\partial t} + [D_x V(t,x,i)] F(x,u^1,i) + \sum_{j \in \mathcal{S}} \lambda_{iaj} V(t,x,j)$$

and

$$G^{(u^1,a)} V(x,i) := [D_x V(x,i)] F(x,u^1,i) + \sum_{j \in \mathcal{S}} \lambda_{iaj} V(x,j),$$

with

$$F(x,u^1,i) := f(x,i) + B(x,i)u^1,$$
$$L(x,i,u^1) := Q(x,i) + \langle u^1, R(x,i)u^1 \rangle$$

where the roles of various terms introduced will be clarified shortly, with precise technical conditions given in the next section. The coupled PDE's (1.1) and (1.2) are the dynamic programming equations for the following optimum stochastic control problem with piecewise deterministic dynamics: Consider a dynamic system which is nonlinear in the state and linear in the piecewise continuous control u^1:

$$\frac{dx}{dt}(t) = f(x(t), \theta(t)) + B(x(t), \theta(t))u^1(t) \qquad (1.3)$$
$$x(0) = x_0$$

where $x \in \mathbb{R}^n$, x_0 is a fixed (known) initial state, u^1 is a control, taking values in $U_1 = \mathbb{R}^r$, and $\theta(t)$ is a controlled, continuous time Markov process, taking values in a finite state space $\mathcal{S}$, of cardinality s. Transitions from state $i \in \mathcal{S}$ to $j \in \mathcal{S}$ occur at a rate controlled by a second controller, who chooses at time t an action $u^2(t)$ from a finite set $U_2(i)$ of actions available at state i. Let $U_2 := \cup_{i \in \mathcal{S}} U_2(i)$. The controlled rate matrix (of transitions within $\mathcal{S}$) is

$$\Lambda = \{\lambda_{i,a,j}\}, \quad i,j \in \mathcal{S}, \quad a = u^2(t) \in U_2(i)$$

where henceforth we drop the "commas" in the subscripts of λ. The λ_{iaj}'s are real numbers such that for any $i \neq j$, and $a \in U_2(i)$, $\lambda_{iaj} \geq 0$, and for all $a \in U_2(i)$ and $i \in \mathcal{S}$, $\lambda_{iai} = -\sum_{j \neq i} \lambda_{iaj}$. Fix some initial state i_0 of the controlled Markov chain $\mathcal{S}$, and the final time t_f (which may be infinite). Consider the class of policies $\mu^k \in \mathcal{U}_k$ for controller ($k = 1, 2$), whose elements (taking values in U_k) are of the form

$$u^k(t) = \mu^k(t, x(t), \theta(t)), \quad t \in [0, t_f). \qquad (1.4)$$

For the finite-horizon case, μ^k is taken to be piecewise continuous in the first argument and local Lipschitz in second argument and measurable in the third argument. In the infinite-horizon case, the dependence of μ^k on t is dropped, but otherwise it is defined the same way. Define $\mathcal{X} = \mathbb{R}^n \times \mathcal{S}$ to be the combined state space of the system and $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$ to be the class of admissible multi-strategies $\mu := (u^1, u^2)$, appropriately defined depending on whether t_f is finite or infinite. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product. Define a running cost $L : \mathcal{X} \times \mathcal{U}_1 \to \mathbb{R}$ as

$$L(x, i, u^1) = Q(x, i) + \langle u^1, R(x, i)u^1 \rangle,$$

where the definitions of Q and R will be made precise later in Section 2.

To any fixed initial state (x_0, i_0) and a multi-strategy $\mu \in \mathcal{U}$, there corresponds a unique probability measure $P^{\mu}_{x_0,i_0}$ on the canonical probability space of the states and actions of the players, equipped with the standard Borel σ-algebra. Denote by $E^{\mu}_{x_0,i_0}$ the expectation operator corresponding to $P^{\mu}_{x_0,i_0}$.

For each fixed initial state (x_0, i_0), multi-strategy $\mu \in \mathcal{U}$, and a finite horizon of duration t_f, the discounted (expected) cost function is defined as

$$J_\beta(x_0,i_0,\mu;t_f):=E^{\mu}_{x_0,i_0}\left\{g(x(t_f),\theta(t_f))e^{-\beta t_f}+\int_0^{t_f} e^{-\beta t}L(x(t),\theta(t),u^1(t))dt\right\} \tag{1.5}$$

where g is a terminal cost function, $\beta \geq 0$ is the discount factor, and the expectation is over the joint process $\{x, \theta\}$. For t_f infinite, a corresponding discounted cost function is defined as:

$$J_\beta(x_0, i_0, \mu) := E^{\mu}_{x_0,i_0}\left\{ \int_0^{\infty} e^{-\beta t}L(x(t), \theta(t), u^1(t))dt\right\} \tag{1.6}$$

We further denote the *cost-to-go* from any time-state pair $(t; x, i)$, under a multi-strategy $\mu \in \mathcal{U}$ by

$$J_\beta(t; x, i, \mu; t_f) := E^{\mu}_{x,i}\left\{g(x(t_f), \theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)}L(x(\tau), \theta(\tau), u^1(\tau))d\tau\right\}$$

and

$$J_\beta(t; x, i, \mu) := E^{\mu}_{x,i}\left\{ \int_t^{\infty} e^{-\beta(\tau-t)}L(x(\tau), \theta(\tau), u^1(\tau))d\tau\right\},$$

for the finite-horizon and infinite-horizon cases, respectively. The optimal *value functions* are then defined by:

$$V(t; x, i; t_f) = \inf_{\mu^1 \in \mathcal{U}_1} \inf_{\mu^2 \in \mathcal{U}_2} J_\beta(t; x, i, \mu; t_f) \quad \text{finite-horizon case}$$

$$V(x, i) = \inf_{\mu^1 \in \mathcal{U}_1} \inf_{\mu^2 \in \mathcal{U}_2} J_\beta(t; x, i, \mu) \quad \text{infinite-horizon case}$$

Dynamic programming arguments (for background on the approach that can be used here, see [13, 19]), lead to the two coupled HJB equations (1.1) and (1.2), corresponding to the finite and infinite-horizon cases, respectively. More precisely, if these equations admit unique *viscosity solutions* on $\mathbb{R}^n$ – a concept that will be introduced in precise terms later in Section 2, then $V(t; x, i; t_f)$ and $V(t; x)$ thus defined constitute the optimal value functions for the finite-horizon and infinite-horizon cases, respectively. It is a verification of this existence of unique viscosity solutions that is the main goal of this paper.

Remark 1.1 If $R(x, i)$ is strictly positive definite for all x and i, then (1.1) can be written as

$$
\begin{aligned}
&-\frac{\partial V(t,x,i)}{\partial t} + \beta V(t,x,i) - \langle D_x V(t,x,i), f(x,i)\rangle \\
&- \inf_a \left\{ \sum_{j \in S} \lambda_{iaj} V(t,x,j) \right\} - Q(x,i) \\
&+ \frac{1}{4}\langle D_x V(t,x,i), B(x,i) R^{-1}(x,i) B(x,i)^T D_x V(t,x,i)\rangle = 0 \\
&V(t_f, x, i) = g(x,i),
\end{aligned}
\tag{1.7}
$$

with the associated minimizing controller u^1 being (since $D_x V$ may not be well defined, this expression is quite informal at this point. It will be made more precise in Section 5):

$$
\begin{aligned}
u^1(t) &= \mu^1(t, x(t), \theta(t)) \\
&= -R^{-1}(x(t), \theta(t)) B^T(x(t), \theta(t)) D_x V(t, x(t), \theta(t))
\end{aligned}
$$

Under the same condition, (1.2) can be written as

$$
\begin{aligned}
&\beta V(x,i) - \langle D_x V(x,i), f(x,i)\rangle - \inf_a \left\{ \sum_{j \in S} \lambda_{iaj} V(x,j) \right\} \\
&\quad - Q(x,i) + \frac{1}{4}\langle D_x V(x,i), B(x,i) R^{-1}(x,i) B(x,i)^T D_x V(x,i)\rangle = 0
\end{aligned}
\tag{1.8}
$$

with the corresponding minimizing controller u^1 in this case being

$$
\begin{aligned}
u^1(t) &= \mu^1(x(t), \theta(t)) \\
&= -R^{-1}(x(t), \theta(t)) B^T(x(t), \theta(t)) D_x V(x(t), \theta(t)).
\end{aligned}
$$

For future use, we introduce the Hamiltonian for both finite- and infinite-horizon cases by $\overline{H}(x,r,p) = \{\hat{H}_i(x,r,p)\}_{i=1}^s$ where $\hat{H}_i$ is given by

$$\hat{H}_i(x,r,p) = \beta r_i - \inf_a \left\{ \sum_{j \in S} \lambda_{iaj} r_j \right\} + H_i(x,p) \tag{1.9}$$

where $r=(r_1,\ldots,r_s)$, and

$$H_i(x,p) = -\langle p, f(x,i)\rangle - Q(x,i) + \frac{1}{4}\langle p, B(x,i)R^{-1}(x,i)B(x,i)^T p\rangle. \tag{1.10}$$

HJB equations are nonlinear partial differential equations, and it is well-known that in general such equations do not admit global classical solutions. Furthermore, the value functions of optimal control problems (and differential games as well) are not differentiable in general, and they are not even continuous for some classes of problems (*e.g.*, *cf.* [18, 21]). The theory of viscosity solutions, first introduced by Crandall and Lions, provides a convenient framework to study solutions of HJB equations. The questions of existence and uniqueness for viscosity solutions of HJB equations have been studied by a number of authors, and in particular, by Crandall and Lions [7, 8], Lions [16], Crandall [4], Evans [10], Ishii [14], Crandall, Evans and Lions [6], Fleming and Soner [13], Ball and Helton [3], and Evans and Souganidis [11].

The class of HJB equations (1.1) and (1.2) to be studied in this paper are different from the ones treated in all these earlier references, first because they are (linearly) coupled, and second because the Hamiltonian does not satisfy the "structure condition", which is:

$$|\hat{H}_i(x,r,p) - \hat{H}_i(y,r,q)| \le m_R(|p-q| + |x-y|) \tag{1.11}$$

where (x,p), $(y,q) \in \mathbb{R}^n \times B_R(0)$ and m_R is a modulus function (*cf.* [4, 7]). When the control set, U_1, is bounded, the structure condition (1.11) generally guarantees that HJB equations admit Lipschitz continuous viscosity solutions (see *e.g.* [4, 13, 17]). Another type of assumption in which the value function is required to satisfy certain growth condition is essentially to require U_1 to be compact [19]. Clearly the boundedness assumption on U_1 could be overly restrictive, a case in point being the linear-quadratic regulator problem (with piecewise deterministic dynamics), where the control cannot be taken to be bounded *a priori*, and hence (1.11) does not generally hold. Other assumptions that are commonly made to replace (1.11), for example polynomial growth assumptions on value functions [19], will also result in restrictions that are not natural in the framework adopted here. Moreover since the Hamiltonian is also a coupled system, the monotonicity property with respect to r no longer holds in general, *i.e.*, $\hat{H}(x,r,p)$ is not monotone with respect to r. Without this property and (1.11), the standard comparison theorem of [4] and [7] does not apply, and the proof of uniqueness becomes much more challenging.

Accordingly, in this paper we generalize the standard comparison theorem to cover the cases of such coupled HJB equations where the Hamiltonian does not satisfy the structure condition and does not have the monotonicity property, but instead, has a quadratic structure, which corresponds to a large class of nonlinear systems in the form (1.3). We prove the existence of solutions to (1.1) and (1.2), under assumptions quite different from the standard ones.

The structure of the paper is as follows. In Subsection 2.1, we list a set of assumptions on the HJB equations (1.1) and (1.2), necessary to be able to develop a reasonable theory. Subsection 2.2 provides the definition of a viscosity solution for coupled HJB equations (1.1) and (1.2). In Section 3, we show the existence of viscosity solutions to (1.1) and (1.2) by using a variation of Dynkin's formula. Generalized comparison theorems for the coupled HJB equations (1.1) and (1.2) are provided in Section 4. The proof of the uniqueness is totally different from that of [19], where a stronger assumption is imposed on the value functions. The paper ends with the concluding remarks of Section 5.

2. ASSUMPTIONS AND DEFINITIONS

2.1. Assumptions

(A1) For each i, f is an n-vector, and there exists a constant $L_f \geq 0$ such that

$$\sup_i \{|f(x,i) - f(y,i)|\} \leq L_f|x-y|, \quad x \in \mathbb{R}^n$$

(A2) For each i, $B(x, i)$ is an $n \times r$ matrix, and

$$\sup_i\{|B(x,i)|\} \le C_{b1}, \quad \sup_i\{|B(x,i) - B(y,i)|\} \le C_{b2}|x-y|, \quad \forall x, y \in \mathbb{R}^n$$

for some C_{b1}, $C_{b2} < \infty$.

(A3) For each i, $Q(\cdot, i): \mathbb{R}^n \to [0, +\infty)$, with

$$0 \le \sup_i\{Q(x,i)\} \le C_q|x|^2, \quad \forall x \in \mathbb{R}^n$$

(A4) Foreach$i$,$R(x, i)$isan$n \times n$matrixwith$R(x, i) = R(x, i)^T > 0$,$\forall x \in \mathbb{R}^n$, $\forall x \in \mathbb{R}^n$, and

$$\sup_i\{|R(x,i)|\} \le C_r, \quad \sup_i\{|R(x,i) - R(y,i)|\} \le C_r'|x-y|, \quad x, y \in \mathbb{R}^n$$

for some C_r, $C_r' > 0$, and there exists $L_R > 0$ such that

$$\sup_i\{|R^{-1}(x,i) - R^{-1}(y,i)|\} \le L_R|x-y|, \quad \forall x, y \in \mathbb{R}^n$$

(A5) For $i \neq j$, $0 \le \lambda_{iaj} \le C_\lambda$, where C_λ is a positive constant, and $\lambda_{iai} + \sum_{j \neq i} \lambda_{iaj} \equiv 0$, $1 \le i \le K$.

(A6) For each i and any $g(\cdot, i): \mathbb{R}^n \to [0, \infty)$,

$$\sup_i\{|g(x,i)|\} \le (1 + C_g)|x|^2,$$
$$\sup_i\{|g(x,i) - g(y,i)|\} \le C_g'(1 + |x| + |y|)|x-y|$$

for all x, $y \in \mathbb{R}^n$, where C_g, C_g' are positive constants.

(A7) β is a nonnegative constant.

(A8) For any $z \in \mathbb{R}^n$, there exists a nondecreasing function $\omega: \{0\} \cup \mathbb{R}^+ \to \{0\} \cup \mathbb{R}^+$ such that $\omega(0) = 0$, $\lim_{r \to +\infty} \omega(r)/r = +\infty$ and

$$\langle z, B(x,i)R^{-1}(x,i)B(x,i)^* z\rangle \ge \omega(|z|), \quad \forall x \in \mathbb{R}^n, \ \forall i \in \mathcal{S}.$$

Remark 2.1 When

$$f(x,i) = A(i)x, \ B(x,i) = B(i), \ Q(x,i) = x^T Q(i)x, \ R(x,i) = R(i),$$

and $g(x,i) = x^T Q_{t_f} x$, where $A(i)$, $B(i)$, $Q(i)$, $R(i)$ and $Q_{t_f}(i)$ are appropriate dimensional matrices which are dependent only on i, then Assumptions (A1)–(A4) and (A6) are automatically satisfied.

Remark 2.2 By Assumptions (A8) and (A4), we can see that for fixed $x \in \mathbb{R}^n$

$$H_i(x,p) \to +\infty \quad \text{as } |p| \to +\infty.$$

Throughout the paper, the following conventions will be adopted, unless otherwise indicated: (1) u^2 and a are used interchangeably to denote the second control; (2) by an abuse of notation $\mu^1(t)$ will be used to denote $\mu^1(x(t), \theta(t))$.

2.2. Definitions

DEFINITION 2.1 Let $\overline{V}$ be a vector function

$$\overline{V} = (V(\cdot,\cdot,1), V(\cdot,\cdot,2), \dots, V(\cdot,\cdot,s)) : ([0,t_f] \times \Omega)^s \to (\mathbb{R}^n)^s$$

(a) $\overline{V}$ is a *viscosity subsolution* of (1.1), if for any i, $V(\cdot,\cdot,i)$ is upper semi-continuous and

$$\beta\Phi(t_0,x_0,i) + \sup_{u^1,u^2} \left[-\mathcal{A}^{(u^1,u^2)}\Phi(t_0,x_0,i) - L(x_0,i,u^1)\right] \le 0 \text{ on } \Omega$$

$$\Phi(t_f,x,i) \le g(t_f,x,i) \text{ on } \Omega$$

whenever $\Phi(\cdot,\cdot,i) \in C^1([0,t_f] \times \Omega)$ is such that $V(t,x,i) - \Phi(t,x,i)$ attains a local maximum at (t_0,x_0) with $\Phi(t_0,x_0,j) = V(t_0,x_0,j)$ for each $j \in \mathcal{S}$, and $(u^1,u^2) \in U_1 \times U_2$;

(b) $\overline{V}$ is a *viscosity supersolution* of (1.1) if for any i, $V(\cdot,\cdot,i)$ is lower semi-continuous and

$$\beta\Phi(t_0,x_0,i) + \sup_{u^1,u^2} \left[-\mathcal{A}^{(u^1,u^2)}\Phi(t_0,x_0,i) - L(x_0,i,u^1)\right] \ge 0 \text{ on } \Omega$$

$$\Phi(t_f,x,i) \ge g(t_f,x,i)$$

whenever $\Phi(\cdot,\cdot,i) \in C^1([0,t_f] \times \Omega)$ is such that $V(t,x,i) - \Phi(t,x,i)$ attains a local minimum at (t_0,x_0) with $\Phi(t_0,x_0,j) = V(t_0,x_0,j)$ for each $j \in \mathcal{S}$, and $(u^1,u^2) \in U_1 \times U_2$;

(c) $\overline{V}$ is a *viscosity solution* of (1.1) if $\overline{V}$ is both a viscosity supersolution and a viscosity subsolution.

Remark 2.3 The statement "$V(t,x,i)-\Phi(t,x,i)$ attains a local maximum (respectively, minimum) at (t_0,x_0), with $i=i_0$" means that there exist $\eta_1, \eta_2>0$ and a subset $S\subseteq \mathcal{S}$, $i_0\in S$, such that when $(t,x)\in [B(t_0,\eta_1)\times B(x_0,\eta_2)]\cap[[0,t_f]\times\Omega]$ and $i\in S$, we have

$$V(t,x,i)-\Phi(t,x,i)\geq V(t_0,x_0,i_0)-\Phi(t_0,x_0,i_0)$$
$$(\textit{respectively}, V(t,x,i)-\Phi(t,x,i)\leq V(t_0,x_0,i_0)-\Phi(t_0,x_0,i_0))$$

The notion of a viscosity solution for (1.2) can be introduced analogously.

3. THE EXISTENCE OF A VISCOSITY SOLUTION

We first provide two propositions for convenience in later developments.

PROPOSITION 3.1 *For any $0\leq t\leq\tau\leq t_f$ and with $x(t)=x$, $\theta(t)=i$,*

$$V(t;x,i;t_f)\geq \inf_{\mu} E^{\mu}_{x,i}\left\{e^{-\beta(\tau-t)}\left[\int_t^{\tau} L(x(s),\theta(s),u^1(s))ds + V(\tau;x(\tau),\theta(\tau);t_f)\right]\right\} \quad (3.1)$$

In the case that $t_f=+\infty$, for any $t\geq 0$

$$V(x,i)\geq \inf_{\mu} E^{\mu}_{x,i}\left\{e^{-\beta t}\left[\int_0^t L(x(s),\theta(s),u^1(s))ds+V(x(t),\theta(t))\right]\right\} \quad (3.2)$$

Proof The results are the direct consequences of the definition of V as well as the assumption of $\beta\geq 0$. ■

The following proposition is a variant of Dynkin's formula.

PROPOSITION 3.2 *Define $\Phi:[0,t_f]\times\mathbb{R}^n\times\mathcal{S}\to\mathbb{R}$ such that the partial derivatives Φ_t, Φ_{x_i}, $i=1,\ldots,n$ are continuous. Then, for $x(t)=x$ and $\theta(t)=i$,*

$$E^{\mu}_{x,i}\Phi(s,x(s),\theta(s))-\Phi(t,x,i)=E^{\mu}_{x,i}\int_t^s \mathcal{A}^{(u^1,u^2)}\Phi(r,x(r),\theta(r))dr$$

where $\mathcal{A}^{(u^1,u^2)}$ *is defined as*

$$\mathcal{A}^{(u^1,u^2)}\Phi(t,x,i) := \Phi_t(t,x,i) + \Phi_x(t,x,i)F(x,u^1,i) + \sum_{j\in S}\lambda_{iaj}\Phi(t,x,j) \tag{3.3}$$

with $u^1 = \mu^1(t,x,i)$, $u^2 = a = \mu^2(t,x,i)$.

Proof See Appendix B of [13]

Remark 3.1 Suppose that the value function V is differentiable with respect to x and t. By Proposition 3.1, we have

$$0 \geq \inf_{\mu} E^{\mu}_{x,i}\left\{ e^{-\beta(\tau-t)}\int_t^{\tau} L(x(s),\theta(s),u^1(s))ds + e^{-\beta(\tau-t)}V(\tau;x(\tau),\theta(\tau);t_f) - V(t;x,i;t_f)\right\} \tag{3.4}$$

Since

$$\begin{aligned} &e^{-\beta(\tau-t)}V(\tau;x(\tau),\theta(\tau);t_f) - V(t;x,i;t_f) \\ &\quad = e^{-\beta(\tau-t)}[V(\tau;x(\tau),\theta(\tau);t_f) - V(t;x,i;t_f)] \\ &\quad - [1-e^{-\beta(\tau-t)}]V(t;x,i;t_f) \end{aligned} \tag{3.5}$$

using this in (3.4), and applying Proposition 3.2, we arrive at

$$0 \geq \inf_{\mu} E^{\mu}_{x,i}\left\{ \int_t^{\tau} e^{-\beta(s-t)}\left[L(x(s),\theta(s),\mu^1(s)) + \mathcal{A}^{(u^1,u^2)}V(s,x(s),\theta(s);t_f)\right]ds - [1-e^{-\beta(\tau-t)}]V(t;x,i;t_f)\right\} \tag{3.6}$$

Dividing (3.6) by $\tau - t$, and letting $\tau \downarrow t$, we have

$$\beta V(t;x,i;t_f) + \sup_{u^1,u^2}\left[-\mathcal{A}^{(u^1,u^2)}V(t;x,i;t_f) - L(x,i,u^1)\right] \geq 0.$$

On the other hand, according to the definition of V and Proposition 3.2,

$$\beta V(t,x,i) + \sup_{\mu}\left[-\mathcal{A}^{(u^1,u^2)}V(t,x,i)\right] \leq 0,$$

and since $L \geq 0$, we have

$$\beta V(t,x,i) + \sup_{u^1,u^2} \left[-\mathcal{A}^{(u^1,u^2)} V(t,x,i) - L(x,i,u^1) \right] \leq 0.$$

Therefore V satisfies the coupled first order partial differential equations (1.1). In general, the value function V is not differentiable with respect to either x or t, and hence we would like to explore the connection between the value functions $\{V(\cdot,\cdot,i,t_f)\}_{i=1}^{s}$ and the coupled HJB equations (1.1) in the viscosity sense. This brings us to the following:

THEOREM 3.3 *Under the Assumptions* $(A1)-(A5)$, *for each* i, *let*

$$V(t;x,i;t_f) = \inf_{\mu \in \mathcal{U}} E^{\mu}_{x,i} \left\{ g(x(t_f),\theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)} L(x(\tau),\theta(\tau),u^1(\tau))d\tau \right\}.$$

If $V(\cdot;\cdot,i;t_f) \in C([0,t_f] \times \mathbb{R}^n)$, *then* $\overline{V} = \{V(t;x,i;t_f)\}_{i\in\mathcal{S}}$ *is a viscosity solution of* (1.1).

Proof Suppose that to the contrary $\{V(t;x,i;t_f)\}_{i\in\mathcal{S}}$ is not a viscosity supersolution of (1.1). Then there would exist at least one $i_0 \in \mathcal{S}$, a pair $(t_0,x_0) \in [0,t_f] \times \mathbb{R}^n$, and a function $\Phi : [0,t_f] \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^n$, such that $V(\cdot;\cdot,i_0;t_f) - \Phi(\cdot,\cdot,i_0)$ has a local minimum. But there exists an $\varepsilon > 0$ such that

$$\beta\Phi(t_0,x_0,i_0) + \sup_{u^1,u^2} \left[-\mathcal{A}^{(u^1,u^2)} \Phi(t_0,x_0,i_0) - L(x_0,i_0,u^1) \right] \leq -\varepsilon \quad (3.7)$$

For any admissible control $\mu = (\mu^1,\mu^2)$, we now consider the following system:

$$\begin{aligned} \frac{dx}{dt}(t) &= f(x(t),\theta(t)) + B(x(t),\theta(t))\mu^1(t,x(t),\theta(t)) \\ x(t_0) &= x_0 \\ \theta(t_0) &= i_0 \end{aligned} \quad (3.8)$$

Note that $x(t) \to x_0$ as $t \downarrow t_0$. Hence for sufficiently small t,

$$\beta\Phi(t,x(t),\theta(t)) - \mathcal{A}^{(u^1,u^2)}\Phi(t,x(t),\theta(t)) - L(x(t),\theta(t),u^1) \leq -\varepsilon \quad (3.9)$$

Multiply both sides of (3.9) by $e^{-\beta(t-t_0)}$. Noting that $L \geq 0$, integrating both sides from t_0 to t, and applying Proposition 3.2, we have

$$\Phi(t_0, x_0, i_0) - E^{\mu}_{x_0, i_0}\{e^{-\beta(t-t_0)}\Phi(t, x(t), \theta(t))\} \leq -\varepsilon(t - t_0) \tag{3.10}$$

Since (t_0, x_0) is a local minimizer of $V(\cdot;\cdot, i_0; t_f) - \Phi(\cdot, \cdot, i_0)$,

$$V(t_0; x_0, i_0; t_f) - V(t; x(t), \theta(t); t_f) \leq \Phi(t_0, x_0, i_0) - \Phi(t, x(t), \theta(t)) \tag{3.11}$$

Therefore we arrive at

$$V(t_0; x_0, i_0; t_f) - E^{\mu}_{x_0, i_0}\{e^{-\beta(t-t_0)}V(t; x(t), \theta(t); t_f)\} \leq -\varepsilon(t - t_0), \tag{3.12}$$

which contradicts the statement of Proposition 3.1 (where we let $t = t_0$, and $\tau = t$).

For the subsolution case, let $(t_0, y_0) \in [0, t_f] \times \mathbb{R}^n$, $j_0 \in \mathcal{S}$ and $\psi(\cdot, \cdot, i) \in C^1([0, t_f] \times \mathbb{R}^n)$ be such that (t_0, y_0) is a local maximizer of function $V - \psi$, and $V(t_0; y_0, j_0; t_f) - \psi(t_0, y_0, j_0) = 0$. By Proposition 3.1, for any $\mu = (u^1, u^2) \in \mathcal{U}$ and $t > t_0$,

$$\begin{aligned}\psi(t_0, y_0, j_0) &= V(t_0; y_0, j_0; t)\\ &\leq E^{\mu}_{y_0, j_0}\left\{\int_{t_0}^{t} e^{-\beta s}L(x(s), \theta(s), u^1(s))ds\right.\\ &\qquad \left. + e^{-\beta t}J_{\beta}(x(t), \theta(t), \mu; t_f)\right\}\end{aligned} \tag{3.13}$$

Observing that $\{x(t, \theta(t), y_0, \mu)\}$ is continuous in t, when $t > t_0$ is sufficiently close to t_0, by Proposition 3.2, (3.13) becomes

$$\begin{aligned}E^{\mu_1}_{y_0, j_0}&\left\{\frac{e^{-\beta(t-t_0)}}{t - t_0}\int_{t_0}^{t} -\mathcal{A}^{(u^1-u^2)}\psi(r, x(r, \theta(r), y_0, \mu), \theta(r))dr\right\}\\ &+ \left(\frac{1 - e^{-\beta(t-t_0)}}{t - t_0}\right)\psi(t_0, y_0, j_0)\\ &\leq E^{\mu_1}_{y_0, j_0}\left\{\frac{1}{t - t_0}\int_{t_0}^{t} e^{-\beta s}L(x(s, \theta(s), y_0, \mu), \theta(s), \mu(s))ds\right\}\end{aligned} \tag{3.14}$$

Taking limits when $t \downarrow t_0$, observing that the integrand is continuous in s or r, and $x(t_0, j_0, y_0, u^1) = y_0$, we obtain

$$\beta\psi(t_0, y_0, j_0) - \mathcal{A}^{(u^1,u^2)}\psi(t_0, y_0, j_0) - L(y_0, j_0, u^1) \leq 0 \tag{3.15}$$

Since (u^1, u^2) was arbitrary, we have

$$\beta\psi(t_0, y_0, j_0) + \sup_{u^1,u^2} \left[-\mathcal{A}^{(u^1,u^2)}\psi(t_0, y_0, j_0) - L(y_0, j_0, u^1) \right] \leq 0, \tag{3.16}$$

that is to say, $\overline{V}$ is a viscosity subsolution of (1.1), and this completes the proof. ■

Remark 3.3 In Theorem 3.3, we did not need R to be strictly positive definite as required by Assumption (A4). (A4) will be needed, however, for uniqueness.

We provide below two sufficient conditions for V to be continuous.

THEOREM 3.4 *When the control space U_1 is compact (instead of being the entire $\mathbb{R}^r$), and Assumptions (A1)–(A6) hold, we have, for each i,* $V(\cdot\,;\cdot, i; t_f) \in C([0, t_f]; \mathbb{R}^n)$.

Proof Suppose that x and y are solutions of (1.3) with initial conditions $x(s) = x_0$ and $y(s) = y_0$, respectively. Since U_1 is compact, we may assume that there exists a constant M such that $|u^1| \leq M$ for any $u^1 \in U_1$. By Assumptions (A1)–(A5), we have

$$|x(t) - y(t)| \leq |x_0 - y_0| + (L_f + C_{b2}M) \int_s^t |x(\tau) - y(\tau)| d\tau \tag{3.17}$$

Using Gronwall's inequality, one can see that, for given $\varepsilon > 0$, there is $\delta > 0$ such that $|x_0 - y_0| < \delta$ implies $|x(t) - y(t)| < \varepsilon$. Noting that L and g are locally Lipschitz, it follows from the definition of V that

$$\begin{aligned} &|V(s; x_0, i; t_f) - V(s; y_0, i; t_f)| \\ &\leq \sup_{u \in U_1 \times U_2} \left| E\left\{ e^{-\beta(t_f - s)}[g(x(t_f), \theta(t), i) - g(y(t_f), \theta(t), i)] \right.\right. \\ &\qquad \left.\left. + \int_s^{t_f} e^{-\beta(\tau - s)}[L(x(\tau), \theta(\tau), u^1) - L(y(\tau), \theta(\tau), u^1)] d\tau \right\} \right| \\ &\leq F(R)|x_0 - y_0|, \quad \text{where } R < \infty \end{aligned} \tag{3.18}$$

for some locally bounded function F. This leads to continuity of V with respect to the space variable x_0. Now we show the continuity of V with respect to the time variable. Let $0 \le s_1 \le s_2 \le t_f$, and $u_0^1 \in U_1$. By (A1)–(A6), $L(x, i, u_0^1) < \infty$ for all $x \in \mathcal{X}$ and $i \in \mathcal{S}$. Consider function $v : [s_1, t_f] \to U_1 \times U_2$ defined by

$$v(t) = \begin{cases} u_0 & s_1 \le t < s_2 \\ u^* & s_2 \le t \le t_f \end{cases}$$

where $u_0 = (u_0^1, u_0^2) \in U_1 \times U_2$ and $u^* \in U_1 \times U_2$ is such that

$$V(s_2; x_0, i; t_f) = J(s_2; x_0, i, u^*; t_f).$$

Note that

$$\begin{aligned} &V(s_1; x_0, i; t_f) - V(s_2; x_0, i; t_f) \\ &\quad \le E_{x,i}^{u_0}\left\{ \int_{s_1}^{s_2} e^{-\beta(\tau - s_1)} L(x_{s_1}^{u_0}(\tau), \theta(\tau), u^0)\, d\tau \right\} \\ &\qquad + E_{x,i}^{v}\left\{ \int_{s_2}^{t_f} e^{-\beta(\tau - s_2)} (L(x_{s_1}^{v}(\tau), \theta(\tau), v(\tau)) \right. \\ &\qquad\qquad \left. - L(x_{s_2}^{u^*}(\tau), \theta(\tau), u^*(\tau)))\, d\tau \right\} \\ &\qquad + e^{-\beta(t_f - s_1)} E_{x,i}^{v}\{g(x_{s_1}^{v}(t_f), \theta(t_f))\} \\ &\qquad - e^{-\beta(t_f - s_2)} E\{g(x_{s_2}^{u^*}(t_f), \theta(t_f))\} \end{aligned} \tag{3.19}$$

where $x_{s_1}^{v}(t) = x(s_1, x_0, v)$ is the solution of (1.3) with $x(s_1) = x_0$, $(u^1, \theta) = v$, and $x_{s_2}^{u^*}(t) = x(s_2, x_0, u^*)$ is the solution of (1.3) with $x(s_2) = x_0$, $(u^1, \theta) = u^*$. On the other hand, $x_{s_1}^{v}(t)$ can be viewed as the solution of

$$\begin{aligned} \frac{dx_{s_1}^{v}}{dt}(t) &= f(x_{s_1}^{v}(t), \theta(t)) + B(x_{s_1}^{v}(t), \theta(t)) v(t) \\ x_{s_1}^{v}(s_2) &= x_{s_1}^{\mu_0}(s_2) \end{aligned}$$

Hence, we have

$$\begin{aligned} |x_{s_1}^{v}(t) - x_{s_2}^{\mu^*}(t)| &\le |x_{s_1}^{\mu_0}(s_2) - x_0| \\ &\quad + (L_f + M_{\mu_1} C_{b2}) \int_{s_2}^{t} |x_{s_1}^{v}(\tau) - x_{s_2}^{\mu^*}(\tau)|\, d\tau \end{aligned} \tag{3.20}$$

By Assumption (A1),

$$
\begin{aligned}
|x_{s_1}^{\mu_0}(s_2) - x_0| &\leq \int_{s_1}^{s_2} |f(x_{s_1}^{\mu_0}(\tau), \theta(\tau)) + B(x_{s_1}^{\mu_0}(\tau), \theta(\tau))\mu_0 d\tau \\
&\leq \int_{s_1}^{s_2} [L_f|x_{s_1}^{\mu_0}(\tau) - x_0| + |f(x_0, \theta(\tau))| + C_{b1}|\mu_0|]d\tau
\end{aligned}
\tag{3.21}
$$

Thus by Gronwall's inequality, we have

$$|x_{s_1}^{\mu_0}(s_2) - x_0| \leq Ce^{s_2 - s_1}(s_2 - s_1) \tag{3.22}$$

where C is a constant. Since g and L are locally Lipschitz,

$$|V(s_1; x_0, i; t_f) - V(s_2; x_0, i; t_f)| \leq C_{x_0}|s_1 - s_2| \tag{3.23}$$

where C_{x_0} is independent of s_1 and s_2, which completes our proof. ■

Remark 3.4 From the above proof we can see that for each i, $V(\cdot, \cdot, i)$ is uniformly locally Lipschitz in x. From elementary analysis, we know that given $R > 0$, there exists $K_R < \infty$ such that

$$
\begin{aligned}
&|V(t; x, i; t_f) - V(t; y, i; t_f)| \leq K_R|x - y|, \quad \forall (t, x), (t, y) \in [0, t_f] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \{x \in \mathbb{R}^n : |x| \leq R\} \\
&\|\nabla V(t; \cdot, i; t_f)\|_{L^\infty(\{|x| \leq R\})} \leq K_R \quad \forall t \in [0, t_f].
\end{aligned}
$$

This fact has been used in the earlier literature to prove the uniqueness of viscosity solutions by assuming the control space to be compact. We can not, however, make use of this fact because here the control space is allowed to be unbounded.

THEOREM 3.5 *Let $f(x, i) = A(i)x$, $Q(x, i) = x^T Q(i)x$, where $A(i)$ and $Q(i)$ are $n \times n$ matrices, and let B, R be independent of x, and $g(x, i) = x^T Q_f(i)x$. Then we have for each i, $V(\cdot; \cdot, i) \in C([0, t_f]; \mathbb{R}^n)$.*

Proof It is not difficult to see that if f, B are linear with respect to x, $L(\cdot, i, \cdot)$ is convex jointly in (x, μ) and g is convex with respect to x, then $J(t; \cdot, i, \cdot; t_f)$ is convex jointly in (x, μ) for each $i \in \mathcal{S}$. This implies that $V(t; \cdot, i; t_f)$ is convex in $\mathbb{R}^n$, and thus is continuous with respect to the space variable x (in fact, it is locally Lipschitz in x; for a proof, see [29]). Now following the proof of Theorem 3.4 by using the fact that B

is independent of x (thus the condition on boundedness of U_1 can be removed), we arrive at the desired result. ■

For the infinite-horizon case, an argument similar to that used in the proof of Theorem 3.3 leads to the following result for the general case:

THEOREM 3.6 *Under Assumptions* $(A1)-(A6)$, *for each* $i \in S$, *let*

$$V(x,i) = \inf_{\mu \in \mathcal{U}} E^{\mu}_{x,i}\left\{ \int_0^{\infty} e^{-\beta\tau} L(x(\tau), \theta(\tau), u^1(\tau)) d\tau \right\}$$

Then, if $V \not\equiv +\infty$ *and* $V(\cdot, i) \in C(\mathbb{R}^n)$, $\overline{V} = \{V(x,i)\}_{i=1}^{s}$ *is a viscosity solution of* (1.2).

4. UNIQUENESS OF THE VISCOSITY SOLUTION

In this section, we show that the viscosity solution of (1.1) is unique, and when $\beta > 0$ the viscosity solution of (1.2) is also unique. For the finite-horizon case, suppose that $\overline{V}$, $\overline{W}$ are a viscosity supersolution and a viscosity subsolution, respectively, of (1.1) on $Q^{\Omega}_{t_f} = [0, t_f] \times \Omega$, and introduce $\overline{V}$, $\overline{W}$ as

$$\overline{V}(t,x) = \{V(t,x,1), V(t,x,2), \dots, V(t,x,s)\}$$
$$\overline{W}(t,x) = \{W(t,x,1), W(t,x,2), \dots, W(t,x,s)\}$$

Furthermore, assume that

$$\overline{W} \le \overline{V} \quad \text{on} \quad (\{t = t_f\} \times \Omega) \cup ([0, t_f] \times \partial\Omega). \tag{4.1}$$

and adopt the convention that $\overline{W} \le \overline{V}$ on $Q^{\Omega}_{t_f}$ if and only if for each i, $W(t,x,i) \le V(t,x,i)$ for any $(t,x) \in Q^{\Omega}_{t_f}$. Ω may be all of $\mathbb{R}^n$, in which case the boundary of Ω (denoted by $\partial\Omega$) is empty. Now we are ready to state the following lemma:

LEMMA 4.1 *Let* V, W *be as above and assume that* $(A1)-(A8)$ *hold. Let* $R < \infty$ *and introduce a function* $\Lambda \in C^1(Q^{\Omega}_{t_f})$, *such that* $\Lambda \ge 0$, $\Lambda(t,x) = 0$ *if* $|x| \ge R$, *and*

$$-\Lambda_t + \beta\Lambda > 0 \quad on \quad (supp\Lambda)^o \cap (Q^{\Omega}_{t_f})^o. \tag{4.2}$$

where the superscript o indicates interior. Then $\overline{W} \ge \overline{V}$ *on* $(supp\Lambda) \cap Q^{\Omega}_{t_f}$.

Proof Suppose that

$$M_0^i = \Lambda(t_0, x_0)[W(t_0, x_0, i_0) - V(t_0, x_0, i_0)]$$
$$= \max_{Q_{t_f}^{\Omega}} \Lambda(t, x)[W(t, x, i) - V(t, x, i)] > 0, \tag{4.3}$$

since otherwise result has already been established.

Let a function $\Phi^{\varepsilon,\delta}: [0, t_f] \times \Omega \times [0, t_f] \times \Omega \times I_K \to \mathbb{R}^n$ be introduced by

$$\Phi^{\varepsilon,\delta}(t, x, s, y, i) = \Lambda(s, y)W(t, x, i) - \Lambda(t, x)V(s, y, i)$$
$$- \frac{1}{2\varepsilon}|x - y|^2 - \frac{1}{2\delta}|t - s|^2 \tag{4.4}$$

Since $\Phi^{\varepsilon,\delta}$ is upper semi-continuous, Λ has a compact support and $\mathcal{S}$ is a finite set, there exist

$$(t_\delta, x_\varepsilon, s_\delta, y_\varepsilon, k_{\varepsilon,\delta}) \in Q_{t_f}^{\Omega} \times Q_{t_f}^{\Omega} \times \mathcal{S}$$

such that

$$\Phi^{\varepsilon,\delta}(t_\delta, x_\varepsilon, s_\delta, y_\varepsilon, k_{\varepsilon,\delta}) = \max_{Q_{t_f}^{\Omega} \times Q_{t_f}^{\Omega} \times \mathcal{S}} \Phi^{\varepsilon,\delta}(t, x, s, y, i) \tag{4.5}$$

We prove (4.2) in several steps.

1. In this step, we establish the validity of the limits:

$$\frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0 \text{ as } \varepsilon \downarrow 0, \quad \frac{1}{2\delta}|t_\delta - s_\delta|^2 \to 0 \text{ as } \delta \downarrow 0 \tag{4.6}$$

Let $M^{\varepsilon,\delta} = \Phi^{\varepsilon,\delta}(t_\delta, x_\varepsilon, s_\delta, y_\varepsilon, k_{\varepsilon,\delta})$, and consider $0 < \varepsilon_2 \le \varepsilon_1$ and $0 < \delta_2 \le \delta_1$; then,

$$M^{\varepsilon_1,\delta_1} - \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2} - y_{\varepsilon_2}|^2}{2} - \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\frac{|t_{\delta_2} - s_{\delta_2}|^2}{2}$$
$$\ge \Phi^{\varepsilon_1,\delta_1}(t_{\delta_2}, x_{\varepsilon_2}, s_{\delta_2}, y_{\varepsilon_2}, k_{\varepsilon_2,\delta_2}) - \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2} - y_{\varepsilon_2}|^2}{2}$$
$$- \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\frac{|t_{\delta_2} - s_{\delta_2}|^2}{2}$$
$$= \Lambda(s_{\delta_2}, y_{\varepsilon_2})W(t_{\delta_2}, x_{\varepsilon_2}, i) - \Lambda(t_{\delta_2}, x_{\varepsilon_2})V(s_{\delta_2}, y_{\varepsilon_2}, i)$$

$$
\begin{aligned}
&-\frac{1}{2\varepsilon_1}|x_{\varepsilon_2}-y_{\varepsilon_2}|^2-\frac{1}{2\delta_2}|t_{\delta_2}-s_{\delta_2}|^2\\
&-\left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2}-y_{\varepsilon_2}|^2}{2}-\left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right)\frac{|t_{\delta_2}-s_{\delta_2}|^2}{2}\\
&=\Phi^{\varepsilon_2,\delta_2}(t_{\delta_2},x_{\varepsilon_2},s_{\delta_2},y_{\varepsilon_2},k_{\varepsilon,\delta_2})=M^{\varepsilon_2,\delta_2}
\end{aligned}
$$

Hence, we can see that $(\varepsilon,\delta)\mapsto M^{\varepsilon,\delta}$ is nondecreasing. Let $\varepsilon_1=2\varepsilon$, $\varepsilon_2=\varepsilon$ and $\delta_1=\delta_2=\delta$; then

$$
M^{2\varepsilon,\delta}-M^{\varepsilon,\delta}\geq\frac{1}{2\varepsilon}\frac{|x_\varepsilon-y_\varepsilon|^2}{2} \tag{4.7}
$$

Note that $M^{2\varepsilon,\delta}-M^{\varepsilon,\delta}\to 0$ as $\varepsilon\downarrow 0$. Thus the first part of (4.6) holds. For the second part, the proof is similar.

2. Since Λ has a compact support and (4.6) holds, there exist sequences $\{\varepsilon_n\}$ and $\{\delta_m\}$ converging to zero, such that

$$
x_{\varepsilon_n}\to\hat{x}, y_{\varepsilon_n}\to\hat{x}, \text{ as } n\to\infty,\quad t_{\delta_m}\to\hat{t}, s_{\delta_m}\to\hat{t} \text{ as } m\to\infty
$$

where $(\hat{t},\hat{x})\in Q^{\Omega}_{t_f}$. In fact it is easy to see that $\hat{x}=x_0$, $\hat{t}=t_0$. Note that by our assumption $(t_0,x_0)\in(\text{supp}\Lambda)^o\cap(Q^{\Omega}_{t_f})^o$. Therefore, for sufficiently large n and m, we have $(t_{\delta_m},x_{\varepsilon_n})$, $(s_{\delta_m},y_{\varepsilon_n})\in(\text{supp}\Lambda)^o\cap(Q^{\Omega}_{t_f})^o$. By noting that $\mathcal{S}$ is a finite set, without loss of generality, we may assume $k_{\varepsilon_n,\delta_m}=k_0$.

3. Since

$$
\begin{aligned}
W(t,x,k_0)-\frac{1}{\Lambda(s_{\delta_m},y_{\varepsilon_n})}\Bigg[&\Lambda(t,x)V(t_{\delta_m},y_{\varepsilon_n},k_0)+\frac{1}{2\varepsilon_n}|x-y_{\varepsilon_n}|^2\\
&+\frac{1}{2\delta_m}|t-s_{\delta_m}|^2\Bigg]
\end{aligned} \tag{4.8}
$$

attains its maximum at $(t,x)=(t_{\delta_m},x_{\varepsilon_n})$, by the definition of viscosity subsolution, we have

$$
\begin{aligned}
&\frac{\Lambda_t(t_{\varepsilon_n},x_{\varepsilon_n})V(t_{\delta_m},x_{\varepsilon_n},k_0)+(t_{\delta_m}-s_{\delta_m})/\delta_m}{\Lambda(s_{\delta_m},y_{\varepsilon_n})}\\
&\quad+\beta W(t_{\delta_m},x_{\varepsilon_n})-\inf_{u^2}\left\{\sum_{j\in\mathcal{S}}\lambda_{k_0aj}W(t_{\delta_m},x_{\varepsilon_n},j)\right\}\\
&\quad+H_i\left(x_{\varepsilon_n},\frac{\Lambda_x(t_{\delta_m},x_{\varepsilon_n})V(t_{\delta_m},x_{\varepsilon_n},k_0)+(x_{\varepsilon_n}-y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m},y_{\varepsilon_n})}\right)\leq 0
\end{aligned} \tag{4.9}
$$

Similarly,

$$V(s,y,k_0) - \frac{1}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\left[\Lambda(s,y)W(t_{\delta_m}, x_{\varepsilon_n}, k_0) - \frac{1}{2\varepsilon_n}|x_{\varepsilon_n} - y|^2 - \frac{1}{2\delta_m}|t_{\delta_m} - s|^2\right] \tag{4.10}$$

has a minimum at $(s_{\delta_m}, y_{\varepsilon_n})$. Note that $W(\cdot\,,\cdot\,,k_0)$ is a supersolution, which results in

$$\frac{\Lambda_s(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (t_{\delta_m} - s_{\delta_m})/\delta_m}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})} + \beta V(s_{\delta_m}, y_{\varepsilon_n}) - \inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}V(s_{\delta_m}, y_{\varepsilon_n}, j)\right\} + H_i\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right) \geq 0 \tag{4.11}$$

Next we claim that for fixed ε_n the sequence $\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}$ is bounded. If this were not the case, we could then assume that $(t_{\delta_m} - s_{\delta_m})/\delta_m \to -\infty$ as $m \to +\infty$ by (4.9) and Assumption (A8). However, by (4.11) this would imply that

$$H_i\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right) \to +\infty \text{ as } m \to +\infty$$

which is impossible because

$$\lim_{m\to+\infty} H_i\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right) = H_i\left(y_{\varepsilon_n}, \frac{\Lambda_y(\hat{t}, y_{\varepsilon_n})W(\hat{t}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(\hat{t}, x_{\varepsilon_n})}\right) < +\infty$$

Thus there exists a converging subsequence, which we still denote by $\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}$. By (4.9) and Assumption (A8), we have that $(x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n$ is also bounded. Note that

$$H\left(x_{\varepsilon_n}, \frac{\Lambda_x(t_{\delta_m}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})}\right)$$

$$- H\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right)$$

$$\leq C\{L_f + C_{b2}|R^{-1}(x_{\varepsilon_n})B^T(x_{\varepsilon_n})| + L_R|B(x_{\varepsilon_n})||B^T(y_{\varepsilon_n})|$$

$$+ C_{b2}|R^{-1}(y_{\varepsilon_n})B^T(y_{\varepsilon_n})|\}\frac{|x_{\varepsilon_n} - y_{\varepsilon_n}|^2}{\varepsilon_n} + L_Q(R)|x_{\varepsilon_n} - y_{\varepsilon_n}| \qquad (4.12)$$

Let $m \to \infty$ in both (4.9) and (4.11), and subtract (4.11) from (4.9), and let $n \to \infty$. This yields

$$-\frac{\Lambda_t(t_0, x_0)[W(t_0, x_0, k_0) - V(t_0, x_0, k_0)]}{\Lambda(t_0, x_0)} + \beta[W(t_0, x_0, k_0) - V(t_0, x_0, k_0)]$$

$$+ \inf_{u^2}\left\{\sum_{j \in S} \lambda_{k_0 aj} V(t_0, x_0, j)\right\} - \inf_{u^2}\left\{\sum_{j \in S} \lambda_{k_0 aj} W(t_0, x_0, j)\right\} \leq 0 \qquad (4.13)$$

4. Here we establish the inequality

$$\inf_{u^2}\left\{\sum_{j \in S} \lambda_{k_0 aj} W(t_0, x_0, j)\right\} - \inf_{u^2}\left\{\sum_{j \in S} \lambda_{k_0 aj} V(t_0, x_0, j)\right\} \leq 0 \qquad (4.14)$$

Note that by (4.3), we have

$$\Lambda(t_0, x_0)[W(t_0, x_0, i) - V(t_0, x_0, i)]$$

$$\leq \Lambda(t_0, x_0)[W(t_0, x_0, k_0) - V(t_0, x_0, k_0)] \quad \forall i \in S \qquad (4.15)$$

By using Assumption (A5), and (4.15), we have

$$\begin{aligned}
&\inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}W(t_0,x_0,j)\right\}-\inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}V(t_0,x_0,j)\right\}\\
&=\inf_{u^2}\left\{\sum_{j\neq k_0}\lambda_{k_0aj}W(t_0,x_0,j)+\lambda_{k_0ak_0}W(t_0,x_0,k_0)\right\}\\
&\quad-\inf_{u^2}\left\{\sum_{j\neq k_0}\lambda_{k_0aj}V(t_0,x_0,j)+\lambda_{k_0ak_0}V(t_0,x_0,k_0)\right\}\\
&=\inf_{u^2}\left\{\sum_{j\neq k_0}\lambda_{k_0aj}[W(t_0,x_0,j)-W(t_0,x_0,k_0)]\right\}\\
&\quad-\inf_{u^2}\left\{\sum_{j\neq k_0}\lambda_{k_0aj}[V(t_0,x_0,j)-V(t_0,x_0,k_0)]\right\}\\
&\leq\sup_{u^2}\left\{\sum_{j\neq k_0}\lambda_{k_0aj}(W(t_0,x_0,j)-W(t_0,x_0,k_0)\right.\\
&\qquad\left.-V(t_0,x_0,j)+V(t_0,x_0,k_0))\right\}\\
&\leq 0
\end{aligned}\tag{4.16}$$

5. (4.3), (4.13) and (4.16) yield that

$$-\Lambda_t(t_0,x_0)+\beta\Lambda(t_0,x_0)\leq 0 \tag{4.17}$$

which contradicts the hypothesis of the lemma. Therefore

$$\max_{Q_{t_f}^{\Omega}\times Q_{t_f}^{\Omega}}\Lambda(t,x)[W(t,x,i)-V(t,x,i)]\leq 0\quad\forall i\in S \tag{4.18}$$

and this completes the proof of Lemma 4.1. ■

Now we are ready to state the following comparison theorem:

THEOREM 4.2 *Under Assumptions* $(A1)-(A7)$, *if* (4.1) *holds, then we have*

$$\overline{W}\leq\overline{V}\quad on\ Q_{t_f}^{\Omega} \tag{4.19}$$

Proof We are interested in finding Λ such that the conditions of Lemma 4.1 are satisfied. A natural choice for the function Λ is:

$$\Lambda(t,x) = \begin{cases} \exp\{(R^2/|x|^2 - R^2) + (\beta - 1)t\}, & |x| < R \\ 0, & |x| \geq R \end{cases}$$

Suppose that there were $(t_0, x_0, i_0) \in Q_{t_f}^{\Omega}$ such that

$$W(t_0, x_0, i_0) > V(t_0, x_0, i_0) \tag{4.20}$$

Let $R > |x_0|$, and Λ be as above. Clearly, (4.2) is satisfied under the choice of the Λ. Applying Lemma 4.1, we know that (4.20) could not hold. Therefore (4.19) must be true. ∎

Under our assumptions, the comparison theorem leads to uniqueness of viscosity solution of (1.1).

COROLLARY 4.3 *Let $\overline{V}$, $\overline{W}$ be two viscosity solutions of* (1.1) *with boundary and terminal conditions*

$$V(t,x,i) = W(t,x,i) = \varphi(t,x,i) \quad on\ [0,t_f] \times \partial\Omega \tag{4.21}$$

$$V(t_f,x,i) = W(t_f,x,i) = g(x,i) \quad on\ \Omega \tag{4.22}$$

Under Assumptions $(A1)-(A7)$, we have

$$\overline{V} = \overline{W} \quad on\ [0,t_f] \times \Omega \tag{4.23}$$

Remark 4.1 In the case $\Omega = \mathbb{R}^n$, the boundary condition (4.21) is no longer there, and we only consider the terminal condition (4.22).

For the infinite-horizon case, when $\beta > 0$, we have a similar result:

THEOREM 4.4 *Suppose that $\beta > 0$. Assume that both $\overline{V}_1$ and $\overline{V}_2$ are viscosity solutions of* (1.2). *If*

(1) $\Omega = \mathbb{R}^n$, *or*
(2) $V_1(x,i) = V_2(x,i) = g(x,i)$ *on* $\partial\Omega$,

then $\overline{V}_1 = \overline{V}_2$ on Ω.

Proof Since the proof is similar to that in the finite-horizon case, we only give an outline of the proof of part (1). Let function Λ be defined as

$$\Lambda(x) = \begin{cases} \exp\{(R^2/|x|^2 - R^2)\}, & |x| < R \\ 0, & |x| \geq R \end{cases}$$

Assume that there exist $x_0 \in \mathbb{R}^n$ and $i_0 \in \mathcal{S}$ such that $V_1(x_0, i_0) - V_2(x_0, i_0) > 0$. Let R be sufficiently large so that $R > |x_0|$. Let a function $\Phi^\varepsilon : \Omega \times \Omega \times I_K \to \mathbb{R}^n$ be

$$\Phi^\varepsilon(x, y, i) = \Lambda(y)V_1(x, i) - \Lambda(x)V_2(y, i) - \frac{1}{2\varepsilon}|x - y|^2 \tag{4.24}$$

Since Λ has a compact support and $\mathcal{S}$ is a finite set, there exist $(x_\varepsilon, y_\varepsilon, k_\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}$ such that

$$\Phi(x_\varepsilon, y_\varepsilon, k_\varepsilon) = \max_{\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}} \Phi(x, y, i) \tag{4.25}$$

1. As in Lemma 4.1, we have

$$\frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{4.26}$$

2. Since $\{x_\varepsilon\}$, $\{y_\varepsilon\} \in \text{supp}(\Lambda)$, there exists a sequence $\varepsilon_n \downarrow 0$ such that

$$x_{\varepsilon_n} \to x_0, \quad y_{\varepsilon_n} \to x_0 \tag{4.27}$$

3. Since

$$V_1(x, k_0) - \frac{1}{\Lambda(y_{\varepsilon_n})}\left[\Lambda(x)V_2(y_{\varepsilon_n}, k_0) + \frac{1}{2\varepsilon_n}|x - y_{\varepsilon_n}|^2\right] \tag{4.28}$$

attains its maximum at $x = x_{\varepsilon_n}$, by the definition of viscosity subsolution, we have

$$\beta V_1(x_{\varepsilon_n}, k_0) - \inf_{u^2}\left\{\sum_{j \in \mathcal{S}} \lambda_{k_0 aj} V_1(x_{\varepsilon_n}, j)\right\} + H_i\left(x_{\varepsilon_n}, \frac{\Lambda'(x_{\varepsilon_n})V_2(x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(y_{\varepsilon_n})}\right) \leq 0 \tag{4.29}$$

Similarly,

$$V_2(y,k_0) - \frac{1}{\Lambda(x_{\varepsilon_n})}\left[\Lambda(y)V_1(x_{\varepsilon_n},k_0) - \frac{1}{2\varepsilon_n}|x_{\varepsilon_n} - y|^2\right] \tag{4.30}$$

has a minimum at (y_{ε_n}). Thus

$$\beta V_2(y_{\varepsilon_n},k_0) - \inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}V_2(y_{\varepsilon_n},j)\right\} + H_i\left(y_{\varepsilon_n}, \frac{\Lambda'(y_{\varepsilon_n})V_1(y_{\varepsilon_n},k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(x_{\varepsilon_n})}\right) \geq 0 \tag{4.31}$$

Inequality (4.29) and Assumption (A8) imply that the sequence $\{(x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n\}$ is bounded. Note that

$$\begin{aligned} &H_i\left(x_{\varepsilon_n}, \frac{\Lambda'(x_{\varepsilon_n})V_2(x_{\varepsilon_n},k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(y_{\varepsilon_n})}\right) \\ &\quad - H\left(y_{\varepsilon_n}, \frac{\Lambda'(y_{\varepsilon_n})V_1(y_{\varepsilon_n},k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(x_{\varepsilon_n})}\right) \\ &\leq C\{L_f + C_{b2}|R^{-1}(x_{\varepsilon_n})B^T(x_{\varepsilon_n}) + L_R|B(x_{\varepsilon_n})|^2 \\ &\quad + C_{b2}|R^{-1}(y_{\varepsilon_n})B^T(y_{\varepsilon_n})|\}\frac{|x_{\varepsilon_n} - y_{\varepsilon_n}|^2}{\varepsilon_n} + L_Q(R)|x_{\varepsilon_n} - y_{\varepsilon_n}| \end{aligned} \tag{4.32}$$

Subtracting (4.31) from (4.29), and letting $n \to \infty$, yields

$$\begin{aligned} \beta(V_1(x_0,k_0) - V_2(x_0,k_0)) &\leq \inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}V_1(x_0,j)\right\} \\ &\quad - \inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}V_2(y_0,j)\right\} \end{aligned} \tag{4.33}$$

4. Following the same lines as in proof of the Proposition (4.14),

$$\inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}V_1(x_0,j)\right\} - \inf_{u^2}\left\{\sum_{j\in S}\lambda_{k_0aj}V_2(y_0,j)\right\} \leq 0 \tag{4.34}$$

5. In view of the assumption $V_1(x_0, i_0) - V_2(x_0, i_0) > 0$, and (4.25), we have

$$\beta(V_1(x_0, i_0) - V_2(x_0, i_0)) \leq 0, \tag{4.35}$$

which contradicts the hypothesis that $\beta > 0$. Therefore we arrive at

$$V_1(x, i) \leq V_2(x, i), \quad \forall x \in \mathbb{R}^n, \ \forall i \in \mathcal{S} \tag{4.36}$$

The reverse inequality can be shown similarly. ■

5. CONCLUDING REMARKS

In this paper, we have shown the existence of a viscosity solution to *a set of linearly coupled* Hamilton-Jacobi-Bellman (HJB) equations and have also generalized the standard comparison theorem from a single HJB equation to a set of coupled HJB equations. The uniqueness of the viscosity solution for the coupled system is also established, and it is shown to hold even if the value functions do not meet the growth conditions invoked in most of the current literature on the topic.

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