

Approximating Solution of Nonlinear Variational Inclusions by Ishikawa Iterative Process with Errors in Banach Spaces

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In this paper, we introduce and study the existence of solutions and convergence of Ishikawa iterative processes with errors for a class of nonlinear variational inclusions with accretive type mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of [4–9, 11, 16–17, 19].

Keywords: Nonlinear variational inclusion; Accretive mapping; ϕ -hemicontractive mapping; Ishikawa type iterative sequence; Convergence

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1. INTRODUCTION

Throughout this paper we suppose that X is a real Banach space, X^* is its dual space, $\langle \cdot, \cdot \rangle$ is the pairing of X and X^* . Let $D(T)$ and $R(T)$ denote the domain and the range of T , respectively.

Let $B: X \rightarrow X$, $g: X \rightarrow X^*$ be two mappings, and $\varphi: X^* \rightarrow \mathbf{R} \cup \{+\infty\}$ a proper convex lower semi-continuous function. For any given $f \in X$, we consider the following problem:

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Find an $u \in X$ such that

$$\begin{cases} g(u) \in D(\partial\varphi) \\ \langle Bu - f, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \quad \forall v \in X^*, \end{cases} \quad (1.1)$$

where $\partial\varphi$ denotes the subdifferential of φ . The problem (1.1) is called a nonlinear variational inclusion in Banach space.

Special cases:

1. If $B = T - A$, where T and A are two mappings from X to X , then the problem (1.1) is equivalent to finding an $u \in X$ for given $f \in X$ such that

$$\begin{cases} g(u) \in D(\partial\varphi) \\ \langle Tu - Au - f, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \quad \forall v \in X^*. \end{cases} \quad (1.2)$$

2. If X is a Hilbert space H and $B = T - A$, where T and A are two mappings from H to H , then the problem (1.1) is equivalent to finding an $u \in H$ for given $f \in H$ such that

$$\begin{cases} g(u) \in D(\partial\varphi) \\ \langle Tu - Au - f, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \quad \forall v \in H, \end{cases} \quad (1.3)$$

which is called the variational inclusion problem in Hilbert space studied by Hassouni–Moudafi [6], Ding [4, 5], Huang [8, 9], Kazmi [11] and Zeng [19].

3. If X is a Hilbert space H , $B = T - A$, where T and A are two mappings from H to H and $\varphi = \delta_K$, where K is a nonempty closed convex subset of H and δ_K is the indicator function of K , i.e.,

$$\delta_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases}$$

then the problem (1.1) is equivalent to finding an $u \in K$ for given $f \in H$ such that

$$\begin{cases} g(u) \in K \\ \langle Tu - Au - f, v - g(u) \rangle \geq 0, \quad \forall v \in K, \end{cases} \quad (1.4)$$

which is called the strongly nonlinear quasi-variational inequality problem studied by Huang [7], Noor [13, 14], Siddiqi *et al.* [16, 17].

The purpose of this paper is to study the existence and uniqueness of solutions and the convergence problem of Ishikawa iterative process with errors for the nonlinear variational inclusion problem (1.1) with strongly accretive mapping and ϕ -hemicontractive mapping in Banach spaces. The results presented in this paper extend and improve the corresponding results in Ding [4, 5], Hassouni–Moudafi [6], Huang [7–9], Kazmi [11], Siddiqi *et al.* [16, 17] and Zeng [19].

2. PRELIMINARIES

A mapping $J : X \rightarrow 2^{X^*}$ is said to be a normalized duality mapping, if it is defined by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad x \in X.$$

It is well known (see, *e.g.*, [18]) that J is bounded and if X is uniformly smooth, then X is smooth and reflexive, and J is single-valued and is uniformly continuous on bounded subsets of X .

A mapping $T : D(T) \subset X \rightarrow X$ is called accretive, if for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq 0.$$

T is called strongly accretive, if there exists a constant $k \in (0, 1)$ such that for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \geq k \cdot \|x - y\|^2.$$

The constant k in above inequality is called the strongly accretive constant.

T is called ϕ -strongly accretive if, for all $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq \phi(\|x - y\|)\|x - y\|.$$

It is known (see, *e.g.*, [15]) that the class of strongly accretive mappings is a proper subset of the class of ϕ -strongly accretive mappings. Closely related to the class of strongly accretive (respectively, ϕ -strongly accretive) mappings is the class of strongly pseudocontractive (respectively, ϕ -strongly pseudocontractive) mappings.

A mapping $A : D(A) \subset X \rightarrow X$ is called strongly pseudocontractive if, for each $x, y \in D(A)$, there exist $j(x-y) \in J(x-y)$ and a constant $t > 1$ such that

$$\langle Ax - Ay, j(x-y) \rangle \leq \frac{1}{t} \|x-y\|^2.$$

A is called ϕ -strongly pseudocontractive if, for each $x, y \in D(A)$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

Furthermore, A is said to be ϕ -hemicontractive if the fixed point set $F(A)$ of A is nonempty, and for each $x \in D(A)$ and $x^* \in F(A)$, there exist $j(x-x^*) \in J(x-x^*)$ and a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - x^*, j(x-x^*) \rangle \leq \|x-x^*\|^2 - \phi(\|x-x^*\|)\|x-x^*\|.$$

It was shown in [15] that the class of strongly pseudocontractive mappings is a proper subset of the class of ϕ -strongly pseudocontractive mappings. The example in [2] shows that the class of ϕ -strongly pseudocontractive mappings is a proper subset of the class of the class of ϕ -hemicontractive mappings. It is easy to see that A is a strongly (respectively, ϕ -strongly) pseudocontractive mapping if and only if $T = I - A$ is strongly (respectively, ϕ -strongly) accretive where I is the identity mapping.

In the sequel we need the following Lemmas.

LEMMA 2.1 *Let X be a real uniformly smooth Banach space, $T : X \rightarrow X$ a ϕ -strongly accretive mapping and $S : X \rightarrow X$ an accretive mapping. Then $T+S : X \rightarrow X$ is also a ϕ -strongly accretive mapping.*

Proof Since X is uniformly smooth, we know that the normalized duality mapping J is a single-valued mapping. Hence, for any $x, y \in X$, we have

$$\begin{aligned} \langle (T+S)x - (T+S)y, J(x-y) \rangle &= \langle Tx - Ty, J(x-y) \rangle \\ &\quad + \langle (Sx - Sy), J(x-y) \rangle \\ &\geq \phi(\|x-y\|)\|x-y\|. \end{aligned}$$

So $T+S$ is ϕ -strongly accretive mapping.

LEMMA 2.2 (See [1]) *Let X be a real Banach space, then for any $x, y \in X$ the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

LEMMA 2.3 (See [12]) *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three sequences of nonnegative numbers satisfying the following conditions: there exists n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where

$$t_n \in (0, 1), \quad \sum_{n=0}^{\infty} t_n = +\infty, \quad b_n = o(t_n), \quad \sum_{n=0}^{+\infty} c_n < +\infty.$$

Then $a_n \rightarrow 0$ ($n \rightarrow +\infty$).

LEMMA 2.4 *Let X be a real reflexive Banach space, then $x^* \in X$ is a solution of the nonlinear variational inclusion problem (1.1) if and only if $x^* \in X$ is a fixed point of the mapping $S: X \rightarrow 2^X$ defined as follows:*

$$S(x) = f - (Bx + \partial\varphi(g(x))) + x.$$

Proof Let x^* be a solution of the nonlinear variational inclusion problem (1.1), then $g(x^*) \in D(\partial\varphi)$ and

$$\langle Bx^* - f, v - g(x^*) \rangle \geq \varphi(g(x^*)) - \varphi(v), \quad \forall v \in X^*.$$

By the definition of subdifferential of φ it follows from the above expression that

$$f - Bx^* \in \partial\varphi(g(x^*)).$$

This implies that

$$x^* \in f - (Bx^* + \partial\varphi(g(x^*))) + x^* = Sx^*,$$

and so x^* is a fixed point of S in X .

Conversely, suppose that x^* is a fixed point of S in X . We have

$$x^* \in Sx^* = f - (Bx^* + \partial\varphi(g(x^*))) + x^*.$$

This implies that

$$f - Bx^* \in \partial\varphi(g(x^*)).$$

From the definition of $\partial\varphi$, it follows that

$$\varphi(v) - \varphi(g(x^*)) \geq \langle f - Bx^*, v - g(x^*) \rangle, \quad \forall v \in X^*,$$

i.e.,

$$\langle Bx^* - f, v - g(x^*) \rangle \geq \varphi(g(x^*)) - \varphi(v), \quad \forall v \in X^*.$$

This implies that x^* is a solution of the nonlinear variational inclusion problem (1.1). This completes the proof of Lemma 2.4.

3. MAIN RESULTS

THEOREM 3.1 *Let X be a real uniformly smooth Banach space, $B: X \rightarrow X$ be semi-continuous (i.e., $x_n \rightarrow x$ implies that $Tx_n \rightarrow Tx$), $g: X \rightarrow X^*$ be continuous and $\varphi: X^* \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function with a continuous Gâteaux differential $\partial\varphi$. Suppose that*

- (i) $B: X \rightarrow X$ is a strongly accretive mapping with strongly accretive constant $k \in (0, 1)$;
- (ii) $\partial\varphi \circ g: X \rightarrow X$ is accretive.

Then the nonlinear variational inclusion problem (1.1) has a unique solution $x^ \in X$. Moreover, for any given $f \in X$, define a mapping $S: X \rightarrow X$ by*

$$Sx = f - (Bx + \partial\varphi(g(x))) + x.$$

If the range $R(S)$ of S is bounded, then for any given $x_0 \in X$, the following Ishikawa iterative sequence $\{x_n\}$ with errors defined by

$$\left. \begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n S y_n + \alpha_n u_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n S x_n + \beta_n v_n, \end{aligned} \right\} \quad n = 0, 1, 2, \dots \quad (3.1)$$

converges strongly to x^ , where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in X satisfying the following*

conditions:

$$\lim_{n \rightarrow \infty} \|u_n\| = \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty. \quad (3.2)$$

Proof First we prove that the nonlinear variational inclusion problem (1.1) has a unique solution $x^* \in X$.

From the conditions (i), (ii) and Lemma 2.1, the mapping $B + \partial\varphi \circ g : X \rightarrow X$ is a strongly accretive semi-continuous mapping with a strongly accretive constant $k \in (0, 1)$. By Theorem 13.1 of Deimling [3], we know that $B + \partial\varphi \circ g$ is surjective. Therefore, for any given $f \in X$, the equation $f = (B + \partial\varphi \circ g)(x)$ has a solution x^* , and so x^* is a fixed point of S , i.e., $x^* = Sx^*$. Since X is reflexive, it follows from Lemma 2.4 that x^* is also a solution of the nonlinear variational inclusion (1.1). Now we prove that x^* is the unique solution of the nonlinear variational inclusion (1.1) in X . Suppose the contrary, $u^* \in X$ is also a solution of (1.1), then u^* is also a fixed point of S and so

$$\begin{aligned} \|x^* - u^*\|^2 &= \langle x^* - u^*, J(x^* - u^*) \rangle \\ &= \langle Sx^* - Su^*, J(x^* - u^*) \rangle \\ &= \langle f - (B + \partial\varphi \circ g)(x^*) \\ &\quad + x^* - (f - (B + \partial\varphi \circ g)(u^*) + u^*), J(x^* - u^*) \rangle \\ &= \|x^* - u^*\|^2 - \langle (B + \partial\varphi \circ g)(x^*) \\ &\quad - (B + \partial\varphi \circ g)(u^*), J(x^* - u^*) \rangle \\ &\leq \|x^* - u^*\|^2 - k\|x^* - u^*\|^2. \end{aligned}$$

Since $k \in (0, 1)$, this implies that $\|x^* - u^*\|^2 = 0$. Hence $x^* = u^*$. This proves that x^* is the unique solution of (1.1).

Next we prove that the Ishikawa iterative sequence $\{x_n\}$ with errors converges strongly to x^* .

Since S has bounded range $R(S)$ and $\{u_n\}, \{v_n\}$ are two bounded sequences in X , we set

$$\begin{aligned} M &= \sup\{\|Sx - x^*\| + \|x_0 - x^*\| : x \in X\} \\ &\quad + \sup\{\|u_n\| : n \geq 0\} + \sup\{\|v_n\| : n \geq 0\}. \end{aligned} \quad (3.3)$$

Obviously, $M < \infty$. Now we prove that for all $n \geq 0$,

$$\|x_n - x^*\| \leq M, \quad \|y_n - x^*\| \leq M. \quad (3.4)$$

In fact, for $n=0$, it follows from (3.3) that $\|x_0 - x^*\| \leq M$. Therefore we have

$$\begin{aligned} \|y_0 - x^*\| &= \|(1 - \beta_0)(x_0 - x^*) + \beta_0(Sx_0 - x^*) + \beta_0 v_0\| \\ &\leq (1 - \beta_0)\|x_0 - x^*\| + \beta_0\|Sx_0 - x^*\| + \beta_0\|v_0\| \\ &\leq M. \end{aligned}$$

Suppose that (3.4) is true for $n=k \geq 0$, then for $n=k+1$, we have

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|(1 - \alpha_k)(x_k - x^*) + \alpha_k(Sy_k - x^*) + \alpha_k u_k\| \\ &\leq (1 - \alpha_k)\|x_k - x^*\| + \alpha_k\|Sy_k - x^*\| + \alpha_k\|u_k\| \\ &\leq M \end{aligned}$$

and

$$\begin{aligned} \|y_{k+1} - x^*\| &= \|(1 - \beta_{k+1})(x_{k+1} - x^*) \\ &\quad + \beta_{k+1}(Sx_{k+1} - x^*) + \beta_{k+1} v_{k+1}\| \\ &\leq (1 - \beta_{k+1})\|x_{k+1} - x^*\| \\ &\quad + \beta_{k+1}\|Sx_{k+1} - x^*\| + \beta_{k+1}\|v_{k+1}\| \\ &\leq M. \end{aligned}$$

From the above discussion, we can conclude that (3.4) is true.

Since the normalized duality mapping J is single-valued, it follows from (3.1) and Lemma 2.2 that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*) + \alpha_n u_n\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Sy_n - x^* \\ &\quad + u_n, J(x_{n+1} - x^*) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Sy_n - x^*, J(y_n - x^*) \rangle \\ &\quad + 2\alpha_n \langle Sy_n - x^* + u_n, J(x_{n+1} - x^*) - J(y_n - x^*) \rangle \\ &\quad + 2\alpha_n \langle u_n, J(y_n - x^*) \rangle \\ &= (1 - \alpha_n)^2 \|x_n - x^*\|^2 \\ &\quad + 2\alpha_n \langle Sy_n - x^*, J(y_n - x^*) \rangle + 2\alpha_n \langle e_n + f_n, \end{aligned} \quad (3.5)$$

where

$$e_n = |\langle Sy_n - x^* + u_n, J(x_{n+1} - x^*) - J(y_n - x^*) \rangle|, \quad f_n = |\langle u_n, J(y_n - x^*) \rangle|.$$

Since $B + \partial\varphi \circ g$ is a strongly accretive mapping with a strongly accretive constant $k \in (0, 1)$, we have

$$\begin{aligned} \langle Sy_n - x^*, J(y_n - x^*) \rangle &= \langle Sy_n - Sx^*, J(y_n - x^*) \rangle \\ &= \langle f - (B + \partial\varphi \circ g)(y_n) + y_n - (f - (B + \partial\varphi \circ g)x^* + x^*), J(y_n - x^*) \rangle \\ &= \|y_n - x^*\|^2 - \langle (B + \partial\varphi \circ g)(y_n) - (B + \partial\varphi \circ g)(x^*), J(y_n - x^*) \rangle \\ &\leq (1 - k)\|y_n - x^*\|^2. \end{aligned} \tag{3.6}$$

On the other hand, from (3.1), (3.3), (3.4) and Lemma 2.2, we have

$$\begin{aligned} \|y_n - x^*\|^2 &= \|(1 - \beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*) + \beta_n v_n\|^2 \\ &\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2\beta_n \langle Sx_n - x^* + v_n, J(y_n - x^*) \rangle \\ &\leq \|x_n - x^*\|^2 + 2\beta_n (\|Sx_n - x^*\| + \|v_n\|) \cdot \|y_n - x^*\| \\ &\leq \|x_n - x^*\|^2 + 2\beta_n M^2. \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\langle Sy_n - x^*, J(y_n - x^*) \rangle \leq (1 - k)\{\|x_n - x^*\|^2 + 2\beta_n M^2\}. \tag{3.8}$$

Now we prove that

$$e_n \rightarrow 0, \quad f_n \rightarrow 0 \quad (n \rightarrow \infty). \tag{3.9}$$

In fact, by (3.3), we have

$$\begin{aligned} e_n &= |\langle Sy_n - x^* + u_n, J(x_{n+1} - x^*) - J(y_n - x^*) \rangle| \\ &\leq M \cdot \|J(x_{n+1} - x^*) - J(y_n - x^*)\|. \end{aligned}$$

Since

$$\begin{aligned} x_{n+1} - x^* - (y_n - x^*) &= x_{n+1} - y_n \\ &= (\beta_n - \alpha_n)x_n \\ &\quad + \alpha_n Sy_n - \beta_n Sx_n + \alpha_n u_n - \beta_n v_n \end{aligned}$$

and $\{x_n\}$, $\{Sy_n\}$, $\{Sx_n\}$, $\{u_n\}$, $\{v_n\}$ are all bounded, by $\alpha_n \rightarrow 0$, $\beta_n \rightarrow 0$ ($n \rightarrow \infty$), we obtain

$$x_{n+1} - x^* - (y_n - x^*) \rightarrow 0 \quad (n \rightarrow \infty).$$

Using the uniformly continuity of J , we know that $\|J(x_{n+1} - x^*) - J(y_n - x^*)\| \rightarrow 0$ ($n \rightarrow \infty$) and so $e_n \rightarrow 0$ ($n \rightarrow \infty$). Furthermore,

$$f_n \leq \|u_n\| \cdot \|y_n - x^*\| \leq \|u_n\| \cdot M \rightarrow 0 \quad (n \rightarrow \infty).$$

Therefore, (3.9) is true.

It follows from (3.5) and (3.8) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq [(1 - \alpha_n)^2 + 2\alpha_n(1 - k)]\|x_n - x^*\|^2 \\ &\quad + 2\alpha_n[(1 - k) \cdot 2\beta_n M^2 + e_n + f_n]. \\ &= (1 + \alpha_n^2 - 2\alpha_n k)\|x_n - x^*\|^2 \\ &\quad + 2\alpha_n[(1 - k)2\beta_n M^2 + e_n + f_n] \\ &= [1 - \alpha_n k + \alpha_n(\alpha_n - k)]\|x_n - x^*\|^2 \\ &\quad + 2\alpha_n[(1 - k)2\beta_n M^2 + e_n + f_n]. \end{aligned} \quad (3.10)$$

Since $\alpha_n \rightarrow 0$ ($n \rightarrow \infty$), there exists an n_0 such that for $n \geq n_0$, $\alpha_n < k$. Hence for any $n \geq n_0$, by (3.10), we have

$$\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n k)\|x_n - x^*\|^2 + 2\alpha_n[(1 - k)2\beta_n M^2 + e_n + f_n]. \quad (3.11)$$

Now let $\|x_n - x^*\|^2 = a_n$, $\alpha_n k = t_n$, $2\alpha_n[(1 - k)2\beta_n M^2 + e_n + f_n] = b_n$ and $c_n = 0$. Then the inequality (3.11) reduces to

$$a_{n+1} \leq (1 - t_n)a_n + b_n.$$

By (3.2), we know that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{t_n\}$ satisfy all conditions in Lemma 2.3. Hence $a_n \rightarrow 0$ ($n \rightarrow \infty$), i.e., $x_n \rightarrow x^*$ ($n \rightarrow \infty$). This completes the proof of Theorem 3.1.

THEOREM 3.2 *Let X be a real reflexive Banach space, $B: X \rightarrow X$, $g: X \rightarrow X^*$ and $\varphi: X^* \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function with a continuous Gâteaux differential $\partial\varphi$. For any given $f \in X$, define the mapping $S: X \rightarrow X$ by*

$$Sx = f - (Bx + \partial\varphi(g(x))) + x.$$

If S is a ϕ -hemicontractive mapping with bounded range $R(S)$, then for any given $x_0 \in X$, the following Ishikawa iterative sequence $\{x_n\}$ with errors defined by

$$\left. \begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n S y_n + \alpha_n u_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n S x_n + \beta_n v_n, \end{aligned} \right\} \quad n = 0, 1, 2, \dots \quad (3.12)$$

converges strongly to the unique solution of the nonlinear variational inclusion problem (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in X satisfying the following conditions:

$$\lim_{n \rightarrow \infty} \|S y_n - S x_{n+1}\| = \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \|u_n\| = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty. \quad (3.13)$$

Proof Since S is ϕ -hemicontractive, then the fixed point set $F(S)$ of S is nonempty. Let $x^* \in F(S)$, i.e., $x^* = Sx^*$. It follows from Lemma 2.4 that x^* is also a solution of the nonlinear variational inclusion problem (1.1). We will show that x^* is the unique solution of (1.1). Suppose that $q \in X$ is also a solution of (1.1) and $q \neq x^*$. Then $\|x^* - q\| > 0$ and $\phi(\|x^* - q\|) > 0$, since ϕ is strictly increasing with $\phi(0) = 0$. Hence,

$$\phi(\|x^* - q\|)\|x^* - q\| > 0. \quad (3.14)$$

Since S is ϕ -hemicontractive and Lemma 2.4 implies that $q \in F(S)$, there exists $j(x^* - q) \in J(x^* - q)$ such that

$$\|x^* - q\|^2 = \langle Sx^* - Sq, j(x^* - q) \rangle \leq \|x^* - q\|^2 - \phi(\|x^* - q\|)\|x^* - q\|,$$

i.e., $\phi(\|x^* - q\|)\|x^* - q\| \leq 0$, contradicting (3.14). Thus, x^* is the unique solution of (1.1).

Let

$$\begin{aligned} M &= \sup\{\|Sx - x^*\| + \|x_0 - x^*\| : x \in X\} \\ &\quad + \sup\{\|u_n\| : n \geq 0\} + \sup\{\|v_n\| : n \geq 0\}. \end{aligned} \quad (3.15)$$

Obviously, $M < \infty$ and it follows from the Proof of Theorem 3.1 that for all $n \geq 0$,

$$\|x_{n+1} - x^*\| \leq M, \quad \|y_{n+1} - x^*\| \leq M. \quad (3.16)$$

It follows from (3.12) and Lemma 2.2 that for all $j \in J(x_{n+1} - x^*)$,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(Sy_n + u_n - x^*)\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Sy_n + u_n - x^*, j \rangle \\ &= (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Sy_n - Sx_{n+1}, j \rangle \\ &\quad + 2\alpha_n \langle u_n, j \rangle + 2\alpha_n \langle Sx_{n+1} - x^*, j \rangle. \end{aligned} \quad (3.17)$$

Since S is ϕ -hemicontractive, we know that there exists a $\tilde{j}_{x_{n+1}, x^*} \in J(x_{n+1} - x^*)$ such that

$$\langle Sx_{n+1} - x^*, \tilde{j}_{x_{n+1}, x^*} \rangle \leq \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|) \|x_{n+1} - x^*\|. \quad (3.18)$$

Substituting (3.18) into (3.17), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Sy_n - Sx_{n+1}, \tilde{j}_{x_{n+1}, x^*} \rangle \\ &\quad + 2\alpha_n \langle u_n, \tilde{j}_{x_{n+1}, x^*} \rangle \\ &\quad + 2\alpha_n \|x_{n+1} - x^*\|^2 - 2\alpha_n \phi(\|x_{n+1} - x^*\|) \|x_{n+1} - x^*\|. \end{aligned} \quad (3.19)$$

Since $\alpha_n \rightarrow 0 (n \rightarrow \infty)$, there exists a positive integer n_0 such that for all $n \geq n_0$, $2\alpha_n < 1$. By (3.15) and (3.16), we rewrite (3.19) as

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \frac{(1 - \alpha_n)^2}{1 - 2\alpha_n} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \cdot a_n - \frac{2\alpha_n}{1 - 2\alpha_n} \\ &\quad \phi(\|x_{n+1} - x^*\|) \|x_{n+1} - x^*\| \\ &\leq \|x_n - x^*\|^2 + \frac{\alpha_n}{1 - 2\alpha_n} (M^2 \alpha_n + 2a_n) - \frac{2\alpha_n}{1 - 2\alpha_n} \\ &\quad \phi(\|x_{n+1} - x^*\|) \|x_{n+1} - x^*\| \end{aligned} \quad (3.20)$$

for all $n \geq n_0$, where

$$a_n = |\langle Sy_n - Sx_{n+1}, \tilde{j}_{x_{n+1}, x^*} \rangle| + |\langle u_n, \tilde{j}_{x_{n+1}, x^*} \rangle|.$$

Since $\{x_n - x^*\}$ is a bounded sequence, we know that $\{J(x_{n+1} - x^*)\}_{n \geq 0}$ is bounded, and so $\{\tilde{j}_{x_{n+1}, x^*}\}_{n \geq 0}$ is a bounded sequence in $\{J(x_{n+1} - x^*)\}_{n \geq 0}$. By (3.13), we have

$$a_n \rightarrow 0, \quad M^2 \alpha_n + 2a_n \rightarrow 0 \quad (n \rightarrow \infty). \quad (3.21)$$

Let $\inf\{\|x_n - x^*\| : n \geq 0\} = \delta$. Then $\delta \geq 0$. Suppose $\delta > 0$, we have $\|x_n - x^*\| \geq \delta, \forall n \geq 0$. Since ϕ is strictly increasing, we get $\phi(\|x_{n+1} - x^*\|) \geq \phi(\delta)$. From (3.21), there exists a positive integer $N \geq n_0$ such that $M^2\alpha_n + 2a_n < \phi(\delta)\delta, \forall n \geq N$. It follows from (3.20) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + \frac{\alpha_n}{1 - 2\alpha_n} \phi(\delta)\delta - \frac{2\alpha_n}{1 - 2\alpha_n} \phi(\delta)\delta \\ &= \|x_n - x^*\|^2 - \frac{\alpha_n}{1 - 2\alpha_n} \phi(\delta)\delta \\ &\leq \|x_n - x^*\|^2 - \alpha_n \phi(\delta)\delta, \quad \forall n \geq N. \end{aligned}$$

This implies that

$$\phi(\delta)\delta \sum_{n=N}^{\infty} \alpha_n \leq \|x_N - x^*\|^2 < \infty,$$

which contradicts condition (3.13). Thus $\inf\{\|x_n - x^*\| : n \geq 0\} = 0$, so that there exists a subsequence $\{\|x_{n_j} - x^*\|\}_{j=0}^{\infty}$ of the sequence $\{\|x_n - x^*\|\}_{j=0}^{\infty}$ such that $\|x_{n_j} - x^*\| \rightarrow 0 (j \rightarrow \infty)$. Let $\varepsilon > 0$ be arbitrary. Then there exists a positive integer n_{j^*} such that

$$\|x_{n_{j^*}} - x^*\| < \varepsilon, \quad M^2\alpha_n + 2a_n < \phi(\varepsilon)\varepsilon, \quad \forall n \geq n_{j^*}. \tag{3.22}$$

We prove by induction that

$$\|x_{n_{j^*}+p} - x^*\| < \varepsilon, \quad p = 1, 2, \dots \tag{3.23}$$

For $p = 1$, we prove that $\|x_{n_{j^*}+1} - x^*\| < \varepsilon$. Suppose $\|x_{n_{j^*}+1} - x^*\| \geq \varepsilon$, then $\phi(\|x_{n_{j^*}+1} - x^*\|) \geq \phi(\varepsilon)$. By (3.20) and (3.22), we have

$$\|x_{n_{j^*}+1} - x^*\| \leq \varepsilon^2 - \frac{\alpha_n}{1 - 2\alpha_n} \phi(\varepsilon)\varepsilon < \varepsilon^2,$$

which contradicts $\|x_{n_{j^*}+1} - x^*\| \geq \varepsilon$. Thus $\|x_{n_{j^*}+1} - x^*\| < \varepsilon$. Assume (3.23) is true for $p = p_0 > 1$. Then we can prove that $\|x_{n_{j^*}+(p_0+1)} - x^*\| < \varepsilon$. In fact, if $\|x_{n_{j^*}+(p_0+1)} - x^*\| \geq \varepsilon$, then $\phi(\|x_{n_{j^*}+(p_0+1)} - x^*\|) \geq \phi(\varepsilon)$. It follows from (3.20) and (3.22) that

$$\|x_{n_{j^*}+(p_0+1)} - x^*\| \leq \varepsilon^2 - \frac{\alpha_n}{1 - 2\alpha_n} \phi(\varepsilon)\varepsilon < \varepsilon^2,$$

a contradiction. So $\|x_{n_n+(p_0+1)} - x^*\| < \varepsilon$. This implies that (3.23) is true for all $p \geq 1$ and therefore $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$. This completes the Proof of Theorem 3.2.

Remark 3.1

- (1) In Theorem 3.2, if S is an uniformly continuous mapping with bounded range, $\{u_n\}$, $\{v_n\}$ are bounded sequences and $\alpha_n \rightarrow 0$, $\beta_n \rightarrow 0 (n \rightarrow \infty)$, then $\|S y_n - S x_{n+1}\| \rightarrow 0 (n \rightarrow \infty)$ (See [10]). In fact, from the Proof of Theorem 3.1, we know that $\|y_n - x_{n+1}\| \rightarrow 0 (n \rightarrow \infty)$ and so $\|S y_n - S x_{n+1}\| \rightarrow 0 (n \rightarrow \infty)$.
- (2) Theorems 3.1 and 3.2 extend and improve the corresponding results in Ding [4, 5], Hassouni–Moudafi [6], Huang [7–9], Kazmi [11], Siddiqi *et al.* [16, 17] and Zeng [19].

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