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Iterative Approximations for Solutions of Nonlinear Equations Involving Non-self-mappings

HAIYUN ZHOU [a], YEOL JE CHO [b,*] and SHIN MIN KANG [b]

[a]*Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, People's Republic of China;* [b]*Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea*

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In this paper, by virtue of some new approach techniques, we prove several strong convergence theorems for iterative approximation of either fixed points or zeros of a class of non-self-mappings in general Banach spaces. Moreover, specific error estimations are also given.

Keywords: Non-self-mapping; ϕ-strongly quasi-accretive operator; ϕ-hemicontraction; Ishikawa iteration process; Error estimation

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1. INTRODUCTION

Let X be a real Banach space with norm $\|\cdot\|$ and X^* be the dual space of X. The normalized duality mapping from X to the family of subsets of X^* is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between X and X^*. A mapping T with domain $D(T)$ in X is said to be accretive if, for

* Corresponding author.

each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0. \tag{1}$$

Furthermore, T is called strongly accretive if there exists a constant $k > 0$ such that, for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ for which the inequality

$$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2 \tag{2}$$

holds. T is said to be ϕ-strongly accretive if there exists a strictly increasing function ϕ: $[0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that, for all x, $y \in D(T)$, there exists $j(x-y) \in J(x-y)$ for which the inequality

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\| \tag{3}$$

holds. Let $F(T) = \{x \in D(T): x = Tx\}$ and $N(T) = \{x \in D(T): 0 \in Tx\}$. If $N(T) \neq \emptyset$ and the inequalities (1), (2) and (3) hold for any $x \in D(T)$ and $y \in N(T)$, then the corresponding operator T is called quasi-accretive, strongly quasi-accretive and ϕ-strongly quasi-accretive, respectively. It was shown in [22] that the class of strongly accretive operators is a proper subclass of ϕ-strongly accretive operators.

A class of mappings closely related to accretive operators is the class of pseudo-contractions. A mapping T: $D(T) \subset X \to X$ is called pseudo-contractive (respectively, strongly pseudo-contractive, ϕ-strongly pseudo-contractive, ϕ-hemicontractive) if and only if $(I - T)$ is accretive (respectively, strongly accretive, ϕ-strongly accretive, ϕ-strongly quasi-accretive), where I denotes the identity operator on X. Such operators have been extensively studied and used by several authors (see [4, 7, 22, 27, 28, 31]).

Recently, several strong convergence theorems for the Mann (steepest descent approximation) and Ishikawa iterative (generalized steepest descent approximation) processes in general Banach spaces have been established for approximating either fixed points of strong pseudo-contractions acted from a nonempty convex subset K into itself or solutions of nonlinear equations with accretive operators acted from a Banach space X into itself (see [4, 7, 11, 19, 29, 30]).

In several practical applications, it is well known that a mapping with domain $D(T)$ and range $R(T)$ need not be a self-mapping. If the domain of T, $D(T)$, is a proper subset of the Banach space X and T

maps $D(T)$ into X, then neither the Mann nor the Ishikawa iterative process may be well defined.

It is our purpose in this paper to establish several strong convergence theorems for the Mann iterative (steepest descent approximation) and Ishikawa iterative (generalized steepest descent approximation) processes involving a class of non-self-mappings in general Banach spaces.

For this purpose, we need the following:

LEMMA 1.1 [29] *Let X be a real Banach space. Then the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

holds for all $x, y \in X$ and all $j(x+y) \in J(x+y)$.

2. MAIN RESULTS

THEOREM 2.1 *Let X be a real Banach space and let $A: D(A) \subset X \to X$ be a ϕ-strongly quasi-accretive and uniformly continuous operator. Suppose that, for some initial value $x_0 \in D(A)$, $\phi^{-1}(\|Ax_0\|)$ is well-defined and there exists a closed ball $B_1 = \{x \in D(A): \|x - x_0\| \leq 3\phi(\|Ax_0\|)\}$ contained in $D(A)$. Then the generalized steepest descent approximation process $(GSDA)_1$ defined by*

$$\begin{cases} x_0 \in B_1, \\ x_{n+1} = x_n - \alpha_n A y_n, \\ y_n = x_n - \beta_n A x_n \end{cases} \qquad (\mathrm{GSDA})_1$$

for all $n \geq 0$ remains in B_1 and converges strongly to $x^* \in N(A)$ provided that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n \leq \min\{((\phi^{-1}(\|Ax_0\|))/(2M)), (\delta/2M)\}$, $n \geq 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and*
(iii) $\alpha_n \to 0$, $\beta_n \to 0$ *as $n \to \infty$, where M and δ are some fixed constants.*

Moreover, if $\inf_{t \geq 0}(\phi(t)/t) > 0$, then we have the error estimation

$$\|x_n - x^*\|^2 \leq r^2 \theta_n,$$

where $r = \max\{\phi^{-1}(\|Ax_0\|), 1\}$ and $\theta_n \leq 1$ for all $n \geq 0$ and $\theta_n \to 0$ as $n \to \infty$.

Proof We first observe that

$$\|x_0 - x^*\| \le \phi^{-1}(\|Ax_0\|). \tag{4}$$

Let $M = \sup\{\|Au\|: u \in B_1\}$. Since $A: D(A) \subset X \to X$ is uniformly continuous, we can choose a positive constant δ such that

$$\|Ax - Ay\| \le \frac{\|Ax_0\|}{2}$$

whenever $\|x-y\| \le \delta$. Now we can choose $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the conditions (i)–(iii).

CLAIM 1 $\|y_n - x^*\| \le \phi^{-1}(\|Ax_0\|)$ *whenever* $\|x_n - x^*\| \le \phi^{-1}(\|Ax_0\|)$.

Let $\|x_n - x^*\| \le \phi^{-1}(\|Ax_0\|)$. *Then* $\|x_n - x_0\| \le 2\phi^{-1}(\|Ax_0\|)$ *and then* $\|Ax_n\| \le M$. *On the other hand, by* $(GSDA)_1$, *we have*

$$\|y_n - x^*\| \le \phi^{-1}(\|Ax_0\|) + \beta_n M \le 2\phi^{-1}(\|Ax_0\|),$$

which shows $y_n \in B_1$. *Now we want to show that* $\|y_n - x^*\| \le \phi^{-1}(\|Ax_0\|)$. *Suppose that* $\|y_n - x^*\| > \phi^{-1}(\|Ax_0\|)$. *Then* $\phi(\|y_n - x^*\|) \ge \|Ax_0\|$. *Observe that*

$$\|y_n - x_n\| \le \beta_n\|Ax_n\| \le \beta_n M \le \delta$$

and so we have

$$\|A(y_n) - A(x_n)\| \le \frac{\|Ax_0\|}{2}. \tag{5}$$

By using Lemma 1.1, $(GSDA)_1$ *and* (5), *we have*

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - x^* - \beta_n A x_n\|^2 \\
&\le \|x_n - x^*\|^2 - 2\beta_n\langle Ax_n, j(y_n - x^*)\rangle \\
&\le \|x_n - x^*\|^2 - 2\beta_n\phi(\|y_n - x^*\|)\|y_n - x^*\| \\
&\quad + 2\beta_n\phi^{-1}(\|Ax_0\|)\|A(y_n) - A(x_n)\| \\
&\le \|x_n - x^*\|^2 - 2\beta_n\|Ax_0\|\phi^{-1}(\|Ax_0\|) \\
&\quad + 2\beta_n\|Ax_0\|\phi^{-1}(\|Ax_0\|) \\
&\le \|x_n - x^*\|^2,
\end{aligned} \tag{6}$$

which implies that $\|y_n - x^*\| \le \|x_n - x^*\| \le \phi^{-1}(\|Ax_0\|)$. *This contradicts the assumption* $\|y_n - x^*\| > \phi^{-1}(\|Ax_0\|)$.

CLAIM 2 $\|x_n - x^*\| \le \phi^{-1}(\|Ax_0\|)$ *for all* $n \ge 0$.

First of all, $\|x_0 - x^*\| \le \phi^{-1}(\|Ax_0\|)$. *Let* $\|x_n - x^*\| \le \phi^{-1}(\|x_0\|)$. *We shall prove that*

$$\|x_{n+1} - x^*\| \le \phi^{-1}(\|Ax_0\|).$$

Assume that it is not the case, i.e., $\|x_{n+1} - x^*\| > \phi^{-1}(\|Ax_0\|)$. *Then we have*

$$\phi(\|x_{n+1} - x^*\|) > \|Ax_0\|.$$

Since $\|x_n - x^*\| \le \phi^{-1}(\|Ax_0\|)$, *by Claim* 1, *we see that*

$$\|y_n - x^*\| \le \phi^{-1}(\|Ax_0\|).$$

Thus it follows from the definition of M *that* $\|Ax_n\| \le M$ *and* $\|Ay_n\| \le M$. *Observe that*

$$\|x_{n+1} - x^*\| \le \phi^{-1}(\|Ax_0\|) + \alpha_n M \le 2\phi^{-1}(\|Ax_0\|),$$

so that $x_{n+1} \in B_1$. *Observing that* $\|y_n - x_{n+1}\| \le (\alpha_n + \beta_n)M \le \delta$, *in view of the uniform continuity of* A, *we have*

$$\|Ay_n - Ax_{n+1}\| \le \frac{\|Ax_0\|}{2}.$$

It follows from Lemma 1.1 *and the above arguments that*

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 - 2\alpha_n \langle Ay_n, j(x_{n+1} - x^*)\rangle \\
&\quad \le \|x_n - x^*\|^2 - 2\alpha_n \langle Ay_n - Ax_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\qquad - 2\alpha_n \langle Ax_{n+1} - Ax^*, j(x_{n+1} - x^*)\rangle \\
&\quad \le \|x_n - x^*\|^2 + 2\alpha_n \|Ax_0\| \phi^{-1}(\|Ax_0\|) \\
&\qquad - 2\alpha_n \phi^{-1}(\|Ax_0\|)\|Ax_0\| \\
&\quad \le \|x_n - x^*\|^2, \qquad (7)
\end{aligned}$$

which implies that

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\| \le \phi^{-1}(\|Ax_0\|).$$

This is a contradiction and so Claim 2 *is true.*

CLAIM 3 *$x_n \to x^*$ as $n \to \infty$.*

Again using Lemma 1.1, we have

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - 2\alpha_n\phi(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| + o(\alpha_n). \quad (8)$$

Set $\limsup_{n\to\infty} \|x_{n+1} - x^*\| = a$. Then $a = 0$. If not, suppose $a > 0$. Then we can do prove that there exists an positive integer N_0 such that for all $n \ge N_0$, $\|x_{n+1} - x^*\| \ge (a/2)$ and hence, for all $n \ge N_0$,

$$\phi(\|x_{n+1} - x^*\|) \ge \phi\left(\frac{a}{2}\right).$$

At this point, we can choose $N_1 \ge N_0$ so large that

$$o(\alpha_n) \le \frac{a}{2}\phi\left(\frac{a}{2}\right)\alpha_n \quad (9)$$

for all $n \ge N_1$. It follows from (8) *and* (9) *that*

$$\frac{a}{2}\phi\left(\frac{a}{2}\right)\sum_{n=N_1}^{\infty} \alpha_n \le \|x_{N_1} - x^*\|^2 < \infty,$$

which contradicts the assumption (ii). *Therefore, we have*

$$\liminf_{n\to\infty} \|x_{n+1} - x^*\| \le \limsup_{n\to\infty} \|x_{n+1} - x^*\| = 0.$$

This implies that $x_n \to x^*$ as $n \to \infty$.

Now we consider an error estimation. For this purpose, assume that

$$\inf_{n \ge 0} \frac{\phi(\|x_{n+1} - x^*\|)}{\|x_{n+1} - x^*\|} = \sigma > 0.$$

Without loss of generality, we assume that $o(\alpha_n) \le 2\sigma\alpha_n$ for all $n \ge 0$. Define iteratively a real sequence $\{\theta_n\}_{n \ge 0}$ as follows:

$$\begin{cases} \theta_0 = 1, \\ \theta_{n+1} = (1 - ((2\sigma\alpha_n)/(1 + 2\sigma\alpha_n)))\theta_n + ((o(\alpha_n))/(1 + 2\sigma\alpha_n)) \end{cases}$$

for all $n \ge 0$. Then we have that $\theta_n \le 1$, $\lim_{n\to\infty}\theta_n = 0$ and $\|x_n - x^*\|^2 \le r^2\theta_n$ for all $n \ge 0$, where $r = \max\{\phi^{-1}(\|Ax_0\|), 1\}$. This completes the proof.

Remark 1 If A is Lipschitz continuous in Theorem 2.1, then we can choose $\delta \le (\|Ax_0\|/2L)$, where $L \ge 1$ is the Lipschitz constant for A.

Remark 2 If $D(A) = X$ and ϕ is surjective, then, for all $x_0 \in X$, $\phi^{-1}(\|Ax_0\|)$ always is well-defined. Moreover, $B_1 \subset X$. In this case, the convergence in Theorem 2.1 is global.

Remark 3 By taking $\beta_n = 0$ for all $n \ge 0$, then we obtain the corresponding convergence theorem for the steepest descent approximation to accretive operator equations in arbitrary Banach spaces.

THEOREM 2.2 *Let X be a real Banach space and let A: $D(A) \subset X \to X$ be a uniformly continuous ϕ-hemicontractive mapping. Set $T = I - A$. Suppose that, for some initial value $x_0 \in D(A)$, $\phi^{-1}(\|Tx_0\|)$ is well-defined and there exists a close ball $B_2 = \{x \in D(A): \|x - x_0\| \le 3\phi(\|Tx_0\|)\}$ contained in $D(A)$. Then the Ishikawa iterative process $(IS)_1$ defined by*

$$\begin{cases} x_0 \in B_2 \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n A y_n, \\ y_n = (1-\beta_n)x_n + \beta_n A x_n \end{cases} \qquad (\mathrm{IS})_1$$

for all $n \ge 0$ remains in B_2 and converges strongly to $x^* \in F(A)$ provided that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1)$ satisfying the following conditions:

(i) $\alpha_n + 2\beta_n \le \min\{((\|Tx_0\|\phi^{-1}(\|Tx_0\|))/(2(M + \phi^{-1}(\|Tx_0\|))), (\delta/(2M)), ((\phi^{-1}(\|Tx_0\|))/(2M))\}$, $n \ge 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and*
(iii) $\alpha_n \to 0$, $\beta_n \to 0$ *as $n \to \infty$, where M and δ are some fixed constants.*

Moreover, if $\inf_{t \ge 0} (\phi(t)/t) > 0$, then we have the error estimation

$$\|x_n - x^*\|^2 \le r^2 \theta_n,$$

where $r = \max\{\phi^{-1}(\|Tx_0\|), 1\}$ and $\theta_n \le 1$ for all $n \ge 0$ and $\theta_n \to 0$ as $n \to \infty$.

Proof We first observe that, if $F(A) \ne \emptyset$, then $F(A)$ must be a singleton. Let $F(A) = \{x^*\}$. By the definition of A, we have

$$\|x_0 - x^*\| \le \phi^{-1}(\|Tx_0\|). \tag{10}$$

Since A: $D(A) \to X$ is uniformly continuous on $D(A)$, so is T. Let $M = \sup\{\|Ty\|: y \in B_2\}$. Then $M < +\infty$. For $\varepsilon = (\|Tx_0\|\phi^{-1}(\|Tx_0\|))/(2(M + \phi^{-1}(\|Tx_0\|)))$, there exists $\delta > 0$ such that $\|Tx - Ty\| \leq \varepsilon$ whenever $\|x - y\| \leq \delta$. At this point we can choose $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the conditions (i)–(iii).

We shall prove that $\|y_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ whenever $\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$. Let $\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$. Then

$$\|x_n - x_0\| \leq 2\phi^{-1}(\|Tx_0\|)$$

and hence $\|Tx_n\| \leq M$ by the definition of M. Observe that

$$\|y_n - x^*\| \leq \phi^{-1}(\|Tx_0\|) + \beta_n M \leq 2\phi^{-1}(\|Tx_0\|)$$

and

$$\|x_n - y_n\| \leq \beta_n M \leq \delta.$$

Thus we have $\|Tx_n - Ty_n\| \leq \varepsilon$. Assume that $\|y_n - x^*\| > \phi^{-1}(\|Tx_0\|)$. Then we have $\phi(\|y_n - x^*\|) > \|Tx_0\|$. Using Lemma 1.1, $(\mathrm{IS})_1$ and the above arguments, we have

$$\begin{aligned} \|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\beta_n\langle Tx_n, j(y_n - x^*)\rangle \\ &\leq \|x_n - x^*\|^2 + 2\beta_n\|Tx_n - Ty_n\|\|y_n - x^*\| \\ &\quad - 2\beta_n\phi(\|y_n - x^*\|)\|y_n - x^*\| \\ &\leq \|x_n - x^*\|^2, \end{aligned} \tag{11}$$

which implies that $\|y_n - x^*\| \leq \|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$. This is a contradiction and so $\|y_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ whenever $\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$. Observe that

$$\|x_{n+1} - x^*\| \leq \alpha_n M + \phi^{-1}(\|Tx_0\|) \leq 2\phi^{-1}(\|Tx_0\|)$$

and

$$\|x_{n+1} - y_n\| \leq (2\beta_n + \alpha_n)M \leq \delta,$$

so that $\|Tx_{n+1} - Ty_n\| \leq \varepsilon$. Again using Lemma 1.1, $(\mathrm{IS})_1$ and the arguments above, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\alpha_n\langle x_n - Ay_n, j(x_{n+1} - x^*)\rangle \\ &\leq \|x_n - x^*\|^2 + 2\alpha_n\|x_n - y_n\|\|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n\|Ty_n - Tx_{n+1}\|\|x_{n+1} - x^*\| \\ &\quad - 2\alpha_n\phi(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| \\ &\leq \|x_n - x^*\|^2, \end{aligned} \tag{12}$$

which implies that $\|x_{n+1} - x^*\| \leq \|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$. This is a contradiction and so $\|x_{n+1} - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ whenever $\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$. By induction, we assert that

$$\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$$

for all $n \geq 0$. Therefore, $\|Tx_n\| \leq M$ and $\|Ty_n\| \leq M$ for all $n \geq 0$. It follows from Lemma 1.1 and $(\mathrm{IS})_1$ that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad \leq \|x_n - x^*\|^2 - 2\alpha_n\phi(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| + o(\alpha_n). \end{aligned} \tag{13}$$

Set $\limsup_{n\to\infty} \|x_{n+1} - x^*\| = \tau$. Then $\tau = 0$. If not, then we can do prove that there exists a positive integer N_0 such that, for all $n \geq N_0$, $\|x_{n+1} - x^*\| \geq (\tau/2)$ and then $\phi(\|x_{n+1} - x^*\|) \geq \phi(\tau/2)$. At this point, we can choose $N_1 \geq N_0$ so large that $o(\alpha_n) \leq (\tau/2)\phi(\tau/2)$. It follows from (13) that

$$\frac{\tau}{2}\phi\left(\frac{\tau}{2}\right) \sum_{n \geq N_1} \alpha_n \leq \|x_{N_1} - x^*\|^2 < \infty,$$

which is a contradiction and so $\tau = 0$. Therefore, we have

$$\liminf_{n\to\infty} \|x_{n+1} - x^*\| \leq \limsup_{n\to\infty} \|x_{n+1} - x^*\| = 0,$$

i.e., $x_n \to x^*$ as $n \to \infty$.

Now we consider an error estimation. For this purpose, assume that

$$\inf_{n \geq 0} \frac{\phi(\|x_{n+1} - x^*\|)}{\|x_{n+1} - x^*\|} = \sigma > 0.$$

Without loss of generality, we assume that $o(\alpha_n) \leq 2\sigma\alpha_n$ for all $n \geq 0$. Define iteratively a real sequence $\{\theta_n\}_{n\geq 0}$ as follows:

$$\begin{cases} \theta_0 = 1, \\ \theta_{n+1} = (1 - ((2\sigma\alpha_n)/(1 + 2\sigma\alpha_n)))\theta_n + ((o(\alpha_n))/(1 + 2\sigma\alpha_n)) \end{cases}$$

for all $n \geq 0$. Then we have that $\theta_n \leq 1$, $\lim_{n\to\infty}\theta_n = 0$ and $\|x_n - x^*\|^2 \leq r^2\theta_n$ for all $n \geq 0$, where $r = \max\{\phi^{-1}(\|Tx_0\|), 1\}$. This completes the proof.

Remark 4 If A is Lipschitz continuous in Theorem 2.2, then we can choose $\delta \leq ((\|Tx_0\|\phi^{-1}(\|Tx_0\|))/(2L(M + \phi^{-1}(\|Tx_0\|)))$, where $L \geq 1$ is the Lipschitz constant for A.

Remark 5 If $D(A) = X$ and ϕ is surjective, then the convergence in Theorem 2.2 is global.

Remark 6 By taking $\beta_n = 0$ for all $n \geq 0$ in Theorem 2.2, then we obtain the corresponding convergence theorems for the Mann iterative process in arbitrary Banach spaces.

THEOREM 2.3 *Let X be a real uniformly smooth Banach space and let $T: D(T) \subset X \to X$ be a ϕ-strongly quasi-accretive operator. Suppose that, for some initial value $x_0 \in D(T)$, $\phi^{-1}(\|Tx_0\|)$ is well-defined and there exists a closed ball $B = \{x \in D(T): \|x - x_0\| \leq 3\phi^{-1}(\|Tx_0\|)\} \subset D(T)$ such that $T(B)$ is bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in* [0, 1] *satisfying the following conditions*:

(i) $\alpha_n, \beta_n \to 0$ *as* $n \to \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n \leq \min\{(\delta/M), ((\phi^{-1}(\|Tx_0\|))/(4M))\}$, *where* $\delta > 0$ *and* M *are some fixed constants satisfying the property*:

$$\|j(x) - j(y)\| \leq \frac{\phi[(1/4)\phi^{-1}(\|Tx_0\|)]\phi^{-1}(\|Tx_0\|)}{16M}$$

whenever $x, y \in B(0, 2\phi^{-1}(\|Tx_0\|))$ *and* $\|x - y\| \leq \delta$.

Define the generalized steepest descent approximation $\{x_n\}_{n\geq 0}$ *as follows*:

$$\begin{cases} x_0 \in D(T), \\ x_{n+1} = x_n - \alpha_n T y_n, \\ y_n = x_n - \beta_n T x_n \end{cases} \qquad (\mathrm{GSDA})_2$$

for all $n \geq 0$. *Then we have the following conclusions*:

(1) T *has a unique zero point in* $D(T)$;
(2) $\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ *for all* $n \geq 0$;
(3) $x_n \to x^*$ *as* $n \to \infty$;
(4) *If* $\inf_{t>0}((\phi(t))/t) > 0$, *then we have also the error estimation*:

$$\|x_n - x^*\| \leq r^2\theta_n$$

for all $n \geq 0$, *where* $r = \max\{\phi^{-1}(\|Tx_0\|), 1\}$ *and* $\theta_n \to 0$ *as* $n \to \infty$.

Proof The proof of (1): If T has two zeros x^*, $x_1^* \in N(T)$, then, by definition of T, we have

$$0 = \langle Tx^* - Tx_1^*, j(x^* - x_1^*)\rangle \geq \phi(\|x^* - x_1^*\|)\|x^* - x_1^*\|,$$

which gives that $x^* = x_1^*$. In the sequel, we denote the unique zero point of T by x^*. Set $M = \sup\{\|Ty\|: y \in B\}$.

The proof of (2): Since X is real uniformly smooth, j is uniformly continuous on the ball $B(0, 2\phi^{-1}(\|Tx_0\|))$ and hence for $\varepsilon = ((\phi((1/4)\phi^{-1}(\|Tx_0\|))\phi^{-1}(\|Tx_0\|)/(16M))$, there exists some fixed $\delta > 0$ such that $\|j(x) - j(y)\| \leq \varepsilon$ whenever $x, y \in B(0, 2\phi^{-1}(\|Tx_0\|))$ and $\|x - y\| \leq \delta$. We finish the proof of (2) by the following two steps:

Step (I) $\|y_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ whenever $\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$.

We first observe that $\|x_0 - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ by the choice of $x_0 \in D(T)$ and the assumption that $\phi^{-1}(\|Tx_0\|)$ is well-defined. Now assume that

$$\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|).$$

Then $x_n \in B$ and so $\|Tx_n\| \leq M$. Observe that

$$\|y_n - x^*\| \leq \phi^{-1}(\|Tx_0\|) + \beta_n M \leq 2\phi^{-1}(\|Tx_0\|)$$

and

$$\|y_n - x_n\| \leq \beta_n M \leq \delta,$$

so that $d_n = \|j(y_n - x^*) - j(x_n - x^*)\| \leq \varepsilon$. We want to prove that

$$\|y_n - x^*\| \leq \phi^{-1}(\|Tx_0\|).$$

If not, *i.e.*, $\|y_n - x^*\| > \phi^{-1}(\|Tx_0\|)$, then it follows from $(GSDA)_2$ and the condition (iii) that $\|x_n - x^*\| \geq (1/2)\phi^{-1}(\|Tx_0\|)$ and hence

$$\phi(\|x_n - x^*\|) \geq \phi\left(\frac{1}{2}\phi^{-1}(\|Tx_0\|)\right).$$

Using Lemma 1.1, $(GSDA)_2$ and the above arguments, we obtain

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\beta_n\langle Tx_n, j(y_n - x^*)\rangle \\ &\leq \|x_n - x^*\|^2 + 2M\beta_n d_n - 2\beta_n\phi(\|x_n - x^*\|)\|x_n - x^*\| \\ &\leq \|x_n - x^*\|^2, \end{aligned} \tag{14}$$

which implies that

$$\|y_n - x^*\| \leq \|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|).$$

This is a contradiction and so Step (I) is true.

Step (II) $\|x_n - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ for all $n \geq 0$.

We have shown that, for $n=0$, the above assertation is true. Now we assume that it is true for $n=k$ and we shall show that it is also true for $n=k+1$. Since $\|x_k - x^*\| \leq \phi^{-1}(\|Tx_0\|)$, we see that $\|y_k - x^*\| \leq \phi^{-1}(\|Tx_0\|)$ by Step (I) and hence $\|Tx_k\| \leq M$ and $\|Ty_k\| \leq M$. On the other hand, observe that

$$\|x_{k+1} - x^*\| \leq \phi^{-1}(\|Tx_0\|) + \alpha_k M \leq 2\phi^{-1}(\|Tx_0\|)$$

and

$$\|x_{k+1} - y_k\| \leq (\alpha_k + \beta_k)M \leq \delta$$

so that $e_k = \|j(x_{k+1} - x^*) - j(y_k - x^*)\| \leq \varepsilon$.

Now we plan to show that $\|x_{k+1} - x^*\| \leq \phi^{-1}(\|Tx_0\|)$. If not, then we have

$$\|x_k - x^*\| \geq \frac{1}{2}\phi^{-1}(\|Tx_0\|), \quad \|y_k - x^*\| \geq \frac{1}{4}\phi^{-1}(\|Tx_0\|)$$

by $(GSDA)_2$ and

$$\phi(\|y_k - x^*\|) \geq \phi\left(\frac{1}{4}\phi^{-1}(\|Tx\|)\right)$$

by the property of ϕ. It follows from Lemma 1.1, $(\mathrm{GSDA})_2$ and the above arguments that

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
&\quad \le \|x_k - x^*\|^2 - 2\alpha_k\langle Ty_k, j(x_{k+1} - x^*)\rangle \\
&\quad \le \|x_k - x^*\|^2 + 2M\alpha_k e_k - 2\alpha_k\phi(\|y_k - x^*\|)\|y_k - x^*\| \\
&\quad \le \|x_k - x^*\|^2, \qquad (15)
\end{aligned}$$

which implies that

$$\|x_{k+1} - x^*\| \le \|x_k - x^*\| \le \phi^{-1}(\|Tx_0\|).$$

This is a contradiction and so Step (II) is true.

The proof of (3): By (2), we see that $\|x_n - x^*\| \le \phi^{-1}(\|Tx_0\|)$ and $\|y_n - x^*\| \le \phi^{-1}(\|Tx_0\|)$ and so $\|Tx_n\| \le M$ and $\|Ty_n\| \le M$. Observe that

$$\|x_{n+1} - y_n\| \le (\alpha_n + \beta_n)M \to 0$$

as $n \to \infty$. Thus, in view of the uniform continuity of j, we assert that $f_n = \|j(x_{n+1} - x^*) - j(y_n - x^*)\| \to 0$ as $n \to \infty$.

Set $\limsup_{n\to\infty}\|y_n - x^*\| = a$. Then $a = 0$. If it is not the case, then we can do prove that there exists a positive integer n_0 such that, for all $n \ge n_0$, $\|y_n - x^*\| \ge a/2$ and then $\phi(\|y_n - x^*\|) \ge \phi(a/2)$. At this point, we can choose $n_1 \ge n_0$ such that, for all $n \ge n_1$, $f_n \le (a\phi(a/2)/2M)$. Using Lemma 1.1, $(\mathrm{GSDA})_2$ and the above arguments, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 - 2\alpha_n\langle Ty_n, j(x_{n+1} - x^*)\rangle \\
&\quad \le \|x_n - x^*\|^2 + 2M\alpha_n f_n - 2\alpha_n\phi(\|y_n - x^*\|)\|y_n - x^*\| \\
&\quad \le \|x_n - x^*\|^2 - \frac{a}{2}\phi\left(\frac{a}{2}\right)\alpha_n, \qquad (16)
\end{aligned}$$

which implies that

$$\frac{a}{2}\phi\left(\frac{a}{2}\right)\sum_{n\ge n_1}\alpha_n \le \|x_{n_1} - x^*\|^2 < \infty.$$

This is a contradiction and so

$$\liminf_{n\to\infty} \|y_n - x^*\| \leq \limsup_{n\to\infty} \|y_n - x^*\| = 0,$$

which implies $y_n \to x^*$ as $n \to \infty$. By $(\text{GSDA})_2$, we see that $x_n \to x^*$ as $n \to \infty$. This completes the proof of (3).

The proof of (4): Assume that $\inf_{n\geq 0}(\phi(\|y_n - x^*\|)/\|y_n - x^*\|) = \sigma > 0$. By using Lemma 1.1 and $(\text{GSDA})_2$, we have

$$\begin{aligned}\|x_n - x^*\|^2 &\leq \|y_n - x^*\|^2 + 2\beta_n\langle Tx_n, j(x_n - x^*)\rangle \\ &\leq \|y_n - x^*\|^2 + 2M\beta_n\|x_n - x^*\| \\ &\leq \|y_n - x^*\|^2 + 2M\beta_n(\|x_n - y_n\| + \|y_n - x^*\|) \\ &\leq \|y_n - x^*\|^2 + 2M\beta_n\phi^{-1}(\|Tx_0\|) + 2M^2\beta_n^2, \end{aligned} \tag{17}$$

which implies that

$$\|y_n - x^*\|^2 \geq \|x_n - x^*\|^2 - 2M\beta_n\phi^{-1}(\|Tx_0\|) - 2M^2\beta_n^2. \tag{18}$$

Substituting (18) in (16) yields to

$$\|x_{n+1} - x^*\|^2 \leq (1 - 2\sigma\alpha_n)\|x_n - x^*\|^2 + o(\alpha_n). \tag{19}$$

Without loss of generality, we assume that $o(\alpha_n) \leq 2\sigma\alpha_n$ for all $n \geq 0$. Define iteratively a real sequence $\{\theta_n\}_{n\geq 0}$ as follows:

$$\begin{cases} \theta_0 = 1, \\ \theta_{n+1} = (1 - 2\sigma\alpha_n)\theta_n + 2\sigma\alpha_n \end{cases}$$

for all $n \geq 0$. Then we have that $\theta_n \leq 1$, $\lim_{n\to\infty}\theta_n = 0$ and $\|x_n - x^*\|^2 \leq r^2\theta_n$ for all $n \geq 0$, where $r = \max\{\phi^{-1}(\|Tx_0\|), 1\}$. This completes the proof.

Remark 7 In Theorem 2.3, the choice of the iteration parameters $\{\alpha_n\}$ and $\{\beta_n\}$ depends on the initial value x_0, but not the smoothness $\rho_X(\tau)$ of X. It is very interesting to determine the size of δ. If X is a s-uniformly smooth Banach space, we can give a actual size of δ.

THEOREM 2.4 *Let X be a real uniformly smooth Banach space and let $T: D(T) \subset X \to X$ be a ϕ-hemicontractive mapping. Suppose that, for* *some initial value* $x_0 \in D(T)$, $\phi^{-1}(\|Ax_0\|)$ *is well-defined and there exists a closed ball* $B = \{x \in D(T): \|x - x_0\| \leq 3\phi^{-1}(\|Ax_0\|)\} \subset D(T)$ *such that* $T(B)$ *is bounded, where* $A = I - T$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $\alpha_n, \beta_n \to 0$ *as* $n \to \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then there exist positive constants M *and* δ *such that the Ishikawa iterative process generated by*

$$\begin{cases} x_0 \in D(T), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \end{cases} \tag{IS$_2$}$$

for all $n \geq 0$ *is well-defined and converges strongly to the unique fixed point* q *of* T *provided that*

$$\alpha_n + \beta_n \leq \min\left\{ \frac{\delta}{2M + \phi^{-1}(\|Ax_0\|)}, \frac{\phi^{-1}(\|Ax_0\|)}{4(2M + \phi^{-1}(\|Ax_0\|))} \right\}.$$

Moreover, if $\inf_{t>0}(\phi(t)/t) > 0$, *then we have also the error estimation*:

$$\|x_n - q\| \leq r^2 \theta_n$$

for all $n \geq 0$, *where* $r = \max\{\phi^{-1}(\|Ax_0\|), 1\}$ *and* $\theta_n \to 0$ *as* $n \to \infty$.

Proof Let $M = \sup\{\|Tx\|: x \in B\}$. Then $M < \infty$ since $T(B)$ is bounded. Since X is uniformly smooth, j is uniformly continuous on bounded subsets of X. Hence, for $\varepsilon = (\phi^{-1}(\|Ax_0\|)\phi(1/4\phi^{-1}(\|Ax_0\|))/8M)$, there exists a positive constant $\delta > 0$ such that

$$\|j(x) - j(y)\| \leq \varepsilon \tag{20}$$

whenever $x, y \in B$ and $\|x - y\| \leq \delta$. Let q denote the unique fixed point of T. Then, by the definition of T and the choice of x_0, we see that

$$\|x_0 - q\| \leq \phi^{-1}(\|Ax_0\|).$$

We finish the proof of Theorem 2.4 by the following three steps:

Step 1 $\|y_n - q\| \le \phi^{-1}(\|Ax_0\|)$ whenever $\|x_n - q\| \le \phi^{-1}(\|Ax_0\|)$. Assume that $\|x_n - q\| \le \phi^{-1}(\|Ax_0\|)$. Then $\|x_n - x_0\| \le 2\phi^{-1}(\|Ax_0\|)$, so that $\|Tx_n\| \le M$ by the definition of M. Now we want to show that

$$\|y_n - q\| \le \phi^{-1}(\|Ax_0\|).$$

If it is not the case, assume that $\|y_n - q\| > \phi^{-1}(\|Ax_0\|)$. Then we have

$$\|x_n - q\| \ge \phi^{-1}(\|Ax_0\|) - \beta_n(2M + \phi^{-1}(\|Ax_0\|)) \ge \frac{1}{2}\phi^{-1}(\|Ax_0\|),$$

so that

$$\phi(\|x_n - q\|) \ge \phi\left(\frac{1}{2}\phi^{-1}(\|Ax_0\|)\right).$$

Let $a_n = \|j(x_n - q) - j(y_n - q)\|$. Observe that

$$\|y_n - q\| \le 2\phi^{-1}(\|Ax_0\|)$$

and

$$\|y_n - x_n\| \le \beta_n(2M + \phi^{-1}(\|Ax_0\|)) \le \delta.$$

Then, by the uniform continuity of j, we assert that $a_n \le \varepsilon$. Using Lemma 1.1 and $(\mathrm{IS})_2$, we have

$$\begin{aligned}
\|y_n - q\|^2 &\le (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\langle Tx_n - q, j(y_n - q)\rangle \\
&\le (1 - \beta_n)^2\|x_n - q\|^2 + 4M\beta_n a_n \\
&\quad + 2\beta_n\|x_n - q\|^2 - 2\beta_n\phi(\|x_n - q\|)\|x_n - q\| \\
&\le \|x_n - q\|^2 + 4M\beta_n a_n - 2\beta_n\phi(\|x_n - q\|)\|x_n - q\| \\
&\le \|x_n - q\|^2, \qquad (21)
\end{aligned}$$

which implies that

$$\|y_n - q\|^2 \le \|x_n - q\|^2 \le [\phi^{-1}(\|Ax_0\|)]^2,$$

i.e., $\|y_n - q\| \le \phi^{-1}(\|Ax_0\|)$. This is a contradiction.

Step 2 $\|x_n - q\| \le \phi^{-1}(\|Ax_0\|)$ for all $n \ge 0$.

First of all, by the choice of $x_0 \in D(T)$, we know that $\|x_0 - q\| \le \phi^{-1}(\|Ax_0\|)$. Assume that $\|x_n - q\| \le \phi^{-1}(\|Ax_0\|)$. Then, by Step 1 we have

$$\|y_n - q\| \leq \phi^{-1}(\|Ax_0\|).$$

Therefore, $\|Tx_n\| \leq M$ and $\|Ty_n\| \leq M$. If $\|x_{n+1} - q\| > \phi^{-1}(\|Ax_0\|)$, then we have

$$\|x_n - q\| \leq \phi^{-1}(\|Ax_0\|) - \alpha_n(2M + \phi^{-1}(\|Ax_0\|)) \geq \frac{1}{2}\phi^{-1}(\|Ax_0\|)$$

and so

$$\|y_n - q\| \geq \frac{1}{2}\phi^{-1}(\|Ax_0\|) - \beta_n(2M + \phi^{-1}(\|Ax_0\|)) \geq \frac{1}{4}\phi^{-1}(\|Ax_0\|)$$

which leads to

$$\phi(\|y_n - q\|) \geq \phi\left(\frac{1}{4}\phi^{-1}(\|Ax_0\|)\right).$$

Let $b_n = \|j(y_n - q) - j(x_{n+1} - q)\|$. Observe that

$$\|y_n - x_{n+1}\| \leq (\alpha_n + \beta_n)(2M + \phi^{-1}(\|Ax_0\|)) \leq \delta.$$

Then, by the uniform continuity of j, we see that $\|j(y_n) - j(x_{n+1})\| \leq \varepsilon$. It follows from Lemma 1.1 and the above discussions that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle Ty_n - q, j(x_{n+1} - q)\rangle \\
&\quad \leq (1 - \alpha_n)^2\|x_n - q\|^2 + 4M\alpha_n b_n + 2\alpha_n\|y_n - q\|^2 \\
&\qquad - 2\alpha_n\phi(\|y_n - q\|)\|y_n - q\|,
\end{aligned} \tag{22}$$

which implies that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 \leq [\phi^{-1}(\|Ax_0\|)]^2,$$

i.e., $\|x_{n+1} - q\| \leq \phi^{-1}(\|Ax_0\|)$. This is a contradiction.

Step 3 $x_n \to q$ as $n \to \infty$.

It follows from the above arguments that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\phi(\|y_n - q\|)\|y_n - q\| + o(\alpha_n) \tag{23}$$

for very large n, which implies that there exist infinite subsequences $\{y_{n_j}\}$ of $\{y_n\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $y_{n_j} \to q$ and $x_{n_j} \to q$ as $j \to \infty$.

By $(\mathrm{IS})_2$ and induction, we can prove that $x_n \to q$ as $n \to \infty$. Assume that

$$\inf_{n \geq 0} \frac{\phi(\|y_n - q\|)}{\|y_n - q\|} = \sigma > 0$$

By using Lemma 1.1 and $(\mathrm{IS})_2$, we have

$$\begin{aligned} \|x_n - q\|^2 &\leq \|y_n - q\|^2 + 2\beta_n \langle Ax_n, j(x_n - q)\rangle \\ &\leq \|y_n - q\|^2 + 2(M + \phi^{-1}(\|Ax_0\|))\beta_n \|x_n - q\| \\ &\leq \|y_n - q\|^2 + 2(M + \phi^{-1}(\|Ax_0\|))\beta_n (\|x_n - y_n\| + \|y_n - q\|) \\ &\leq \|y_n - q\|^2 + 2(M + \phi^{-1}(\|Ax_0\|))\beta_n \phi^{-1}(\|Ax_0\|) \\ &\quad + 2(M + \phi^{-1}(\|Ax_0\|))^2 \beta_n^2, \end{aligned} \tag{24}$$

which gives to

$$\begin{aligned} \|y_n - q\|^2 \geq \|x_n - q\|^2 &- 2(M + \phi^{-1}(\|Ax_0\|))\beta_n \phi^{-1}(\|Ax_0\|) \\ &- 2(M + \phi^{-1}(\|Ax_0\|))^2 \beta_n^2. \end{aligned} \tag{25}$$

Substituting (25) in (23) yields to

$$\|x_{n+1} - q\|^2 \leq (1 - 2\sigma\alpha_n)\|x_n - q\|^2 + o(\alpha_n). \tag{26}$$

Without loss of generality, we assume $o(\alpha_n) \leq 2\sigma\alpha_n$ for all $n \geq 0$. Define iteratively a real sequence $\{\theta_n\}_{n \geq 0}$ as follows:

$$\begin{cases} \theta_0 = 1, \\ \theta_{n+1} = (1 - 2\sigma\alpha_n)\theta_n + 2\sigma\alpha_n \end{cases}$$

for all $n \geq 0$. Then we have that $\theta_n \leq 1$, $\lim_{n \to \infty} \theta_n = 0$ and $\|x_n - x^*\|^2 \leq r^2\theta_n$ for all $n \geq 0$, where $r = \max\{\phi^{-1}(\|Ax_0\|), 1\}$. This completes the proof.

Remark 8 In Theorem 2.4, the choice of the iteration parameters $\{\alpha_n\}$ and $\{\beta_n\}$ depends on the initial value x_0, but not the smoothness $\rho_X(\tau)$ of X. It is very interesting to determine the size of δ. If X is a s-uniformly smooth Banach space, we can give a actual size of δ.

As a direct consequence of Theorem 2.4, we have the following:

COROLLARY 2.1 *Let X be a real uniformly smooth Banach space and let $T: D(T) \subset X \to X$ be a ϕ-strongly quasi-accretive operator. Suppose that, for some initial value $x_0 \in D(T)$, $\phi^{-1}(\|Tx_0\|)$ is well-defined and there exists a closed ball $B = \{x \in D(T): \|x - x_0\| \leq 3\phi^{-1}(\|Tx_0\|)\} \subset D(T)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n, \beta_n \to 0$ *as* $n \to \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Define $Sx = x - Tx$ for each $x \in D(T)$. Then there exist positive constants $M(x_0)$ and δ such that the Ishikawa iteration process generated by

$$\begin{cases} x_0 \in D(T), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n S x_n \end{cases} \tag{IS$_3$}$$

for all $n \geq 0$ is well-defined and converges strongly to the unique zero point of T provided that

$$\alpha_n + \beta_n \leq min\left\{ \frac{\delta}{2M + \phi^{-1}(\|Tx_0\|)}, \frac{\phi^{-1}(\|Tx_0\|)}{2M + \phi^{-1}(\|Tx_0\|)} \right\}.$$

Proof Observe that $S: D(T) \to X$ is ϕ-hemicontractive. Thus the conclusion of Corollary follows from Theorem 2.4.

Remark 9 The iterative scheme used in Corollary 2.1 is different from one used in Theorem 2.3. We don't know which one's rate of convergence is faster. It is very interesting to make some differences between these two kinds of iterative schemes.

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