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Research Article

Some Results on Bellman Equations of Optimal Production Control in a Stochastic Manufacturing System

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The paper studies the production inventory problem of minimizing the expected discounted present value of production cost control in a manufacturing system with degenerate stochastic demand. We establish the existence of a unique solution of the Hamilton-Jacobi-Bellman (HJB) equations associated with this problem. The optimal control is given by a solution to the corresponding HJB equation.

1. Introduction

Many manufacturing enterprisers use a production inventory system to manage fluctuations in consumer demand for the product. Such a system consists of a manufacturing plant and a finished goods warehouse to store those products which are manufactured but not immediately sold. The advantages of having products in inventory are as follows: first, they are immediately available to meet demand; second, by using the warehouse to store excess production during low demand periods to be available for sale during high demand periods. This usually permits the use of a smaller manufacturing plant than would otherwise be necessary, and also reduces the difficulties of managing the system.

We are concerned with the optimization problem to minimize the expected discounted cost control of production planning in a manufacturing systems with degenerate stochastic demand:

$$J(p) = E\left[\int_0^\infty e^{-\rho t}\left\{h(x_t) + p_t^2\right\}dt\right] \tag{1.1}$$

subject to the dynamics of the state equation which says that the inventory at time t is increased by the production rate and decreased by the demand rate can be written according to

$$dx_t = (p_t - y_t)dt, \quad x_0 = x, \ x > 0, \ 0 \le p_{\min} \le p_{\max}, \tag{1.2}$$

and the demand equation with the production rate is described by the Brownian motion

$$dy_t = Ay_t \, dt + \sigma y_t \, dw_t, \quad y_0 = y, \ y > 0, \tag{1.3}$$

in the class $\mathcal{P}$ of admissible controls of production processes p_t with nonnegative constant

$$p_t \ge 0 \tag{1.4}$$

defined on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with the natural filtration $\mathcal{F}_t$ generated by $\sigma(w_s, s \le t)$ carrying a one-dimensional standard Brownian motion w_t, x_t is the inventory level for production rate at time t (state variable), y_t is the demand rate at time t, p_t is the production rate at time t (control variable), $\rho > 0$ is the constant nonnegative discount rate, A is the nonzero constant, σ is nonzero constant diffusion coefficient, x_0 is the initial value of inventory level, and y_0 is the initial value of demand rate.

This optimization control problem of production planning in manufacturing systems has been studied by many authors like Fleming et al. [1], Sethi and Zhang [2], Sprzeuzkouski [3], Hwang et al. [4], Hartl and Sethi [5], and Feichtinger and Hartl [6]. The Bellman equation associated with production inventory control problem is quite different from them and it is treated by Bensoussan et al. [7] for the one-dimensional manufacturing systems with the unbounded control region. Generally speaking, the similar type of linear control problems has been investigated for the stochastic deferential systems with invariant measures like Bensoussan [8], and Borkar [9]. The works of Bensoussan and Frehse [10] Da Prato and Ichikawa [11] on the Bellman equation of ergodic control without convex and polynomial growth hypothesis and the linear quadratic case are done for the linear ergodic control problem. This type of optimization problem has been studied also by Morimoto and Kawaguchi [12] for renewable resources as well as Baten and Sobhan Miah [13] for one-sector neoclassical growth model with the CES function. The optimality can be shown by an extension of the results given in Fujita and Morimoto [14], and for another setting of optimal control in manufacturing systems they are available in Morimoto and Okada [15] and Sethi et al. [16]. These papers treat the cases with bounded control regions. On the contrary, our control region is unbounded as in (1.4).

The purpose of the paper is to give an optimal production cost control by an existence unique solution associated with the two-dimensional HJB equation. We apply the technique of dynamic programming principle [17] for the Riccati-based solution of the reduced (one-dimensional) HJB equation corresponding to production inventory control problem. This paper is organized as follows. In Section 2 by the principle of optimality Bellman [17], we have obtained the HJB equation and then the two-dimensional HJB equation has been reduced to one-dimensional second-order differential equation. We have derived the dynamics of inventory-demand ratio that evolves according to stochastic neoclassical differential equation through Itô's lemma. We have finally found the Riccati-based solution of production inventory control problem that is satisfied by the value function of this optimization problem. In Section 3 we have established the properties of the value function and have shown the existence of an unique solution associated with the reduced (one-dimensional) Hamilton-Jacobi-Bellman (HJB) equation. Finally in Section 4 we present an application to production control of optimization problem (1.1) subject to (1.2) and (1.3).

2. Riccati-Based Solution of Hamilton-Jacobi Bellman Equation

2.1. The Hamilton-Jacobi-Bellman Equation

Suppose $u(x,y,t) : \mathbf{R}^{\mathbf{n}} \times \mathbf{R}^{\mathbf{n}} \times \mathbf{R} \to \mathbf{R}$ is a function whose value is the minimum value of the objective function of the production inventory control problem for the manufacturing system given that we start it at time t in state x, and y. That is,

$$u(x,y,t) = \inf_{p} J(p), \tag{2.1}$$

where the value function u is finite valued and twice continuously differentiable on $(0,\infty)$. We initially assume that $u(x,y,t)$ exists for all x, y, and t in the relevant ranges.

Since (1.2) and (1.3) is a scalar equation, the subscript t here means only time t. Thus, x and y will not cause any confusion and, at the same time, will eliminate the need of writing many parentheses. Thus, dw_t is a scalar.

To solve the problem defined by (1.1), (1.2), and (1.3), let $u(x,y,t)$, known as the value function, be the expected value of the objective function (1.1) from t to infinity, when an optimal policy is followed from t to infinity, given $x_t = x$, $y_t = y$. Then by the principle of optimality [17],

$$u(x,y,t) = \min_{p}\left[\left\{h(x_t) + p_t^2\right\}dt + u\left(x + dx_t, y + dy_t, t + dt\right)\right]. \tag{2.2}$$

We assume that $u(x,y,t)$ is a continuously differentiable of its arguments. By Taylor's expansion, we have

$$\begin{aligned} u\left(x + dx_t, y + dy_t, t + dt\right) = -\rho u(x,y,t)dt + u_y\, dy_t + u_x\, dx_t + \frac{1}{2}u_{yy}(dy_t)^2 \\ +\ \text{higher order terms}. \end{aligned} \tag{2.3}$$

From (1.2), we can formally write

$$(dx_t)^2 = (p)^2(dt)^2 + (y)^2(dt)^2 - 2py(dt)^2, \tag{2.4}$$

$$(dy_t)^2 = (Ay)^2(dt)^2 + (\sigma y)^2(dw_t)^2 + 2(Ay)(\sigma y)dw_t\, dt, \tag{2.5}$$

$$dx_t\, dt = p(dt)^2 - y(dt)^2, \tag{2.6}$$

$$dy_t\, dt = (Ay)(dt)^2 + (\sigma y)dw_t\, dt. \tag{2.7}$$

The exact meaning of these expressions comes from the theory of stochastic calculus; Arnold [18, chapter 5] and Karatzas and Shreve [19]. For our purposes, it is sufficient to know the multiplication rules of the stochastic calculus:

$$(dw_t)^2 = dt, \qquad dw_t\, dt = 0, \qquad dt^2 = 0. \tag{2.8}$$

Substitute (2.3) into (2.2) and use (2.4), (2.5) (2.6), (2.7), and (2.8) to obtain

$$u = \min\left[-\rho u(x,y,t)dt + \frac{1}{2}\sigma^2 y^2 u_{yy}\, dt + Ayu_y\, dt - yu_x\, dt + \left\{p^2 + pu_x\right\}dt + h(x)dt + \circ(dt)\right]. \tag{2.9}$$

Note that we have suppressed the arguments of the functions involved in (2.3).

Canceling the term u on both sides of (2.9), dividing the remainder by dt, and letting $t \to 0$, we obtain the dynamic programming partial differential equation or Hamilton-Jacobi-Bellman equation

$$-\rho u(x,y) + \frac{1}{2}\sigma^2 y^2 u_{yy} + Ayu_y - yu_x + F^*(u_x) + h(x) = 0, \quad u(0,y) = 0, \quad x, y > 0, \tag{2.10}$$

where $F^*(x)$ is the Legendre transform of $F(x)$, that is, $F^*(x) = \min_{p>0}\{p^2 + px\} = -x^2/4$ and u_x, u_y, u_{xx}, u_{yy} are partial derivatives of $u(x,y,t)$ with respect to x and y.

2.2. A Reduction to 1-Dimensional Case

In this subsection, the general (two-dimensional) HJB equation has been reduced to a one-dimensional second-order differential equation. From the two-dimensional state space form (one state x for inventory level and the other state y for demand rate), it has been reduced to one-dimensional form for ($z = x/y$) the ratio of inventory to demand.

There exists a $v \in C(0,\infty)$ such that $u(x,y) = y^2 v(x/y)$, $y > 0$. Since $u_x = yv'(x/y)$, $u_y = 2yv(x/y) - xv'(x/y)$, $u_{yy} = 2v(x/y) - 2(x/y)v'(x/y) + (x/y)^2 v''(x/y)$. Setting $z = x/y$ and substituting these in (2.10), we have

$$\begin{aligned} &-\rho v(z) + \frac{1}{2}\sigma^2\left[2v(z) - 2v'(z)z + z^2 v''(z)\right] + 2Av(z) - Azv'(z) - v'(z) \\ &\quad + \min_{p\geq 0}\left(p^2 + pyv'(z)\right) + h(z) = 0. \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned} \min_{p\geq 0}\left(p^2 + pyv'(z)\right) &= y^2 \min_{p\geq 0}\left(\left(\frac{p}{y}\right)^2 + \left(\frac{p}{y}\right)v'(z)\right) = y^2 \min_{q\geq 0}\left(q^2 + qv'(z)\right), \\ \min_{q\geq 0}\left(q^2 + qv'(z) - v'(z)\right) &= \min_{k+1\geq 0}\left((k+1)^2 + kv'(z)\right). \end{aligned} \tag{2.12}$$

Then the HJB equation (2.11) becomes

$$-\tilde{\rho}v(z) + \frac{1}{2}\sigma^2 z^2 v''(z) + \tilde{A}zv'(z) + \min_{k\geq -1}\Big((k+1)^2 + kv'(z)\Big) + h(z) = 0, \quad v(0) = 0,\ z > 0, \tag{2.13}$$

where $\tilde{\rho} = -\rho + \sigma^2 + 2A$, $\tilde{A} = -(A + \sigma^2)$, and $F^*(z)$ is the Legendre transform of $F(z)$, that is, $F^*(z) = \min_{(k+1)\geq 0}\{(k+1)^2 + kz\} = -z^2/4 - z$.

The main feature of the HJB equation (2.13) is the vanishing of the coefficient of u_{xx} for $x = 0$ in partial differential equation terminology, then the equation is degenerate elliptic. Generally speaking, the difficulty stems from the degeneracy in the second-order term of the HJB equation (2.13).

2.3. *Value Function*

Let us consider the minimum value of the payoff function is a function of this initial point. The value function can be defined as a function whose value is the minimum value of the objective function of the production inventory control problem (1.1) for the manufacturing system, that is,

$$\begin{aligned} V(z) &= \inf E\left[\int_0^\infty e^{-\tilde{\rho}t}\Big\{h(z_t) + (k_t+1)^2\Big\}dt\right] \\ &= \inf \tilde{J}(k_t); \quad (k+1) \geq 0. \end{aligned} \tag{2.14}$$

The value function $V(z)$ is a solution to the reduced (one-dimensional) HJB equation (2.13) and the solution of this HJB equation is used to test controller for optimality or perhaps to construct a feedback controller. Again the HJB equation (2.13) arises in the production control problem (1.1), (1.2), (1.3) with constraint

$$0 \leq p_t \leq k_t, \quad \forall t \geq 0. \tag{2.15}$$

2.4. *Stochastic Neoclassical Differential Equation for Dynamics of Inventory-Demand Ratio*

As in the certainty optimal production control model, the dynamics of the state equation of inventory level (1.2) can be reduced to a one-dimensional process by working in intensive (per capita) variables. Define

$$\begin{aligned} z_t &\equiv \frac{x_t}{y_t}, \quad \text{inventory-demand ratio}, \\ k_t &\equiv \frac{p_t}{y_t}, \quad \text{per capita production}. \end{aligned} \tag{2.16}$$

To determine the stochastic differential for the inventory-demand ratio, $z \equiv x/y$, we apply Itô's lemma as follows:

$$\begin{aligned} z &= \frac{x}{y} \equiv G(y,t), \\ \frac{\partial G}{\partial y} &= -\frac{x}{y^2} = -\frac{z}{y}, \\ \frac{\partial^2 G}{\partial y^2} &= 2\frac{x}{y^3} = 2\frac{z}{y^2}, \\ \frac{\partial G}{\partial t} = \frac{\dot{x}}{y} &= \frac{(p-y)}{y} = \frac{p}{y} - 1 = k_t - 1. \end{aligned} \tag{2.17}$$

From Itô's lemma,

$$dz = \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(dy)^2. \tag{2.18}$$

From (1.3), we have that $(dy)^2 = \sigma^2 y^2 dt$. Substituting the above expressions into (2.18), we have that the dynamics of z_t to be the inventory-demand ratio at time t which evolves according to the stochastic neoclassical differential equation for demand

$$\begin{aligned} dz_t &= \left(-\frac{z_t}{y_t}\right)(Ay_t\,dt + \sigma y_t\,dw_t) + (k_t - 1)dt + \frac{1}{2}2\frac{z_t}{y_t^2}\sigma^2 y_t^2\,dt \\ &= \left[-Az_t + (k_t - 1) + \sigma^2 z_t\right]dt - \sigma z_t\,dw_t \\ &= \left[\left(-A + \sigma^2\right)z_t + (k_t - 1)\right]dt - \sigma z_t\,dw_t \\ &\le \left[\widetilde{A}z_t + k_t\right]dt - \sigma z_t\,dw_t, \quad z_0 = z,\ z > 0. \end{aligned} \tag{2.19}$$

2.5. Riccati-Based Solution

This subsection deals with the Riccati-based solution of the reduced one-dimensional HJB equation (2.13) corresponding to the production inventory control problem (2.14) subject to (2.19) using the dynamic programming principle [17].

To find the Riccati-based solution of HJB equation (2.13), we refer to Da Prato [20] and Da Prato and Ichikawa [11] for the degenerate linear control problems related to Riccati equation in case of convex function like $h(z) = z^2$.

By taking the derivative of (2.13) with respect to k and setting it to zero, we can minimize the expression inside the bracket of (2.13) (i.e., $F^*(v'(z)) = \min_{k\ge -1}((k+1)^2 + kv'(z)))$ with respect to k. This procedure yields

$$k^* = -\frac{v'(z)}{2} - 1. \tag{2.20}$$

Substituting (2.20) into (2.13) yields the equation

$$-\tilde{\rho}v(z) + \frac{1}{2}\sigma^2 z^2 v''(z) + \tilde{A}zv'(z) - \frac{(v'(z))^2}{4} - v'(z) + z^2 = 0 \tag{2.21}$$

known as the HJB equation. This is a partial differential equation which has a solution form

$$v(z) = a(t)z^2(t). \tag{2.22}$$

Then

$$v'(z) = 2a(t)z(t), \qquad v''(z) = 2a(t). \tag{2.23}$$

Substituting (2.22) and (2.23) into (2.21) yields

$$\left(1 - \tilde{\rho}a + \sigma^2 a - a^2 + 2a\tilde{A}\right)z^2 - 2az = 0. \tag{2.24}$$

Since (2.24) must hold for any value of z, we must have

$$a^2 - a\left(2\tilde{A} + \sigma^2 + \tilde{\rho}\right) - 1 = 0, \tag{2.25}$$

called a Riccati equation from which we obtain

$$a = \frac{-\left(2\tilde{A} + \sigma^2 + \tilde{\rho}\right) \pm \sqrt{(2\tilde{A} + \sigma^2 + \tilde{\rho})^2 + 4}}{2} = K_1 \text{ (say)}. \tag{2.26}$$

So, (2.22) is a solution form of (2.21).

3. Bellman Equations for Discounted Cost Control

3.1. Existence and Uniqueness

To solve the Bellman equation (2.13) let us consider this HJB equation associated with the discounted production control problem in the following form:

$$-\tilde{\rho}v(z) + \frac{1}{2}\sigma^2 z^2 v''(z) + \tilde{A}zv'(z) + F\left(v'(z)\right) + h(z) = 0, \quad 0 < \tilde{\rho} < 1, \tag{3.1}$$

where

$$F(z) = \begin{cases} -\dfrac{z^2}{4} - z & \text{if } z \leq 0, \\ -z & \text{if } z \geq 0. \end{cases} \tag{3.2}$$

We make the following assumptions:

$$\begin{aligned} &h: \text{continuous function on } \mathbf{R}, \\ &h: \text{non-negative, convex on } \mathbf{R}, \end{aligned} \tag{3.3}$$

h satisfies the polynomial growth condition such that

$$0 \le h(z) \le K\left(1 + |z|^n\right), \quad z \in \mathbf{R}, \tag{3.4}$$

for some $K > 0$, $n \ge 2$.

In order to ensure the integrability of $J(p)$, we assume that

$$-\tilde{\rho} + 2n\tilde{A} + n(2n-1)\sigma^2 < 0. \tag{3.5}$$

This condition (3.5) is needed for the integrability of z_t or $J(p)$. Under (3.5), we have Lemmas 3.1, 3.2, and Theorem 3.5, which ensures the finiteness of $J(p)$ and hence the finiteness of an existence unique solution of HJB equation of v.

First we have established the properties of the value function of the optimal control problem.

Lemma 3.1. *Under* (3.5) *and for each* $n \in \mathbf{N}_+$, *there exists* $K > 0$ *such that*

$$E\left[e^{-\tilde{\rho}t}|z_t|^{2n}\right] \le K(1+t). \tag{3.6}$$

Proof. We have given its proof here to need the same kind of calculations in the future. By Itô's formula we have

$$\begin{aligned} e^{-\tilde{\rho}t}|z_t|^{2n} &= |z|^{2n} + \int_0^t (-\tilde{\rho})e^{-\tilde{\rho}s}|z_s|^{2n}ds + 2n\int_0^t e^{-\tilde{\rho}s}|z_s|^{2n-1}\operatorname{sgn}(z_s)dz_s \\ &\quad + \frac{1}{2}2n(2n-1)\sigma^2\int_0^t e^{-\tilde{\rho}s}{z_s}^2|z_s|^{2n-2}ds \\ &= |z|^{2n} + \int_0^t e^{-\tilde{\rho}s}\Big\{-\tilde{\rho}|z_s|^{2n} + 2n\tilde{A}z_s|z_s|^{2n-1}\operatorname{sgn}(z_s) + 2nk_s|z_s|^{2n-1}\operatorname{sgn}(z_s) \\ &\qquad\qquad + n(2n-1)\sigma^2{z_s}^2|z_s|^{2n-2}\Big\}ds \\ &\quad - 2n\sigma\int_0^t e^{-\tilde{\rho}s}z_s|z_s|^{2n-1}\operatorname{sgn}(z_s)dw_s \\ &= |z|^{2n} + \int_0^t e^{-\tilde{\rho}s}\Big\{-\tilde{\rho} + 2n\tilde{A} + n(2n-1)\sigma^2\Big\}|z_s|^{2n}ds \\ &\quad + 2n\int_0^t e^{-\tilde{\rho}s}k_s|z_s|^{2n-1}\operatorname{sgn}(z_s)ds - 2n\sigma\int_0^t e^{-\tilde{\rho}s}z_s|z_s|^{2n-1}\operatorname{sgn}(z_s)dw_s. \end{aligned} \tag{3.7}$$

Now by (2.15), (3.5) and taking expectation on the both sides, we obtain

$$\begin{aligned} E\left[e^{-\tilde{\rho}t}|z_t|^{2n}\right] &\leq |z|^{2n} + 2nE\left[\int_0^t e^{-\tilde{\rho}s}|z_s|^{2n}ds\right] \\ &= |z|^{2n} + E\left[\int_0^t e^{-\tilde{\rho}s}\mathbf{Z}(z_s)ds\right], \end{aligned} \tag{3.8}$$

where $\mathbf{Z}(z) = 2n|z|^{2n}$. Obviously, $\mathbf{Z}(z)$ is bounded above. Thus we can deduce (3.6). □

Lemma 3.2. *Under* (3.5) *and for each* $n \in \mathbf{N}_+$*, there exists* $K > 0$ *such that*

$$\tilde{\rho}E\left[\int_0^\infty e^{-\tilde{\rho}t}|z_t|^{2n}\,dt\right] \leq K\left(1 + |z|^{2n+2}\right). \tag{3.9}$$

Proof. By an application of Itô's formula to $e^{-\tilde{\rho}t}|z_t|^{2n+2}$, we have

$$\begin{aligned} e^{-\tilde{\rho}t}|z_t|^{2n+2} &= |z|^{2n+2} + \int_0^t (-\tilde{\rho})e^{-\tilde{\rho}s}{z_s}^{2n+2}\,ds + (2n+2)\int_0^t e^{-\tilde{\rho}s}|z_s|^{2n+1}\,dz_s \\ &\quad + \frac{1}{2}(2n+2)(2n+1)\sigma^2\int_0^t e^{-\tilde{\rho}s}{z_s}^2{z_s}^{2n}\,ds \\ &= |z|^{2n+2} + \int_0^t (-\tilde{\rho})e^{-\tilde{\rho}s}{z_s}^{2n+2}\,ds + (2n+2)\int_0^t e^{-\tilde{\rho}s}|z_s|^{2n+1}\left(\tilde{A}z_s + k_s\right)ds \\ &\quad - 2(n+1)\sigma\int_0^t e^{-\tilde{\rho}s}z_s|z_s|^{2n+1}\,dw_s + \frac{1}{2}(2n+2)(2n+1)\sigma^2\int_0^t e^{-\tilde{\rho}s}{z_s}^2{z_s}^{2n}\,ds \\ &= |z|^{2n+2} + \int_0^t e^{-\tilde{\rho}s}\left\{-\tilde{\rho} + 2(n+1)\tilde{A} + (n+1)(2n+1)\sigma^2\right\}{z_s}^{2n+2}\,ds \\ &\quad + 2(n+1)\int_0^t e^{-\tilde{\rho}s}k_s|z_s|^{2n+1}\,ds - 2(n+1)\sigma\int_0^t e^{-\tilde{\rho}s}z_s|z_s|^{2n+1}\,dw_s. \end{aligned} \tag{3.10}$$

Now by (2.15), (3.5) and taking expectation on the both sides we obtain

$$E\left[e^{-\tilde{\rho}t}|z_t|^{2n+2}\right] \leq |z|^{2n+2} + 2(n+1)E\left[\int_0^t e^{-\tilde{\rho}s}|z_s|^{2n+1}ds\right]. \tag{3.11}$$

Hence

$$E\left[\int_0^\infty e^{-\tilde{\rho}s}\mathbf{G}(z_s)ds\right] \leq |z|^{2n+2}, \tag{3.12}$$

where $\mathbf{G}(z) = 2(n+1)|z|^{2n+1}$.

We choose $\zeta < 0$ such that $|z|^{2n} \leq \mathbf{G}(z)$ for all $z \geq \zeta$. Then

$$\begin{aligned} E\left[\int_0^\infty e^{-\widetilde{\rho}t}|z_t|^{2n}dt\right] &= \int_0^\infty e^{-\widetilde{\rho}t}E\left[|z_t|^{2n}1_{(|z_t|<\zeta)} + |z_t|^{2n}1_{(|z_t|\geq\zeta)}\,dt\right] \\ &\leq \int_0^\infty e^{-\widetilde{\rho}t}\left[\left\{\zeta^{2n} + E[\mathbf{G}(z_t)]\right\}dt\right] \\ &\leq \frac{\zeta^{2n}}{\widetilde{\rho}} + |z|^{2n+2}. \end{aligned} \tag{3.13}$$

Thus we get (3.9) with $K > 0$ independent of sufficiently small $\widetilde{\rho}$. □

Proposition 3.3. *We assume* (3.3), (3.4). *Then the value function* $v(z)$ *is convex.*

Proof. For any $\epsilon > 0$, there exist $k, \widehat{k} \in \mathcal{D}_k$ such that

$$\begin{aligned} E\left[\int_0^\infty e^{-\widetilde{\rho}t}\left\{h(z_t) + (k_t+1)^2\right\}dt\right] &< v(z) + \epsilon, \\ E\left[\int_0^\infty e^{-\widetilde{\rho}t}\left\{h(\widehat{z}_t) + \left(\widehat{k}_t+1\right)^2\right\}dt\right] &< v(\widehat{z}) + \epsilon, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} dz_t &= \left[\widetilde{A}z_t + k_t\right]dt - \sigma z_t\,dw_t, \qquad z_0 = z \in \mathbf{R}, \\ d\widehat{z}_t &= \left[\widetilde{A}\widehat{z}_t + \widehat{k}_t\right]dt - \sigma \widehat{z}_t\,dw_t, \qquad \widehat{z}_0 = \widehat{z} \in \mathbf{R}. \end{aligned} \tag{3.15}$$

We set

$$\begin{aligned} \widetilde{k}_t &= \xi k_t + (1-\xi)\widehat{k}_t, \\ \widetilde{z}_t &= \xi z_t + (1-\xi)\widehat{z}_t, \\ \widetilde{z}_0 &= \xi z + (1-\xi), \\ \widehat{z} &\equiv \widetilde{z}, \end{aligned} \tag{3.16}$$

for $0 < \xi < 1$. Clearly,

$$d\widetilde{z}_t = \left[\widetilde{A}\widetilde{z}_t + \widetilde{k}_t\right]dt - \sigma\widetilde{z}_t\,dw_t. \tag{3.17}$$

Hence, by convexity

$$\begin{aligned}
v(\tilde{z}) &\le E\left[\int_0^\infty e^{-\tilde{\rho}t}\left\{h(\tilde{z}_t) + (\tilde{k}_t + 1)^2\right\}dt\right] \\
&\le \xi E\left[\int_0^\infty e^{-\tilde{\rho}t}\left\{h(z_t) + (k_t + 1)^2\right\}dt\right] + (1-\xi)E\left[\int_0^\infty e^{-\tilde{\rho}t}\left\{h(\hat{z}_t) + (\hat{k}_t + 1)^2\right\}dt\right] \\
&\le \xi(v(z) + \epsilon) + (1-\xi)(v(\hat{z}) + \epsilon).
\end{aligned} \tag{3.18}$$

Letting $\epsilon \to 0$, we get

$$v(\tilde{z}) = v(\xi z + (1-\xi)\hat{z}) \le \xi v(z) + (1-\xi)v(\hat{z}), \tag{3.19}$$

which completes the convexity of the value function $v(z)$. □

Theorem 3.4. *Assume* (3.4) *and* (3.5). *Choose $g_r(z) = \delta + |z|^r$ for any $2 \le r \le 2n$, then there exist $\delta \ge 1$ and $\zeta > 0$ depending on M such that*

$$-\tilde{\rho}g_r(z) + \frac{1}{2}\sigma^2 z^2 g_r''(z) + \tilde{A}zg_r'(z) + \max_{|q|\le M}\left(q^2 + qg_r'(z)\right) + \zeta g_r(z) \le 0. \tag{3.20}$$

Further

$$E\left[\int_0^\tau e^{-\tilde{\rho}s}\zeta g_r(z_s)ds + e^{-\tilde{\rho}\tau}g_r(z_\tau)\right] \le g_r(z) \quad for\ 2 \le r \le 2n, \tag{3.21}$$

where τ is any stopping time and z_t is the response to $(p_t) \in \mathcal{P}_M$.

Then one has

$$0 \le V(z) \le \overline{K}(1 + |z|^r), \tag{3.22}$$

where $\overline{K} = K\delta/\zeta$, for some positive constant K.

Proof. By (3.5), we choose $\zeta \in (0, \widetilde{\rho})$ such that

$$-\widetilde{\rho} + r\widetilde{A} + \frac{1}{2}r(r-1)\sigma^2 + \zeta < 0, \tag{3.23}$$

and then $\delta \geq 1$ such that

$$\left(-\widetilde{\rho} + r\widetilde{A} + \frac{1}{2}r(r-1)\sigma^2 + \zeta\right)|z|^r + Mr|z|^{r-1} + \left(M^2 + \zeta\delta - \widetilde{\rho}\delta\right) \leq 0. \tag{3.24}$$

Then (3.20) is immediate.

Applying Itô's formula to $e^{-\widetilde{\rho}\tau} g_r(z_\tau)$, we have

$$\begin{aligned}
e^{-\widetilde{\rho}\tau} g_r(z_\tau) &= g_r(z) + \int_0^\tau (-\widetilde{\rho}) e^{-\widetilde{\rho}s} g_r(z_s) ds + \int_0^\tau e^{-\widetilde{\rho}s} g_r'(z_s) dz_s \\
&\quad + \frac{1}{2}\sigma^2 \int_0^\tau e^{-\widetilde{\rho}s} {z_s}^2 g_r''(z_s) ds \\
&= g_r(z) + \int_0^\tau e^{-\widetilde{\rho}s} \left\{ -\widetilde{\rho} g_r(z_s) + \widetilde{A} z_s g_r'(z_s) + q g_r'(z_s) + \frac{1}{2}\sigma^2 z_s^2 g_r''(z_s) \right\} ds \\
&\quad - \sigma \int_0^\tau e^{-\widetilde{\rho}s} z_s g_r'(z_s) dw_s \\
&= g_r(z) + \int_0^\tau e^{-\widetilde{\rho}s} \left\{ -\widetilde{\rho} g_r(z_s) + \widetilde{A} z_s g_r'(z_s) + \frac{1}{2}\sigma^2 z_s^2 g_r''(z_s) \right. \\
&\qquad\qquad \left. + \max_{|q|\leq M}\left(q^2 + q g_r'(z_s)\right) + \zeta g_r(z_s) \right\} ds \\
&\quad - \max_{|q|\leq M} \int_o^\tau q^2 e^{-\widetilde{\rho}s} ds - \int_0^\tau e^{-\widetilde{\rho}s} \zeta g_r(z_s) ds - \sigma \int_0^\tau e^{-\widetilde{\rho}s} z_s g_r'(z_s) dw_s.
\end{aligned} \tag{3.25}$$

Now by (3.20) and taking expectation on the both sides, we obtain

$$E\left[\int_0^\tau e^{-\widetilde{\rho}s} \zeta g_r(z_s) ds + e^{-\widetilde{\rho}\tau} g_r(z_\tau)\right] \leq g_r(z) - \frac{M^2}{\widetilde{\rho}}, \tag{3.26}$$

from which we deduced (3.21).

The convexity of the value function V follows from the same line as (Proposition 3.3). Let (z_t^0) be the unique solution of

$$dz_t^0 = \widetilde{A} z_t^0 \, dt - \sigma z_t^0 \, dw_t, \qquad z_0^0 = z. \tag{3.27}$$

Then by (3.4) and (3.21),

$$\begin{aligned}
V(z) &\leq E\left[\int_0^\infty e^{-\tilde{\rho}t} h\left(z_t^0\right) dt\right] \\
&\leq KE\left[\int_0^\infty e^{-\tilde{\rho}t} g_r\left(z_t^0\right) dt\right] \\
&\leq \frac{K g_r(z)}{\zeta} \\
&= \frac{K(\delta + |z|^r)}{\zeta} \\
&= \frac{K\delta}{\zeta}\left(1 + \frac{|z|^r}{\delta}\right) \\
&\leq \overline{K}(1 + |z|^r); \quad 2 \leq r \leq 2n,
\end{aligned} \tag{3.28}$$

which implies (3.22) and satisfies (3.4). Hence this completes the proof. □

Theorem 3.5. *Assume* (3.3), (3.4), (3.5). *Then there exists a unique solution* $v \in C^2(\mathbf{R})$ *of* (3.1) *such that*

$$\tilde{\rho} v(z) \leq K\left(1 + |z|^{n+3}\right), \quad z \in \mathbf{R}, \tag{3.29}$$

for some constant $K > 0$. *Moreover,* v *admits a representation*

$$v(z) = \inf E\left[\int_0^\infty e^{-\tilde{\rho}t}\left\{h(z_t) + (k_t + 1)^2\right\} dt\right]; \quad (k_t + 1) \geq 0. \tag{3.30}$$

Proof. Since $F^*(z) = \min\{(k+1)^2 + kz\}$ is Lipschitz continuous, this follows from Bensoussan [21] in case of Assumption (3.4) except convexity. For the general case, we take a nondecreasing sequence $h_n \in C(\mathbf{R})$ convergent to h with $0 \leq h_n \leq h$. It is well known (Bensoussan [21]) that, for every $n \in \mathbf{N}_+$, (3.1) has a unique solution v_n for h_n of the form

$$v_n(z) = \inf E\left[\int_0^\infty e^{-\tilde{\rho}t}\left\{h_n(z_t) + (k_t + 1)^2\right\} dt\right]; \quad k_t + 1 \geq 0 \tag{3.31}$$

in the class $C^2(\mathbf{R})$ of continuous functions vanishing at infinitely, where z_t is a solution of (2.19).

To prove (3.29), we recall (3.30). Hence by (3.4) and **Lemma 3.2** we have

$$\begin{aligned}
0 \leq \tilde{\rho} v_n(z) &\leq \tilde{\rho} E\left[\int_0^{\infty} e^{-\tilde{\rho} t}\left\{h_n(z_t) + (k_t+1)^2 \mid\right\} dt\right] \\
&\leq \tilde{\rho} E\left[\int_0^{\infty} e^{-\tilde{\rho} t}\left\{h(z_t) + (k_t+1)^2 \mid\right\} dt\right] \\
&\leq K\left(1 + \tilde{\rho} E\left[\int_0^{\infty} e^{-\tilde{\rho} t}\left\{\left(1+|z_t|^n\right) + (k_t+1)^2\right\} dt\right]\right) \\
&\leq K\left(1 + |z|^{n+3}\right).
\end{aligned} \tag{3.32}$$

This implies that v satisfies (3.29).

To estimate v_n' on $[-s,s]$ for $s \in \mathbf{R}_+$, we remember the Taylor expansions of $v \in C^2(\mathbf{R})$:

$$\begin{gathered}
\varepsilon v'(b) = v(b+\varepsilon) - v(b) - \frac{\varepsilon^2}{2} v''(b + \vartheta_1 \varepsilon), \\
v'(z) = v'(b) + v''(b + \vartheta_2(z-b))(z-b), \quad x \in V_\varepsilon, 0 < \vartheta_1, \vartheta_2 < 1,
\end{gathered} \tag{3.33}$$

where $V_\varepsilon = V_\varepsilon(b)$ is any ε-neighborhood of $b \in [-s,s]$. Then wa can obtain the Landau-Kolmogorov inequality:

$$\sup_{V_\varepsilon}\left|v'(z)\right| \leq \frac{2}{\varepsilon}\sup_{V_\varepsilon}|v(z)| + \frac{3\varepsilon}{2}\sup_{V_\varepsilon}\left|v''(z)\right|. \tag{3.34}$$

Choosing $0 < \varepsilon < 1 \wedge (1/3(k + 2\tilde{A}/\sigma^2))$, and by (3.1), (3.2), and (3.34) we have

$$\begin{aligned}
\sup_{V_\varepsilon}\left|v_n'(z)\right| &\leq \frac{2}{\varepsilon}\sup_{V_\varepsilon}|v_n(z)| + \frac{3\varepsilon}{2}\sup_{V_\varepsilon}\left|v_n''(z)\right| \\
&\leq \frac{2}{\varepsilon}\sup_{V_\varepsilon}|v_n(z)| + \frac{3\varepsilon}{2}\sup_{V_\varepsilon}\left[\frac{2\tilde{\rho}}{\sigma^2}\left|\frac{v_n(z)}{z^2}\right| + \frac{2\tilde{A}}{\sigma^2}\left|\frac{v_n'(z)}{z}\right| + \frac{2}{\sigma^2}\left|\frac{F(v_n'(z))}{z^2}\right| + \frac{2\tilde{A}}{\sigma^2}\left|\frac{h_n(z)}{z}\right|\right] \\
&\leq \left(\frac{2}{\varepsilon} + \frac{3\varepsilon\tilde{\rho}}{\sigma^2}\right)\sup_{V_\varepsilon}|v_n(z)| + 3\varepsilon\left(k + \frac{2\tilde{A}}{\sigma^2}\right)\sup_{V_\varepsilon}\left|v_n'(z)\right| + \frac{3\varepsilon\tilde{A}}{\sigma^2}\sup_{V_\varepsilon}|h_n(z)|,
\end{aligned} \tag{3.35}$$

from which

$$\begin{aligned}
\sup_{V_\varepsilon}\left|v_n'(z)\right| \leq \frac{1}{1 - 3\varepsilon\left(k + 2\tilde{A}/\sigma^2\right)} &\left(\frac{2}{\varepsilon} + \frac{3\varepsilon\tilde{\rho}}{\sigma^2}\right)\sup_{V_\varepsilon}|v_n(z)| \\
&+ 3\varepsilon\left(k + \frac{2\tilde{A}}{\sigma^2}\right)\sup_{V_\varepsilon}\left|v_n'(z)\right| + \frac{3\varepsilon\tilde{A}}{\sigma^2}\sup_{V_\varepsilon}|h_n(z)|.
\end{aligned} \tag{3.36}$$

Now by (3.34) and (3.4), we have

$$\sup_{V_\varepsilon}\left|v_n'(z)\right| \le \frac{K}{1-3\varepsilon\left(k+2\widetilde{A}/\sigma^2\right)}\left(\frac{2}{\varepsilon}+\frac{3\varepsilon\widetilde{\rho}}{\sigma^2}\right)\left(1+(s+1)^{n+3}\right). \tag{3.37}$$

Thus, taking the finite covering $V_\varepsilon(b_i)$, $i = 1,2,\dots,j$, of $[-s,s]$, we deduce

$$\begin{aligned}\sup_{-s\le z\le s}\left|v_n'(z)\right| &\le \sum_{i=1}^{j}\sup_{V_\varepsilon(b_i)}\left|v_n'(z)\right| \\ &\le \frac{jK}{1-3\varepsilon\left(k+2\widetilde{A}/\sigma^2\right)}\left(\frac{2}{\varepsilon}+\frac{3\varepsilon\widetilde{\rho}}{\sigma^2}\right)\left(1+(s+1)^{n+3}\right),\end{aligned} \tag{3.38}$$

and hence

$$\sup_n \sup_{-s\le z\le s}\left(\left|v_n'(z)\right| + \left|v_n''(z)\right|\right) < \infty \quad \text{for every } s \in \mathbf{R}_+. \tag{3.39}$$

By the Ascoli-Arzelà theorem, we have

$$v_n \longrightarrow v, \quad v_n' \longrightarrow v' \quad \text{uniformly on } [-s,s], \tag{3.40}$$

taking a subsequence if necessary. Passing to the limit, we can obtain (3.1) and (3.30).

Following the inequality (3.29), we have

$$E\left[e^{-\widetilde{\rho}t}v(z_t)\right] \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.41}$$

Hence by Itô's formula to $e^{-\widetilde{\rho}t}v(z_t)$, for convex function [19, page 219], we have

$$\begin{aligned} e^{-\widetilde{\rho}t}v(z_t) = v(z) &+ \int_0^\infty e^{-\widetilde{\rho}t}\left\{-\widetilde{\rho}v(z) + \widetilde{A}zv'(z) + k_t v'(z) + \frac{1}{2}\sigma^2 z^2 v''(z)\right\}dt \\ &- \int_0^\infty e^{-\widetilde{\rho}t}\sigma z_t v'(z)dw_t. \end{aligned} \tag{3.42}$$

By virtue of (3.1) and taking expectation on the both sides we have

$$E\left[e^{-\widetilde{\rho}t}v(z_t)\right] = v(z) - E\left[\int_0^\infty e^{-\widetilde{\rho}t}\left\{h\left(z_t + (k_t+1)^2\right)\right\}dt\right]. \tag{3.43}$$

Now by (3.41), we obtain

$$v(z) = \inf E\left[\int_0^\infty e^{-\widetilde{\rho}t}\left\{h(z_t) + (k_t+1)^2\right\}dt\right]; \quad k_t \ge -1, \tag{3.44}$$

which is similar to (3.30) and here the infimum is attained by the feedback law $\beta(v'(z))$ with $\beta(\theta) = \arg\min F(\theta)$. The proof is complete. □

Corollary 3.6. *Assume* (3.3), (3.4), *and* (3.5). *Then there exists a unique solution* $u \in C^2(\mathbf{R})$ *of* (2.10) *such that*

$$\rho u(x,y) \leq K\left(1+|x|^{n+3}\right), \quad (x,y) \in \mathbf{R}, \tag{3.45}$$

for some constant $K > 0$. *Moreover, u admits a representation*

$$u(x,y) = \inf E\left[\int_0^\infty e^{-\rho t}\left\{h(x_t) + p_t^2\right\}dt\right]; \quad p \geq 0. \tag{3.46}$$

4. An Application to Production Control

In this section we will study the production control problem to minimize the cost (2.14) over the class $\mathcal{P}_k$ of all progressively measurable processes p_t such that $0 \leq p_t \leq k_t$ and

$$\lim_{t\to\infty} \frac{1}{t} E\left[e^{-\tilde{\rho}t}|z_t|^n\right] = 0 \tag{4.1}$$

for the response z_t to p_t.

Let us consider the stochastic differential equation

$$dz_t^* = \left[\tilde{A}z_t^* + \beta_k(v'(z_t^*))\right]dt - \sigma z_t^*\, dw_t, \quad z_0^* = z, \quad z > 0, \tag{4.2}$$

where $\beta_k(z) = \arg\min F(z)$, that is,

$$\beta_k(z) = \begin{cases} k-1 & \text{if } z \leq -2k, \\ -\dfrac{z}{2} - 1 & \text{if } -2k < z \leq 0, \\ -1 & \text{if } 0 < z. \end{cases} \tag{4.3}$$

We need to establish the following lemmas.

Lemma 4.1. *Under* (3.5) *and for each* $n \in \mathbf{N}_+$, *there exists* $K > 0$ *such that*

$$E\left[e^{-\tilde{\rho}t}|z_t^*|^{2n}\right] \leq K(1+t). \tag{4.4}$$

Proof. Since, we have by Itô's formula

$$\begin{aligned} d(z_t^*)^2 &= 2z_t^*\,dz_t^* + \frac{1}{2}\sigma^2(z_t^*)^2 2\,dt \\ &= 2z_t^*\left[\left(\widetilde{A}z_t^* + \beta_k v'(z_t^*)\right)dt - \sigma z_t^*\,dw_t\right] + \sigma^2(z_t^*)^2\,dt, \\ d(\widetilde{z}_t)^2 &= 2\widetilde{z}_t\,dz_t + \frac{1}{2}\sigma^2(\widetilde{z}_t)^2 2\,dt \\ &\le 2\widetilde{z}_t\left[\widetilde{A}\widetilde{z}_t\,dt - \sigma\widetilde{z}_t\,dw_t\right] + \sigma^2(\widetilde{z}_t)^2 dt. \end{aligned} \tag{4.5}$$

Now by the assumptions of comparison theorem we have $b_1(z) = \widetilde{A}z + \beta_k(v'(z_t)) + \sigma^2(z)^2$, $b_2(z) = \widetilde{A}z + \sigma^2(z)^2$, then we have $b_1(z) < b_2(z)$ and $|\sigma z - \sigma\widetilde{z}| = |\sigma||z - \widetilde{z}|$, where $\sigma z = \eta(z)$, so $\int_0^{\epsilon}\eta^{-2}(z)dx = \int_0^{\epsilon}(\sigma z)^{-2}dz = \infty$. Thus we can see $(z_t^*)^2 \le (\widetilde{z}_t)^2$ by the comparison theorem of Ikeda and Watanabe [22]. Since the explosion time $\sigma = \inf\{t : |z_t^*| = \infty\}$, we have $\infty = (z_\sigma^*)^2 \le (\widetilde{z}_\sigma)^2$. Hence $\sigma = \infty$.

By the monotonicity of $\beta(v'(z))$, we have $\beta_k(v'(\widetilde{z})) < \beta(v'(z))$ for $\widetilde{z} < z$. Then

$$\begin{aligned} d\widetilde{z}_t &= \left[\widetilde{A}\widetilde{z}_t + \beta\left(v'(\widetilde{z}_t)\right)\right]dt - \sigma\widetilde{z}_t\,dw_t, \qquad \widetilde{z}_0 = z^*, \\ dz_t^* &= \left[\widetilde{A}z_t^* + \beta\left(v'(z_t^*)\right)\right]dt - \sigma z_t^* dw_t, \qquad z_0^* = z^*, \\ d(\widetilde{z}_t - z_t^*) &= \left[\widetilde{A}(\widetilde{z}_t - z_t^*) + \beta_k\left(v'(\widetilde{z}_t)\right) - \beta_k\left(v'(z_t^*)\right)\right]dt - \sigma(\widetilde{z}_t - z_t^*)dw_t, \end{aligned} \tag{4.6}$$

by Itô's formula,

$$\begin{aligned} d(\widetilde{z}_t - z_t^*)^2 &= 2(\widetilde{z}_t - z_t^*)d(\widetilde{z}_t - z_t^*) + \frac{1}{2}\sigma^2(\widetilde{z}_t - z_t^*)^2 2\,dt \\ &= 2\widetilde{A}(\widetilde{z}_t - z_t^*)^2 dt + 2(\widetilde{z}_t - z_t^*)\left(\beta_k\left(v'(\widetilde{z}_t)\right) - \beta_k\left(v'(z_t^*)\right)\right)dt \\ &\quad - 2\sigma(\widetilde{z}_t - z_t^*)^2 dw_t + \sigma^2(\widetilde{z}_t - z_t^*)^2 dt \\ &\le \left(2\widetilde{A} + \sigma^2\right)(\widetilde{z}_t - z_t^*)^2 dt - 2\sigma(\widetilde{z}_t - z_t^*)^2 dw_t. \end{aligned} \tag{4.7}$$

Hence

$$\begin{aligned} (\widetilde{z}_t - z_t^*)^2 &\le \left(2\widetilde{A} + \sigma^2\right)\int_0^t (\widetilde{z}_s - z_s^*)^2 ds - 2\sigma\int_0^t (\widetilde{z}_s - z_s^*)^2 dw_s, \\ E\left[(\widetilde{z}_t - z_t^*)^2\right] &\le \left(2\widetilde{A} + \sigma^2\right)\int_0^t E\left[(\widetilde{z}_s - z_s^*)^2\right]ds. \end{aligned} \tag{4.8}$$

Set $\xi(t) = E[(\widetilde{z}_t - z_t^*)^2]$, we obtain $\xi(t) \le K\int_0^t \xi(s)ds$, $\forall t \ge 0$ where $K > 2A + \sigma^2$. By Gronwall Lemma, we have $\xi(t) = 0$ a.s. $\forall t \ge 0$. Therefore, $E[(\widetilde{z}_t - z_t^*)^2] = 0$ a.s. $\forall t \ge 0$, from which we have $\tilde{z}_t \leq z_t^*$. So, the uniqueness of (4.2) holds. Thus we conclude that (4.2) admits a unique strong solution (z_t^*), Ikeda and Watanabe [22, Chapter 4, Theorem 1.1], with $E[|z_t^*|^{2n} < \infty]$.

By (3.5) and Itô's formula,

$$\begin{aligned} E\left[e^{-\tilde{\rho}t}|z_t^*|^{2n}\right] &= |z|^{2n} + E\left[\int_0^t e^{-\tilde{\rho}s}\left\{-\tilde{\rho} + 2n\tilde{A} + n(2n-1)\sigma^2\right\}|z_s^*|^{2n}ds\right] \\ &\quad + 2nE\left[\int_0^t e^{-\tilde{\rho}s}\beta_k(v'(z_s^*))|z_s^*|^{2n-1}\operatorname{sgn}(z_s^*)ds\right] \\ &\leq |z|^{2n} + 2nE\left[\int_0^t e^{-\tilde{\rho}s}\beta_k(v'(z_s^*))|z_s^*|^{2n-1}\operatorname{sgn}(z_s^*)ds\right] \\ &= |z|^{2n} + E\left[\int_0^t e^{-\tilde{\rho}s}\mathbf{Z}^{(\mathbf{n})}(s)ds\right], \end{aligned} \tag{4.9}$$

where $\mathbf{Z}^{(\mathbf{n})}(\mathbf{s}) = 2n\beta_k(v'(z_s^*))|z_s^*|^{2n-1}\operatorname{sgn}(z_s^*)$.

By (4.3) it is easily seen that $z\beta_k(v'(z)) \leq (k-1)|z|$ if $|z| \geq a$, for sufficiently large $a > 0$. Clearly

$$\sup_s E\left[e^{-\tilde{\rho}s}\mathbf{Z}^{(\mathbf{n})}(s)1_{(|z_s^*|<a)}\right] < \infty. \tag{4.10}$$

Also

$$E\left[e^{-\tilde{\rho}s}\mathbf{Z}^{(\mathbf{n})}(s)1_{(|z_s^*|\geq a)}\right] \leq E\left[e^{-\tilde{\rho}s}(k-1)|z|1_{(|z_s^*|\geq a)}\right]. \tag{4.11}$$

By the same line as the proof of Lemma 3.1, we see that the right-hand side is bounded from above. This completes the proof. □

Theorem 4.2. *One assumes* (3.3), (3.4), (3.5). *Then the optimal cost control k_t^* is given by*

$$k_t^* = \beta_k(v'(z_t^*)), \tag{4.12}$$

and the minimum value by

$$J(k^*) = v(z), \tag{4.13}$$

where z_t^* is defined by (4.2).

Proof. We first note by (4.3) that $F_k(z) = \min\{(k+1)^2 + kz;\ k+1 \geq 0\}$, and the minimum is attained by $\beta_k(z)$. We apply Itô's formula for convex functions [19, page 219] to obtain

$$\begin{aligned} e^{-\tilde{\rho}t}v(z_t^*) = v(z) &+ \int_0^t e^{-\tilde{\rho}s}\left(-\tilde{\rho}v(z_s) + \tilde{A}zv'(z_s) + \beta_k(v'(z_s)) + \frac{1}{2}\sigma^2 z^2 v''(z_s)\right)\Big|_{z=z_s^*} ds \\ &- \int_0^t e^{-\tilde{\rho}s}\sigma z_s^* v'(z_s^*)dw_s. \end{aligned} \tag{4.14}$$

Taking expectation on the both sides,

$$E\left[e^{-\tilde{\rho}t}v(z_t^*)\right] = v(z) + E\left[\int_0^t e^{-\tilde{\rho}s}\left(-\tilde{\rho}v(z_s^*) + \tilde{A}z^*v'(z_s^*) + k_s^*v'(z_s^*) + \frac{1}{2}\sigma^2 z^{*2}v''(z_s^*)\right)ds\right]. \tag{4.15}$$

By virtue of (2.13),

$$E\left[e^{-\tilde{\rho}t}v(z_t^*)\right] = v(z) - E\left[\int_0^t e^{-\tilde{\rho}s}\left\{h(z_s^*) + (k+1)^2\right\}ds\right]. \tag{4.16}$$

Choose $n \in \mathbf{N}_+$ such that $2n > m$. By (3.4) and Lemma 3.1, we have

$$\begin{aligned} \frac{1}{t}E\left[e^{-\tilde{\rho}t}v(z_t^*)\right] &\leq \frac{K}{t}\left(1 + E\left[e^{-\tilde{\rho}mt}|z_t^*|^m\right]\right) \\ &\leq \frac{K}{t}\left(1 + E\left(\left[e^{-2n\tilde{\rho}t}|z_t^*|^{2n}\right]\right)^{m/2n}\right) \\ &\leq \frac{K}{t}\left(1 + (K(1+t))^{m/2n}\right) < \infty, \end{aligned} \tag{4.17}$$

which implies that

$$\liminf_{t\to\infty} \frac{1}{t}E\left[e^{-\tilde{\rho}t}v(z_t^*)\right] = 0. \tag{4.18}$$

Hence z_t^* satisfies (4.1). Then we get $E[\int_0^t e^{-\tilde{\rho}s}\{h(z_s^*) + (k+1)^2\}ds] \leq v(z)$ from which $J(k^*) \leq v(z)$. By (3.4), we have $J(k^*) \leq v(z) < \infty$, hence $k^* = (k_t^*) \in \mathcal{D}_k$.

Clearly $F_k(z) \leq (k+1)^2 + kz$ for every $k \in \mathcal{D}_k$. Again following the same construction of (3.45) and by the HJB equation (2.13), we have

$$E\left[e^{-\tilde{\rho}t}v(z_t)\right] \geq v(z) - E\left[\int_0^t e^{-\tilde{\rho}s}\left\{h(z_s) + (k+1)^2\right\}ds\right], \quad k = (k_t) \in \mathcal{D}_k. \tag{4.19}$$

By (4.1) we have

$$E\left[\int_0^\infty e^{-\tilde{\rho}s}\left\{h(z_s)+(k+1)^2\right\}ds\right] \geq v(z). \tag{4.20}$$

Thus we deduce $J(k^*) \geq v(z)$. The proof is complete. □

Lemma 4.3. *Under* (3.5)*, there exists a unique solution* $x_t^* \geq 0$ *of*

$$dx_t^* = \left[\varphi_k(u'(x_t^*, y_t)) - y_t\right]dt, \qquad x_0^* = x > 0, \tag{4.21}$$

where

$$\varphi_k(x) = \begin{cases} -\dfrac{x}{2} & \text{if } x \leq 0, \\ 0 & \text{if } 0 < x, \end{cases}$$
$$\varphi(x) := -\frac{x}{2} \text{ is the minimizer of } \min_{p\geq 0}\left\{p^2 + pu_x\right\}. \tag{4.22}$$

Proof. Set

$$k(x,y) = \varphi(u_x(x,y)) \quad \text{if } x, y > 0. \tag{4.23}$$

Since $k(x,y)$ is continuous in x and y, there exists a nonexplosive solution x_t^* of (4.21).

Now we will show $x_t^* \geq 0$ $\forall t \geq 0$ a.s. Suppose $0 < v'(0+) < \infty$. Then L'Hospital's rule gives

$$\lim_{z\to 0+}\frac{z^2 v'(z)}{z} = \lim_{z\to 0+} z^2 v''(z) = 0. \tag{4.24}$$

Letting $z \to 0+$ in (3.1), we have $F(v'(0+)) = 0$, and hence $v'(0+) = -(k+1)^2/k \leq 0$. This contradicts the assumption. Thus we get $v'(0+) = \infty$, which implies that $\varphi(u_x(0+, y)) = 0$. In case $T = \inf\{t \geq 0 : x_t^* = 0\} = \infty$, we have at $t = T$,

$$\frac{dx_t^*}{dt} = \varphi(u_x(x_t^*, y_t)) - y_t = 0. \tag{4.25}$$

Therefore $x_t^* \geq 0$.

To prove uniqueness, let $x_i^*(t)$, $i = 1, 2$ be two solutions of (4.21). Then $x_1^*(t) - x_2^*(t)$ satisfies

$$d(x_1^*(t) - x_2^*(t)) = \left[(\varphi_p(u_x(x_1^*(t), y_t)) - y_t) - (\varphi_p(u_x(x_2^*(t), y_t)) - y_t)\right]dt, \quad x_1^*(0) - x_2^*(0) = 0. \tag{4.26}$$

We have

$$d(x_1^*(t)-x_2^*(t))^2 = 2\big(x_1^*(t)-x_2^*(t)\big)d\big(x_1^*(t)-x_2^*(t)\big)$$
$$= 2\big(x_1^*(t)-x_2^*(t)\big)\big[\psi\big(u_x\big(x_1^*(t),y_t\big)\big)-\psi\big(u_x\big(x_2^*(t),y_t\big)\big)\big]dt. \tag{4.27}$$

Note that the function $x \to \psi(u_x(x,y))$ is increasing. Hence

$$(x_1^*(t)-x_2^*(t))^2 \le 2\int_0^t (x_1^*(s)-x_2^*(s))^2 ds. \tag{4.28}$$

By Gronwall's lemma, we have

$$x_1^*(t) = x_2^*(t), \quad \forall t > 0. \tag{4.29}$$

So, the uniqueness of (4.21) holds. The proof is complete. □

Theorem 4.4. *Under* (3.5)*, the optimal production inventory cost control p_t^* is given by*

$$p_t^* = \psi_p\big(u'\big(x_t^*,y_t\big)\big), \tag{4.30}$$

where x_t^* is defined by (4.21).

Proof. By Lemma 4.3, we observe that (p_t^*) belongs to $\mathcal{D}$. Now we apply Itô's formula

$$e^{-\rho t}u\big(x_t^*,y_t\big) = u(x,y) + \int_0^t e^{-\rho s}\Big\{-\rho u(x,y) + u_x(x,y)(p_s-y_s) + Ayu_y(x,y)$$
$$+\frac{1}{2}\sigma^2y^2u_{yy}(x,y)\Big\}\Big|_{(x=x_s^*,y=y_s)} ds \tag{4.31}$$
$$+\int_0^t e^{-\rho s}\sigma y_s u_y(x^*,y)dw_s.$$

By the HJB equation (2.10), we have

$$e^{-\rho t}u\big(x_t^*,y_t\big) = u(x,y) - \int_0^t e^{-\rho s}\Big\{h(x^*)+p_s^{*2}\Big\}ds + \int_0^t e^{-\rho s}\sigma y_s u_y(x^*,y)dw_s, \tag{4.32}$$

from which

$$E\Big[e^{-\rho(t\wedge T)}u\big(x_{t\wedge T}^*,y_{t\wedge T}\big)\Big] + E\bigg[\int_0^{t\wedge T} e^{-\rho s}\Big\{h(x^*)+p_s^{*2}\Big\}ds\bigg] = u(x,y). \tag{4.33}$$

Then we obtain by (4.18)

$$E\left[\int_0^\infty e^{-\rho s}\left\{h(x^*)+p_s^{*2}\right\}ds\right]=u(x,y). \tag{4.34}$$

Let p_t belong to $\mathcal{P}$. By the same line as above, we have

$$\begin{aligned}e^{-\rho t}u(x_t,y_t)=&u(x,y)+\int_0^t e^{-\rho s}\Big\{-\rho u(x,y)+u_x(x,y)(p_s-y_s)\\&+Ayu_y(x,y)+\frac{1}{2}\sigma^2y^2u_{yy}(x,y)\Big\}\Big|_{(x=x_s,y=y_s)}ds\\&+\int_0^t e^{-\rho s}\sigma y_s u_y(x,y)dw_s.\end{aligned} \tag{4.35}$$

Again by the HJB equation (2.10), we can obtain

$$E\left[e^{-\rho t}u(x_t,y_t)\right]\geq u(x,y)-E\left[\int_0^t e^{-\rho s}\left\{h(x_s)+p_s^2\right\}ds\right]. \tag{4.36}$$

By the same line as Lemma 4.1 and by (4.1),

$$u(x,y)\leq E\left[\int_0^\infty e^{-\rho s}\left\{h(x_s)+p_s^2\right\}ds\right],\quad \text{for any } (p_t)\in\mathcal{P}. \tag{4.37}$$

Combining (4.34) with (4.37), we have

$$J(p^*)\leq J(p). \tag{4.38}$$

Therefore the optimal production cost control p^* which minimizes the production control problem (1.1) subject to (1.2) and (1.3). The proof is complete. □

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