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Research Article

Complete Convergence for Weighted Sums of Sequences of Negatively Dependent Random Variables

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Applying to the moment inequality of negatively dependent random variables the complete convergence for weighted sums of sequences of negatively dependent random variables is discussed. As a result, complete convergence theorems for negatively dependent sequences of random variables are extended.

1. Introduction and Lemmas

Definition 1.1. Random variables X and Y are said to be negatively dependent (ND) if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \tag{1.1}$$

for all $x, y \in \mathrm{R}$. A collection of random variables is said to be pairwise negatively dependent (PND) if every pair of random variables in the collection satisfies (1.1).

It is important to note that (1.1) implies that

$$P(X > x, Y > y) \le P(X > x)P(Y > y) \tag{1.2}$$

for all $x, y \in \mathrm{R}$. Moreover, it follows that (1.2) implies (1.1), and, hence, (1.1) and (1.2) are equivalent. However, (1.1) and (1.2) are not equivalent for a collection of 3 or more random variables. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

Definition 1.2. The random variables $X_1, \ldots, X_n$ are said to be negatively dependent (ND) if, for all real $x_1, \ldots, x_n$,

$$P\left(\bigcap_{j=1}^{n}(X_j \le x_j)\right) \le \prod_{j=1}^{n} P(X_j \le x_j),$$
$$P\left(\bigcap_{j=1}^{n}(X_j > x_j)\right) \le \prod_{j=1}^{n} P(X_j > x_j). \quad (1.3)$$

An infinite sequence of random variables $\{X_n; n \ge 1\}$ is said to be ND if every finite subset $X_1, \ldots, X_n$ is ND.

Definition 1.3. Random variables $X_1, X_2, \ldots, X_n$, $n \ge 2$, are said to be negatively associated (NA) if, for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{cov}(f_1(X_i; i \in A_1),\ f_2(X_j; j \in A_2)) \le 0, \quad (1.4)$$

where f_1 and f_2 are increasing in every variable (or decreasing in every variable), provided this covariance exists. A random variables sequence $\{X_n; n \ge 1\}$ is said to be NA if every finite subfamily is NA.

The definition of PND is given by Lehmann [1], the concept of ND is given by Bozorgnia et al. [2], and the definition of NA is introduced by Joag-Dev and Proschan [3]. These concepts of dependence random variables have been very useful in reliability theory and applications.

First, note that by letting $f_1(X_1, X_2, \ldots, X_{n-1}) = I_{(X_1 \le x_1, X_2 \le x_2, \ldots, X_{n-1} \le x_{n-1})}$, $f_2(X_n) = I_{(X_n \le x_n)}$ and $\overline{f}_1(X_1, X_2, \ldots, X_{n-1}) = I_{(X_1 > x_1, X_2 > x_2, \ldots, X_{n-1} > x_{n-1})}$, $\overline{f}_2(X_n) = I_{(X_n > x_n)}$, separately, it is easy to see that NA implies (1.3). Hence, NA implies ND. But there are many examples which are ND but are not NA. We list the following two examples.

Example 1.4. Let X_i be a binary random variable such that $P(X_i = 1) = P(X_i = 0) = 0.5$ for $i = 1, 2, 3$. Let (X_1, X_2, X_3) take the values $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, and $(1,1,1)$, each with probability $1/4$.

It can be verified that all the ND conditions hold. However,

$$P(X_1 + X_3 \le 1,\ X_2 \le 0) = \frac{4}{8} \not\le P(X_1 + X_3 \le 1) P(X_2 \le 0) = \frac{3}{8}. \quad (1.5)$$

Hence, X_1, X_2, and X_3 are not NA.

In the next example $X = (X_1, X_2, X_3, X_4)$ possesses ND, but does not possess NA obtained by Joag-Dev and Proschan [3].

Example 1.5. Let X_i be a binary random variable such that $P(X_i = 1) = .5$ for $i = 1, 2, 3, 4$. Let (X_1, X_2) and (X_3, X_4) have the same bivariate distributions, and let $X = (X_1, X_2, X_3, X_4)$ have joint distribution as shown in Table 1.

Table 1

		(X_1, X_2)				
		(0, 0)	(0, 1)	(1, 0)	(1, 1)	Marginal
	(0, 0)	.0577	.0623	.0623	.0577	.24
	(0, 1)	.0623	.0677	.0677	.0623	.26
(X_3, X_4)	(1, 0)	.0623	.0677	.0677	.0623	.26
	(1, 1)	.0577	.0623	.0623	.0577	.24
	marginal	.24	.26	.26	.24	

It can be verified that all the ND conditions hold. However,

$$P(X_i = 1,\ i = 1, 2, 3, 4) > P(X_1 = X_2 = 1)P(X_3 = X_4 = 1), \tag{1.6}$$

violating NA.

From the above examples, it is shown that ND does not imply NA and ND is much weaker than NA. In the papers listed earlier, a number of well-known multivariate distributions are shown to possess the ND properties, such as (a) multinomial, (b) convolution of unlike multinomials, (c) multivariate hypergeometric, (d) dirichlet, (e) dirichlet compound multinomial, and (f) multinomials having certain covariance matrices. Because of the wide applications of ND random variables, the notions of ND random variables have received more and more attention recently. A series of useful results have been established (cf. Bozorgnia et al. [2], Amini [4], Fakoor and Azarnoosh [5], Nili Sani et al. [6], Klesov et al. [7], and Wu and Jiang [8]). Hence, the extending of the limit properties of independent or NA random variables to the case of ND random variables is highly desirable and of considerable significance in the theory and application. In this paper we study and obtain some probability inequalities and some complete convergence theorems for weighted sums of sequences of negatively dependent random variables.

In the following, let $a_n \ll b_n$ ($a_n \gg b_n$) denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ ($a_n \geq cb_n$) for sufficiently large n, and let $a_n \approx b_n$ mean $a_n \ll b_n$ and $a_n \gg b_n$. Also, let $\log x$ denote $\ln(\max(e, x))$ and $S_n \widehat{=} \sum_{j=1}^{n} X_j$.

Lemma 1.6 (see [2]). *Let $X_1, \ldots, X_n$ be ND random variables and $\{f_n; n \geq 1\}$ a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n); n \geq 1\}$ is still a sequence of ND r. v. 's.*

Lemma 1.7 (see [2]). *Let $X_1, \ldots, X_n$ be nonnegative r. v. 's which are ND. Then*

$$E\left(\prod_{j=1}^{n} X_j\right) \leq \prod_{j=1}^{n} EX_j. \tag{1.7}$$

In particular, let $X_1, \ldots, X_n$ be ND, and let $t_1, \ldots, t_n$ be all nonnegative (or non-positive) real numbers. Then

$$E\left(\exp\left(\sum_{j=1}^{n} t_j X_j\right)\right) \leq \prod_{j=1}^{n} E\left(\exp\left(t_j X_j\right)\right). \tag{1.8}$$

Lemma 1.8. *Let $\{X_n; n \geq 1\}$ be an ND sequence with $EX_n = 0$ and $E|X_n|^p < \infty$, $p \geq 2$. Then for $B_n = \sum_{i=1}^{n} EX_i^2$,*

$$E|S_n|^p \leq c_p \left\{\sum_{i=1}^{n} E|X_i|^p + B_n^{p/2}\right\}, \tag{1.9}$$

$$E\left(\max_{1\leq i\leq n}|S_i|^p\right) \leq c_p \log^p n \left\{\sum_{i=1}^{n} E|X_i|^p + B_n^{p/2}\right\}, \tag{1.10}$$

where $c_p > 0$ depends only on p.

Remark 1.9. If $\{X_n; n \geq 1\}$ is a sequence of independent random variables, then (1.9) is the classic Rosenthal inequality [9]. Therefore, (1.9) is a generalization of the Rosenthal inequality.

Proof of Lemma 1.8. Let $a > 0$, $X_i' = \min(X_i, a)$, and $S_n' = \sum_{i=1}^{n} X_i'$. It is easy to show that $\{X_i'; i \geq 1\}$ is a negatively dependent sequence by Lemma 1.6. Noting that $(e^x - 1 - x)/x^2$ is a nondecreasing function of x on R and that $EX_i' \leq EX_i = 0$, $tX_i' \leq ta$, we have

$$\begin{aligned} E\left(e^{tX_i'}\right) &= 1 + tEX_i' + E\left(\frac{e^{tX_i'} - 1 - tX_i'}{t^2 X_i'^2} t^2 X_i'^2\right) \\ &\leq 1 + \left(e^{ta} - 1 - ta\right) a^{-2} EX_i'^2 \\ &\leq 1 + \left(e^{ta} - 1 - ta\right) a^{-2} EX_i^2 \\ &\leq \exp\left\{\left(e^{ta} - 1 - ta\right) a^{-2} EX_i^2\right\}. \end{aligned} \tag{1.11}$$

Here the last inequality follows from $1 + x \leq e^x$, for all $x \in$ R.

Note that $B_n = \sum_{i=1}^{n} EX_i^2$ and $\{X_i'; i \geq 1\}$ is ND, we conclude from the above inequality and Lemma 1.7 that, for any $x > 0$ and $h > 0$, we get

$$\begin{aligned} e^{-hx} E\left(e^{hS_n'}\right) &= e^{-hx} E\left(\prod_{i=1}^{n} e^{hX_i'}\right) \leq e^{-hx} \prod_{i=1}^{n} E\left(e^{hX_i'}\right) \\ &\leq \exp\left\{-hx + \left(e^{ha} - 1 - ha\right) a^{-2} B_n\right\}. \end{aligned} \tag{1.12}$$

Letting $h = \ln((xa)/B_n + 1)/a > 0$, we get

$$\left(e^{ha} - 1 - ha\right) a^{-2} B_n = \frac{x}{a} - \frac{B_n}{a^2} \ln\left(\frac{xa}{B_n} + 1\right) \leq \frac{x}{a}. \tag{1.13}$$

Putting this one into (1.12), we get furthermore

$$e^{-hx} E\left(e^{hS'_n}\right) \leq \exp\left\{\frac{x}{a} - \frac{x}{a} \ln\left(\frac{xa}{B_n} + 1\right)\right\}. \tag{1.14}$$

Putting $x/a = t$ into the above inequality, we get

$$\begin{aligned} P(S_n \geq x) &\leq \sum_{i=1}^{n} P(X_i > a) + P(S'_n \geq x) \\ &\leq \sum_{i=1}^{n} P(X_i > a) + e^{-hx} E e^{hS'_n} \\ &\leq \sum_{i=1}^{n} P\left(X_i > \frac{x}{t}\right) + \exp\left\{t - t \ln\left(\frac{x^2}{tB_n} + 1\right)\right\} \\ &= \sum_{i=1}^{n} P\left(X_i > \frac{x}{t}\right) + e^t \left(1 + \frac{x^2}{tB_n}\right)^{-t}. \end{aligned} \tag{1.15}$$

Letting $-X_i$ take the place of X_i in the above inequality, we can get

$$\begin{aligned} P(-S_n \geq x) = P(S_n \leq -x) &\leq \sum_{i=1}^{n} P\left(-X_i > \frac{x}{t}\right) + e^t \left(1 + \frac{x^2}{tB_n}\right)^{-t} \\ &= \sum_{i=1}^{n} P\left(X_i < \frac{-x}{t}\right) + e^t \left(1 + \frac{x^2}{tB_n}\right)^{-t}. \end{aligned} \tag{1.16}$$

Thus

$$P(|S_n| \geq x) = P(S_n \geq x) + P(S_n \leq -x) \leq \sum_{i=1}^{n} P\left(|X_i| < \frac{x}{t}\right) + 2e^t \left(1 + \frac{x^2}{tB_n}\right)^{-t}. \tag{1.17}$$

Multiplying (1.17) by px^{p-1}, letting $t = p$, and integrating over $0 < x < +\infty$, according to

$$E|X|^p = p \int_0^{+\infty} x^{p-1} P(|X| \geq x) \mathrm{d}x, \tag{1.18}$$

we obtain

$$\begin{aligned}
E|S_n|^p &= p\int_0^{+\infty} x^{p-1}P(|S_n| \geq x)\mathrm{d}x \\
&\leq p\sum_{i=1}^{n}\int_0^{+\infty} x^{p-1}P\left(|X_i| \geq \frac{x}{p}\right)\mathrm{d}x + 2pe^p\int_0^{+\infty} x^{p-1}\left(1+\frac{x^2}{pB_n}\right)^{-p}\mathrm{d}x \\
&= p^{p+1}\sum_{i=1}^{n}E|X_i|^p + pe^p(pB_n)^{p/2}\int_0^{+\infty}\frac{u^{p/2-1}}{(1+u)^p}\mathrm{d}u \\
&= p^{p+1}\sum_{i=1}^{n}E|X_i|^p + p^{p/2+1}\mathrm{e}^pB\left(\frac{p}{2},\frac{p}{2}\right)B_n^{p/2},
\end{aligned} \tag{1.19}$$

where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \int_0^{+\infty} x^{\alpha-1}(1+x)^{-(\alpha+\beta)}\mathrm{d}x, \alpha,\beta > 0$ is Beta function. Letting $c_p = \max(p^{p+1}, p^{1+p/2}\mathrm{e}^pB(p/2,p/2))$, we can deduce (1.9) from (1.19). From (1.9), we can prove (1.10) by a similar way of Stout's paper [10, Theorem 2.3.1]. □

Lemma 1.10. *Let $\{X_n; n \geq 1\}$ be a sequence of ND random variables. Then there exists a positive constant c such that, for any $x \geq 0$ and all $n \geq 1$,*

$$\left(1 - P\left(\max_{1\leq k\leq n}|X_k| > x\right)\right)^2\sum_{k=1}^{n}P(|X_k| > x) \leq cP\left(\max_{1\leq k\leq n}|X_k| > x\right). \tag{1.20}$$

Proof. Let $A_k = (|X_k| > x)$ and $\alpha_n = 1 - P(\bigcup_{k=1}^n A_k) = 1 - P(\max_{1\leq k\leq n}|X_k| > x)$. Without loss of generality, assume that $\alpha_n > 0$. Note that $\{I_{(X_k>x)} - EI_{(X_k>x)};\ k \geq 1\}$ and $\{I_{(X_k<-x)} - EI_{(X_k<-x)};\ k \geq 1\}$ are still ND by Lemma 1.6. Using (1.9), we get

$$\begin{aligned}
E\left(\sum_{k=1}^{n}(I_{A_k} - EI_{A_k})\right)^2 &= E\left(\sum_{k=1}^{n}\left(I_{(X_k>x)} - EI_{(X_k>x)}\right) + \left(I_{(X_k<-x)} - EI_{(X_k<-x)}\right)\right)^2 \\
&\leq 2E\left(\sum_{k=1}^{n}\left(I_{(X_k>x)} - EI_{(X_k>x)}\right)\right)^2 + 2E\left(\sum_{k=1}^{n}\left(I_{(X_k<-x)} - EI_{(X_k<-x)}\right)\right)^2 \\
&\leq c\sum_{k=1}^{n}P(A_k).
\end{aligned} \tag{1.21}$$

Combining with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\sum_{k=1}^{n}P(A_k) &= \sum_{k=1}^{n}P\left(A_k, \bigcup_{j=1}^{n}A_j\right) = \sum_{k=1}^{n}E\left(I_{A_k}I_{\bigcup_{j=1}^n A_j}\right) \\
&= E\left(\sum_{k=1}^{n}(I_{A_k} - EI_{A_k})\right)I_{\bigcup_{j=1}^n A_j} + \sum_{k=1}^{n}P(A_k)P\left(\bigcup_{j=1}^{n}A_j\right)
\end{aligned}$$

$$\leq \left(E\left(\sum_{k=1}^{n}(I_{A_k}-EI_{A_k})\right)^2 EI_{\bigcup_{j=1}^{n}A_j}\right)^{1/2}+(1-\alpha_n)\sum_{k=1}^{n}P(A_k)$$
$$\leq \left(\frac{c(1-\alpha_n)}{\alpha_n}\alpha_n\sum_{k=1}^{n}P(A_k)\right)^{1/2}+(1-\alpha_n)\sum_{k=1}^{n}P(A_k)$$
$$\leq \frac{1}{2}\left(\frac{c(1-\alpha_n)}{\alpha_n}+\alpha_n\sum_{k=1}^{n}P(A_k)\right)+(1-\alpha_n)\sum_{k=1}^{n}P(A_k). \tag{1.22}$$

Thus

$$\alpha_n^2\sum_{k=1}^{n}P(A_k)\leq c(1-\alpha_n), \tag{1.23}$$

that is,

$$\left(1-P\left(\max_{1\leq k\leq n}|X_k|>x\right)\right)^2\sum_{k=1}^{n}P(|X_k|>x)\leq cP\left(\max_{1\leq k\leq n}|X_k|>x\right). \tag{1.24}$$

□

2. Main Results and the Proofs

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [11] as follows. A sequence $\{Y_n; n\geq 1\}$ of random variables converges completely to the constant c if $\sum_{n=1}^{\infty}P(|X_n-c|>\varepsilon)<\infty$, for all $\varepsilon>0$. In view of the Borel-Cantelli lemma, this implies that $Y_n\to 0$ almost surely. Therefore, complete convergence is one of the most important problems in probability theory. Hsu and Robbins [11] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Baum and Katz [12] proved that if $\{X, X_n; n\geq 1\}$ is a sequence of i.i.d. random variables with mean zero, then $E|X|^{p(t+2)}<\infty(1\leq p<2,\ t\geq -1)$ is equivalent to the condition that $\sum_{n=1}^{\infty}n^tP(\sum_{1=1}^{n}|X_i|/n^{1/p}>\varepsilon)<\infty$, for all $\varepsilon>0$. Recent results of the complete convergence can be found in Li et al. [13], Liang and Su [14], Wu [15, 16], and Sung [17].

In this paper we study the complete convergence for negatively dependent random variables. As a result, we extend some complete convergence theorems for independent random variables to the negatively dependent random variables without necessarily imposing any extra conditions.

Theorem 2.1. *Let $\{X, X_n;\ n\geq 1\}$ be a sequence of identically distributed ND random variables and $\{a_{nk};\ 1\leq k\leq n,\ n\geq 1\}$ an array of real numbers, and let $r>1$, $p>2$. If, for some $2\leq q<p$,*

$$N(n,m+1)\widehat{=}\sharp\left\{k\geq 1; |a_{nk}|\geq (m+1)^{-1/p}\right\}\approx m^{q(r-1)/p},\quad n,m\geq 1, \tag{2.1}$$
$$EX=0\quad \text{for } 1\leq q(r-1), \tag{2.2}$$
$$\sum_{k=1}^{n}a_{nk}^2\ll n^{\delta}\quad \text{for } 2\leq q(r-1) \text{ and some } 0<\delta<\frac{2}{p}, \tag{2.3}$$

then, for $r \geq 2$,

$$E|X|^{p(r-1)} < \infty \tag{2.4}$$

if and only if

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon n^{1/p} \right) < \infty, \quad \forall \varepsilon > 0. \tag{2.5}$$

For $1 < r < 2$, (2.4) *implies* (2.5), *conversely, and* (2.5) *and* $n^{r-2} P(\max_{1\leq k\leq n}|a_{nk}X_k| > n^{1/p})$ *decreasing on* n *imply* (2.4).

For $p = 2$, $q = 2$, we have the following theorem.

Theorem 2.2. *Let* $\{X, X_n;\ n \geq 1\}$ *be a sequence of identically distributed ND random variables and* $\{a_{nk};\ 1 \leq k \leq n, n \geq 1\}$ *an array of real numbers, and let* $r > 1$. *If*

$$N(n, m+1) \widehat{=} \sharp\left\{ k; |a_{nk}| \geq (m+1)^{-1/2} \right\} \approx m^{r-1}, \quad n, m \geq 1, \tag{2.6}$$

$$EX = 0, \qquad 1 \leq 2(r-1),$$
$$\sum_{k=1}^{n} |a_{nk}|^{2(r-1)} = O(1), \tag{2.7}$$

then, for $r \geq 2$,

$$E|X|^{2(r-1)} \log|X| < \infty \tag{2.8}$$

if and only if

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon n^{1/2} \right) < \infty, \quad \forall \varepsilon > 0. \tag{2.9}$$

For $1 < r < 2$, (2.8) *implies* (2.9), *conversely, and* (2.9) *and* $n^{r-2} P(\max_{1\leq k\leq n}|a_{nk}X_k| > n^{1/2})$ *decreasing on* n *imply* (2.8).

Remark 2.3. Since NA random variables are a special case of ND r. v. 's, Theorems 2.1 and 2.2 extend the work of Liang and Su [14, Theorem 2.1].

Remark 2.4. Since, for some $2 \leq q \leq p$, $\sum_{k\in N} |a_{nk}|^{q(r-1)} \ll 1$ as $n \to \infty$ implies that

$$N(n, m+1) \widehat{=} \sharp\left\{ k \geq 1; |a_{nk}| \geq (m+1)^{-1/p} \right\} \ll m^{q(r-1)/p} \quad \text{as } n \longrightarrow \infty, \tag{2.10}$$

taking $r = 2$, then conditions (2.1) and (2.6) are weaker than conditions (2.13) and (2.9) in Li et al. [13]. Therefore, Theorems 2.1 and 2.2 not only promote and improve the work of Li et al. [13, Theorem 2.2] for i.i.d. random variables to an ND setting but also obtain their necessities and relax the range of r.

Proof of Theorem 2.1. Equation (2.4)⇒(2.5). To prove (2.5) it suffices to show that

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} a_{ni}^{\pm} X_i \right| > \varepsilon n^{1/p} \right) < \infty, \quad \forall \varepsilon > 0, \tag{2.11}$$

where $a_{ni}^{+} = \max(a_{ni}, 0)$ and $a_{ni}^{-} = \max(-a_{ni}, 0)$. Thus, without loss of generality, we can assume that $a_{ni} > 0$ for all $n \ge 1$, $i \le n$. For $0 < \alpha < 1/p$ small enough and sufficiently large integer K, which will be determined later, let

$$\begin{aligned}
X_{ni}^{(1)} &= -n^{\alpha} I_{(a_{ni}X_i < -n^{\alpha})} + a_{ni} X_i I_{(a_{ni}|X_i| \le n^{\alpha})} + n^{\alpha} I_{(a_{ni}X_i > n^{\alpha})}, \\
X_{ni}^{(2)} &= (a_{ni}X_i - n^{\alpha}) I_{(n^{\alpha} < a_{ni}X_i < \varepsilon n^{1/p}/K)}, \\
X_{ni}^{(3)} &= (a_{ni}X_i + n^{\alpha}) I_{(-\varepsilon n^{1/p}/K < a_{ni}X_i < -n^{\alpha})}, \\
X_{ni}^{(4)} &= a_{ni}X_{ni} - X_{ni}^{(1)} - X_{ni}^{(2)} - X_{ni}^{(3)} \\
&= (a_{ni}X_i + n^{\alpha}) I_{(a_{ni}X_i \le -\varepsilon n^{1/p}/K)} + (a_{ni}X_i - n^{\alpha}) I_{(a_{ni}X_i \ge \varepsilon n^{1/p}/K)}, \\
S_{nk}^{(j)} &= \sum_{i=1}^{k} X_{ni}^{(j)}, \quad j = 1,2,3,4;\ 1 \le k \le n,\ n \ge 1.
\end{aligned} \tag{2.12}$$

Thus $S_{nk} \widehat{=} \sum_{i=1}^{k} a_{ni} X_i = \sum_{j=1}^{4} S_{nk}^{(j)}$. Note that

$$\left(\max_{1\le k\le n} |S_{nk}| > 4\varepsilon n^{1/p} \right) \subseteq \bigcup_{j=1}^{4} \left(\max_{1\le k\le n} \left| S_{nk}^{(j)} \right| > \varepsilon n^{1/p} \right). \tag{2.13}$$

So, to prove (2.5) it suffices to show that

$$I_j \widehat{=} \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n} \left| S_{nk}^{(j)} \right| > \varepsilon n^{1/p} \right) < \infty, \quad j = 1,2,3,4. \tag{2.14}$$

For any $q' > q$,

$$\begin{aligned}\sum_{i=1}^{n} a_{ni}^{q'(r-1)} &= \sum_{j=1}^{\infty} \sum_{(j+1)^{-1} \leq a_{ni}^{p} < j^{-1}} a_{ni}^{q'(r-1)} \leq \sum_{j=1}^{\infty} \sum_{(j+1)^{-1} \leq a_{ni}^{p} < j^{-1}} j^{-q'(r-1)/p} \\ &\ll \sum_{j=1}^{\infty} \left(N(n, j+1) - N(n, j)\right) j^{-q'(r-1)/p} \\ &\ll \sum_{j=1}^{\infty} N(n, j) \left(j^{-q'(r-1)/p} - (j+1)^{-q'(r-1)/p}\right) \\ &\ll \sum_{j=1}^{\infty} j^{-1-(q'-q)(r-1)/p} < \infty.\end{aligned} \tag{2.15}$$

Now, we prove that

$$n^{-1/p} \max_{1 \leq k \leq n} \left| ES_{nk}^{(1)} \right| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.16}$$

(i) For $0 < q(r-1) < 1$, taking $q < q' < p$ such that $0 < q'(r-1) < 1$, by (2.4) and (2.15), we get

$$\begin{aligned} &n^{-1/p} \max_{1 \leq k \leq n} \left| ES_{nk}^{(1)} \right| \\ &\leq n^{-1/p} \sum_{i=1}^{n} \left(E|a_{ni}X_i| I_{(|a_{ni}X_i| \leq n^{\alpha})} + n^{\alpha} P(|a_{ni}X_i| > n^{\alpha}) \right) \\ &\leq n^{-1/p} \left(\sum_{i=1}^{n} E|a_{ni}X_i|^{q'(r-1)} |a_{ni}X_i|^{1-q'(r-1)} I_{(|a_{ni}X_i| \leq n^{\alpha})} + n^{\alpha - \alpha q'(r-1)} \sum_{i=1}^{n} E|a_{ni}X_i|^{q'(r-1)} \right) \\ &\ll n^{-1/p + \alpha - \alpha q'(r-1)} \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \tag{2.17}$$

(ii) For $1 \leq q(r-1)$, letting $q < q' < p$, by (2.2), (2.4), and (2.15), we get

$$\begin{aligned} &n^{-1/p} \max_{1 \leq k \leq n} \left| ES_{nk}^{(1)} \right| \\ &\leq n^{-1/p} \sum_{i=1}^{n} \left(E|a_{ni}X_i| I_{(|a_{ni}X_i| > n^{\alpha})} + n^{\alpha} P(|a_{ni}X_i| > n^{\alpha}) \right) \\ &\leq n^{-1/p} \sum_{i=1}^{n} \left(E|a_{ni}X_i| \left(\frac{|a_{ni}X_i|}{n^{\alpha}} \right)^{q'(r-1)-1} I_{(|a_{ni}X_i| \leq n^{\alpha})} + n^{\alpha - \alpha q'(r-1)} E|a_{ni}X_i|^{q'(r-1)} \right) \\ &\ll n^{-1/p + \alpha - \alpha q'(r-1)} \longrightarrow 0. \end{aligned} \tag{2.18}$$

Hence, (2.16) holds. Therefore, to prove $I_1 < \infty$ it suffices to prove that

$$\widetilde{I}_1 \widehat{=} \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}\left|S_{nk}^{(1)} - ES_{nk}^{(1)}\right| > \varepsilon n^{1/p}\right) < \infty, \quad \forall \varepsilon > 0. \tag{2.19}$$

Note that $\{X_{ni}^{(1)}; 1 \le i \le n, n \ge 1\}$ is still ND by the definition of $X_{ni}^{(1)}$ and Lemma 1.6. Using the Markov inequality and Lemma 1.8, we get for a suitably large M, which will be determined later,

$$\begin{aligned}
\widetilde{I}_1 &\ll \sum_{n=1}^{\infty} n^{r-2-M/p} E\left(\max_{1\le k\le n}\left|S_{nk}^{(1)} - ES_{nk}^{(1)}\right|\right)^M \\
&\ll \sum_{n=1}^{\infty} n^{r-2-M/p} \log^M n \left[\sum_{i=1}^{n} E\left|X_{ni}^{(1)}\right|^M + \left(\sum_{i=1}^{n} E\left(X_{ni}^{(1)}\right)^2\right)^{M/2}\right] \\
&\widehat{=} \widetilde{I}_{11} + \widetilde{I}_{12}.
\end{aligned} \tag{2.20}$$

Taking $M > \max(2, p(r-1)(1-\alpha q')/(1-\alpha p))$, then $r - 2 - M/p + \alpha M - \alpha q'(r-1) < -1$, and, by (2.15), we get

$$\begin{aligned}
\widetilde{I}_{11} &\le \sum_{n=1}^{\infty} n^{r-2-M/p} \log^M n \sum_{i=1}^{n}\left(E|a_{ni}X_i|^M I_{(|a_{ni}X_i|\le n^\alpha)} + n^{M\alpha} P(|a_{ni}X_i| > n^\alpha)\right) \\
&\le \sum_{n=1}^{\infty} n^{r-2-M/p} \log^M n \sum_{i=1}^{n}\left(E|a_{ni}X_i|^{q'(r-1)} n^{\alpha(M-q'(r-1))} + n^{\alpha(M-q'(r-1))} E|a_{ni}X_i|^{q'(r-1)}\right) \\
&\ll \sum_{n=1}^{\infty} n^{r-2-M/p+\alpha M-\alpha q'(r-1)} \log^M n \\
&< \infty.
\end{aligned} \tag{2.21}$$

(i) For $q(r-1) < 2$, taking $q < q' < p$ such that $q'(r-1) < 2$ and taking $M > \max(2,\ 2p(r-1)/(2-2\alpha p + \alpha p q'(r-1)))$, from (2.15) and $r - 2 - M/p + \alpha M - M\alpha q'(r-1)/2 < -1$, we have

$$\begin{aligned}
\widetilde{I}_{12} &\le \sum_{n=1}^{\infty} n^{r-2-M/p} \log^M n \left[\sum_{i=1}^{n} E|a_{ni}X_i|^{q'(r-1)} n^{\alpha(2-q'(r-1))} I_{(|a_{ni}X_i|\le n^\alpha)}\right. \\
&\qquad \left. + n^{2\alpha-\alpha q'(r-1)} E|a_{ni}X_i|^{q'(r-1)}\right]^{M/2} \\
&\ll \sum_{n=1}^{\infty} n^{-r-2-M/p+\alpha M-M\alpha q'(r-1)/2} \log^M n \\
&< \infty.
\end{aligned} \tag{2.22}$$

(ii) For $q(r-1) \geq 2$, taking $q < q' < p$ and $M > \max(2,\ 2p(r-1)/(2-p\delta))$, where δ is defined by (2.3), we get, from (2.3), (2.4), (2.15), and $r-2-M/p+\delta M/2 < -1$,

$$\begin{aligned}\widetilde{I}_{12} &\ll \sum_{n=1}^{\infty} n^{r-2-M/p} \log^M n \left[\sum_{i=1}^{n} a_{ni}^2 + n^{2\alpha-\alpha q'(r-1)} E|a_{ni}X_i|^{q'(r-1)}\right]^{M/2} \\ &\ll \sum_{n=1}^{\infty} n^{r-2-M/p+\delta M/2} \log^M n \\ &< \infty. \end{aligned} \tag{2.23}$$

Since

$$\begin{aligned}\left(\sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p}\right) &= \left(\sum_{i=1}^{n} (a_{ni}X_i - n^{\alpha}) I_{(n^{\alpha} < a_{ni}X_i < \varepsilon n^{1/p}/K)} > \varepsilon n^{1/p}\right) \\ &\subseteq (\text{there at least exist } K \text{ indices } k \text{ such that } a_{nk}X_k > n^{\alpha}),\end{aligned} \tag{2.24}$$

we have

$$P\left(\sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p}\right) \leq \sum_{1 \leq i_1 < i_2 < \cdots < i_K \leq n} P(a_{ni_1}X_{i_1} > n^{\alpha}, a_{ni_2}X_{i_2} > n^{\alpha}, \ldots, a_{ni_K}X_{i_K} > n^{\alpha}). \tag{2.25}$$

By Lemma 1.6, $\{a_{ni}X_i; 1 \leq i \leq n, n \geq 1\}$ is still ND. Hence, for $q < q' < p$ we conclude that

$$\begin{aligned}P\left(\sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p}\right) &\leq \sum_{1 \leq i_1 < i_2 < \cdots < i_K \leq n} \prod_{j=1}^{K} P\left(a_{ni_j}X_{i_j} > n^{\alpha}\right) \\ &\leq \left(\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{\alpha})\right)^K \\ &\leq \left(\sum_{i=1}^{n} n^{-\alpha q'(r-1)} E|a_{ni}X_i|^{q'(r-1)}\right)^K \\ &\ll n^{-\alpha q'(r-1)K},\end{aligned} \tag{2.26}$$

via (2.4) and (2.15). $X_{ni}^{(2)} > 0$ from the definition of $X_{ni}^{(2)}$. Hence by (2.26) and by taking $\alpha > 0$ and K such that $r-2-\alpha K q'(r-1) < -1$, we have

$$I_2 = \sum_{n=1}^{\infty} n^{r-2} P\left(\sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p}\right) \ll \sum_{n=1}^{\infty} n^{r-2-\alpha q'(r-1)K} < \infty. \tag{2.27}$$

Similarly, we have $X_{ni}^{(3)} < 0$ and $I_3 < \infty$.

Last, we prove that $I_4 < \infty$. Let $Y = KX/\varepsilon$. By the definition of $X_{ni}^{(4)}$ and (2.1), we have

$$\begin{aligned}
P\left(\max_{1\le k\le n}\left|S_{nk}^{(4)}\right| > \varepsilon n^{1/p}\right) &\le P\left(\sum_{i=1}^{n}\left|X_{ni}^{(4)}\right| > \varepsilon n^{1/p}\right) \\
&\le P\left(\bigcup_{i=1}^{n}\left(a_{ni}|X_i| > \frac{\varepsilon n^{1/p}}{K}\right)\right) \\
&\le \sum_{i=1}^{n} P\left(a_{ni}|X_i| > \frac{\varepsilon n^{1/p}}{K}\right) \\
&= \sum_{j=1}^{\infty} \sum_{(j+1)^{-1}\le a_{ni}^{p} < j^{-1}} P\left(|Y| > (nj)^{1/p}\right) \\
&= \sum_{j=1}^{\infty}(N(n,j+1) - N(n,j))\sum_{l=nj}^{\infty} P(l \le |Y|^p < l+1) \\
&= \sum_{l=n}^{\infty}\sum_{j=1}^{[l/n]}(N(n,j+1) - N(n,j))P(l \le |Y|^p < l+1) \\
&\approx \sum_{l=n}^{\infty}\left(\frac{l}{n}\right)^{q(r-1)/p} P(l \le |Y|^p < l+1).
\end{aligned} \tag{2.28}$$

Combining with (2.15),

$$\begin{aligned}
I_4 &\approx \sum_{n=1}^{\infty} n^{r-2}\sum_{l=n}^{\infty}\left(\frac{l}{n}\right)^{q(r-1)/p} P(l \le |Y|^p < l+1) \\
&= \sum_{l=1}^{\infty}\sum_{n=1}^{l} n^{r-2-q(r-1)/p} l^{q(r-1)/p} P(l \le |Y|^p < l+1) \\
&\approx \sum_{l=1}^{\infty} l^{r-1} P(l \le |Y|^p < l+1) \\
&\approx E|Y|^{p(r-1)} \approx E|X|^{p(r-1)} < \infty.
\end{aligned} \tag{2.29}$$

Now we prove (2.5)⇒(2.4). Since

$$\max_{1\le j\le n}\left|a_{nj}X_j\right| \le \max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| + \max_{1\le j\le n}\left|\sum_{i=1}^{j-1} a_{ni}X_i\right|, \tag{2.30}$$

then from (2.5) we have

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le j\le n}\left|a_{nj}X_j\right| > n^{1/p}\right) < \infty. \tag{2.31}$$

Combining with the hypotheses of Theorem 2.1,

$$P\left(\max_{1\le j\le n}\left|a_{nj}X_j\right| > n^{1/p}\right) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.32}$$

Thus, for sufficiently large n,

$$P\left(\max_{1\le j\le n}\left|a_{nj}X_j\right| > n^{1/p}\right) < \frac{1}{2}. \tag{2.33}$$

By Lemma 1.6, $\{a_{nj}X_j; 1 \le j \le n, n \ge 1\}$ is still ND. By applying Lemma 1.10 and (2.1), we obtain

$$\sum_{k=1}^{n} P\left(|a_{nk}X_k| > n^{1/p}\right) \le 4CP\left(\max_{1\le k\le n}|a_{nk}X_k| > n^{1/p}\right). \tag{2.34}$$

Substituting the above inequality in (2.5), we get

$$\sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\left(|a_{nk}X_k| > n^{1/p}\right) < \infty. \tag{2.35}$$

So, by the process of proof of $I_4 < \infty$,

$$E|X|^{p(r-1)} \approx \sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\left(|a_{nk}X_k| > n^{1/p}\right) < \infty. \tag{2.36}$$

□

Proof of Theorem 2.2. Let $p = 2$, $\alpha < 1/p = 1/2$, and $K > 1/(2\alpha)$. Using the same notations and method of Theorem 2.1, we need only to give the different parts.

Letting (2.7) take the place of (2.15), similarly to the proof of (2.19) and (2.26), we obtain

$$n^{-1/2}\max_{1\le k\le n}\left|ES_{nk}^{(1)}\right| \ll n^{-1/2+\alpha-2\alpha(r-1)} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.37}$$

Taking $M > \max(2, 2(r-1))$, we have

$$\widetilde{I}_{11} \ll \sum_{n=1}^{\infty} n^{-1-(1-2\alpha)(M/2-(r-1))} \log^M n < \infty. \tag{2.38}$$

For $r - 1 \le 1$, taking $M > \max(2, 2(r-1)/(1-2\alpha+2\alpha(r-1)))$, we get

$$\widetilde{I}_{12} \ll \sum_{n=1}^{\infty} n^{-1-(1-2\alpha(r-1)-2\alpha)M/2+(r-1)} \log^M n < \infty. \tag{2.39}$$

For $r-1>1$, $EX_{ni}^2<\infty$ from (2.8). Letting $M>2(r-1)^2$, by the Hölder inequality,

$$\begin{aligned}\widetilde{I}_{12} &\ll \sum_{n=1}^{\infty} n^{r-2-M/2}\log^M n\left[\sum_{i=1}^{n} a_{ni}^2 + n^{2\alpha-2\alpha(r-1)}E(a_{ni}X_i)^{2(r-1)}\right]^{M/2} \\ &\ll \sum_{n=1}^{\infty} n^{r-2-M/2}\log^M n\left[\left(\sum_{i=1}^{n} a_{ni}^{2(r-1)}\right)^{1/(r-1)}\left(\sum_{i=1}^{n}1\right)^{r-2/(r-1)}\right]^{M/2} \\ &\ll \sum_{n=1}^{\infty} n^{-1-M/2(r-1)+(r-1)}\log^M n<\infty.\end{aligned} \tag{2.40}$$

By the definition of K,

$$I_2 \ll \sum_{n=1}^{\infty} n^{-1-(r-1)(2\alpha K-1)}<\infty. \tag{2.41}$$

Similarly to the proof (2.31), we have

$$\begin{aligned}I_4 &\ll \sum_{l=1}^{\infty}\sum_{n=1}^{l} n^{-1}l^{r-1}P\left(l\le |Y|^2<l+1\right) \\ &= \sum_{l=1}^{\infty} l^{r-1}\log l P\left(l\le |Y|^2<l+1\right) \\ &\approx E\left(|Y|^{2(r-1)}\log|Y|\right) \\ &\approx E\left(|X|^{2(r-1)}\log|X|\right) \\ &<\infty.\end{aligned} \tag{2.42}$$

Equation (2.9)⇒(2.8) Using the same method of the necessary part of Theorem 2.1, we can easily get

$$E\left(|X|^{2(r-1)}\log|X|\right)\approx \sum_{n=1}^{\infty} n^{r-2}\sum_{k=1}^{n}P\left(|a_{nk}X_k|>n^{1/2}\right)<\infty. \tag{2.43}$$

□

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