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Research Article

Weighted Strong Law of Large Numbers for Random Variables Indexed by a Sector

Przemysław Matuła[1] and Michał Seweryn[2]

[1] *Institute of Mathematics, Marie Curie-Skłodowska University, Plac Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland*
[2] *Faculty of Mathematics and Computer Science, University of Łódź, Ulica Banacha 22, 90-238 Łódź, Poland*

Correspondence should be addressed to Przemysław Matuła, matula@hektor.umcs.lublin.pl

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We find necessary and sufficient conditions for the weighted strong law of large numbers for independent random variables with multidimensional indices belonging to some sector.

1. Introduction and the Notation

Let $\mathbb{N}^d$, $d \geq 1$, be a d-dimensional lattice. The points of this lattice will be denoted by $\underline{m} = (m_1, \ldots, m_d)$, $\underline{n} = (n_1, \ldots, n_d)$, and so forth. The set $\mathbb{N}^d$ is partially ordered by the relation $\underline{m} \leq \underline{n}$ if and only if for every $i = 1, \ldots, d$ we have $m_i \leq n_i$. We will also write $\underline{m} < \underline{n}$ if for every $i = 1, \ldots, d$, $m_i \leq n_i$ and for at least one i_0 we have $m_{i_0} < n_{i_0}$. Let W be an infinite subset of $\mathbb{N}^d$; moreover, assume that a nonnegative, increasing real function $\delta : W \to \mathbb{R}^+$ is given, and set $T_{W,\delta}(m) := \operatorname{card}\{\underline{k} \in W : \delta(\underline{k}) \leq m\}$, $\tau_{W,\delta}(m) := T_{W,\delta}(m) - T_{W,\delta}(m-1)$, $m \in \mathbb{N}$. Assume that $T_{W,\delta}(m) < \infty$, for each $m \in \mathbb{N}$ and $T_{W,\delta}(m) \to \infty$ as $m \to \infty$. Moreover, put $|\underline{n}| = \prod_{i=1}^{d} n_i$ and $\|\underline{n}\| = \max_{1 \leq i \leq d} |n_i|$. We aim to study the convergence of sequences indexed by lattice points. For this means let us recall that $\underline{n} \to \infty$ may have different meanings; in other words, the term "$\underline{n}$ tends to infinity" may be understood as $|\underline{n}| \to \infty$ (equivalently $\|\underline{n}\| \to \infty$) or $\min_{1 \leq i \leq d}(n_i) \to \infty$, we will be using the first meaning. Thus, for a field $(a_{\underline{n}})_{\underline{n} \in \mathbb{N}^d}$ of real numbers indexed by positive lattice points, we write $a_{\underline{n}} \to a$, $\underline{n} \to \infty$ if and only if for every $\varepsilon > 0$ there exist a $\underline{n}_0 \in \mathbb{N}^d$ such that for each $\underline{n} \nleq \underline{n}_0$ we have $|a_{\underline{n}} - a| < \varepsilon$.

Our setting is an extension of the one investigated by Klesov and Rychlik [1] or Indlekofer and Klesov [2], that is, of the so called *"sectorial convergence"*. Let $f_{i,j}, F_{i,j} : \mathbb{R}_+ \to \mathbb{R}_+$, $1 \leq i < j \leq d$, be nondecreasing nonnegative real functions such that $f_{i,j}(x) \leq x \leq F_{i,j}(x)$,

$1 \le i < j \le d$, $x \in \mathbb{R}_+$. Let us denote by $A_{f,F}$ a d-dimensional sector defined by these functions in the following way:

$$A_{f,F} := \left\{ \underline{n} \in \mathbb{N}^d : f_{i,j}(n_j) \le n_i \le F_{i,j}(n_j),\ 1 \le i < j \le d \right\}. \tag{1.1}$$

Moreover, for $m = 1, 2, \ldots$, let us consider the Dirichlet divisors $\tau_{f,F}(m)$ for the sector $A_{f,F}$ defined by

$$\tau_{f,F}(m) = \operatorname{card}\{ \underline{n} \in A_{f,F} : |\underline{n}| = m \}, \tag{1.2}$$

and set

$$T_{f,F}(m) = \sum_{k=1}^{m} \tau_{f,F}(k) = \operatorname{card}\{ \underline{n} \in A_{f,F} : |\underline{n}| \le m \}. \tag{1.3}$$

For $x \in \mathbb{R}_+$ we extend the function $T_{f,F}$ by defining the step function $T_{f,F}(x) = T_{f,F}([x])$ where $[x]$ denotes the integer part of x.

We consider a modified version of *"sectorial convergence"* in which we say that a field of real numbers $(a_{\underline{n}})$ indexed by lattice points in $\mathbb{N}^d$ converge in the set W to $a \in \mathbb{R}$, and write $\lim_W a_{\underline{n}} = a$ if and only if for every $\varepsilon > 0$ the inequality $|a_{\underline{n}} - a| < \varepsilon$ holds for all but finite number of $\underline{n} \in W$. It was Gut (see [3]), who for the first time considered sectorial convergence for random fields, with the sector defined as

$$A_\theta := \left\{ \underline{n} \in \mathbb{N}^d : \theta n_j \le n_i \le \frac{n_j}{\theta},\ 1 \le i,\ j \le d,\ i \ne j \right\}, \tag{1.4}$$

and $\tau_\theta(m) = \operatorname{card}\{\underline{n} \in A_\theta : |\underline{n}| = m\}$, $T_\theta(m) = \sum_{k=1}^{m} \tau_\theta(k)$, $m = 1, 2, \ldots$, $\theta \in [0, 1)$, where $A_0 = \mathbb{N}^d$. For recent results in this area, see [1, 2, 4]. Our aim is to extend the results of [2] in the spirit considered by Lagodowski and Matuła in [4].

We will be studying necessary and sufficient conditions for the weighted strong law of large numbers (WSLLN for short) for random fields of independent random variables for a general class of weights (defined by Feller in [5] and Jajte in [6]). The case of such summability methods in the multi-index setting was considered in [4].

Let us recall the definition of the class of transformations considered by Lagodowski and Matuła (see [4]). Let $g, h : \mathbb{R}^+ \to \mathbb{R}^+$ be nonnegative real functions; moreover, let g be increasing with range $(0, \infty)$, and set $\phi(x) := g(x)h(x)$, $x > 0$. We will say that the functions g, h satisfy the Feller-Jajte condition if the following two conditions are satisfied.

(A1) There exists $p > 0$ such that the function ϕ is increasing in the interval (p, ∞) with range $(0, \infty)$ and $\lim_{n\to\infty} \phi(n) = \infty$.

(A2) There exists a constant $a > 0$ such that

$$\begin{gathered} \sum_{k=s}^{\infty} \frac{\tau_\theta(k)}{\phi^2(k)} \le a \frac{s}{\phi^2(s)}, \qquad s > p,\ \theta \in (0,1), \\ \sum_{k=s}^{\infty} \frac{\tau_0(k)}{\phi^2(k)} \le a \frac{s \log^{d-1} s}{\phi^2(s)}, \qquad s > p. \end{gathered} \tag{1.5}$$

The first assumption is technical, and it is required in order for the inverse function ϕ^{-1} to exist. In our present work we will consider a modification of this class. Instead of (A2) we will consider the following condition.

(A2′) There exists a constant $a > 0$ such that

$$\sum_{k=s}^{\infty} \frac{\tau_{W,\delta}(k)}{\phi^2(k)} \leq a \frac{T_{W,\delta}(s)}{\phi^2(s)}, \quad s > p. \tag{1.6}$$

Making use of the well-known asymptotics (see [7, 8]) for τ_θ and T_θ, that is, the relations

$$\begin{gathered} c_1 k \leq T_\theta(k) \leq c_2 k, \quad \theta \in (0,1), \\ c_1 k \log^{d-1} k \leq T_0(k) \leq c_2 k \log^{d-1} k, \end{gathered} \tag{1.7}$$

where c_1, c_2 are nonnegative constants, we see that, in the case of the sector considered by Gut, the conditions (A2) and (A2′) coincide. It is worth noting that the number $\tau_{W,\delta}(k)$ does not exceed $\tau_0(k)$.

Let $(X_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ be a field of independent random variables. The aim of the paper is to find the necessary and sufficient conditions for the almost sure convergence of

$$\lim_W \frac{1}{g(\delta(\underline{n}))} \sum \frac{X_{\underline{k}} - a_{\underline{k}}}{h(\delta(\underline{k}))} = 0, \tag{1.8}$$

where the summation is extended over all $\underline{k} \leq \underline{n}$ or $W \ni \underline{k} \leq \underline{n}$, and the centering constants $a_{\underline{k}}$ are either the moments or truncated moments of $X_{\underline{k}}$. Our main results, the necessary and sufficient conditions for the WSLLN for independent random fields, may also be seen as an extension of the previous results of [2, 4].

2. Main Results

Let $(X_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ be a field of independent random variables; moreover, let $W \subset \mathbb{N}^d$ and the functions $\delta : W \to \mathbb{R}^+$, $T_{W,\delta}, \tau_{W,\delta}$ be as before. For simplicity we impose some regularity condition on the function $T_{W,\delta}$; namely,

$$\frac{T_{W,\delta}(k+1)}{T_{W,\delta}(k)} \leq B, \quad \text{for some } B > 0 \text{ and every } k \in \mathbb{N}. \tag{2.1}$$

Note—similarly as in [2]—that, if (2.1) holds, then $ET_{W,\delta}(|X|) < \infty$ is equivalent to

$$\sum_{\underline{n}\in W} P(|X| \geq \delta(\underline{n})) < \infty. \tag{2.2}$$

Furthermore,

$$\sum_{\underline{n}\in W} P(|X| \ge \delta(\underline{n})) = \sum_{k=1}^{\infty} \tau_{W,\delta}(k) P(|X| \ge k)$$
$$= \sum_{k=1}^{\infty} T_{W,\delta}(k) P(k \le |X| < k+1). \tag{2.3}$$

Let us observe that (2.1) is satisfied for standard sectors A_θ and if $A_{\theta_1} \subset W \subset A_{\theta_2}$, for some θ_1 and θ_2. Let the functions g, h satisfy the conditions (A1) and (A2'). We will also be using a well-known truncation technique with

$$Y_{\underline{n}} := X_{\underline{n}} I\{|X_{\underline{n}}| \le \phi(\delta(\underline{n}))\}, \qquad m_{\underline{n}} := EY_{\underline{n}}. \tag{2.4}$$

In the next theorem we give sufficient conditions for the WSLLN of the form:

$$\lim_{W} \frac{1}{g(\delta(\underline{n}))} \sum_{W \ni \underline{k} \le \underline{n}} \frac{X_{\underline{k}} - m_{\underline{k}}}{h(\delta(\underline{k}))} = 0, \quad \text{almost surely.} \tag{2.5}$$

In the first theorem we will not assume that the random variables $X_{\underline{n}}$ have the same distribution. Instead we will use the notion of weak domination.

Definition 2.1. A random field $(X_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ is said to be weakly dominated on the average in the set A by the random variable Y if there exists a constant $C > 0$ such that, for every $k \in \mathbb{N}$ and $t \ge 0$,

$$\sum_{\underline{n}\in A : \delta(\underline{n}) = k} P(|X_{\underline{n}}| \ge t) \le C\tau_{A,\delta}(k) P(|Y| \ge t), \tag{2.6}$$

where $\tau_{A,\delta}(k) := \operatorname{card}\{\underline{n} \in A : \delta(\underline{n}) = k\}$ and obviously $T_{A,\delta}(k) := \tau_{A,\delta}(1) + \tau_{A,\delta}(2) + \cdots + \tau_{A,\delta}(k)$.

This condition, to the best of our knowledge, was introduced in [9], where it is also discussed that this condition is independent of the notion of weak mean domination (see also [10]). In the rest of our work we will write "weakly dominated on the average" without indicating the set A on which we consider the domination condition (unless it causes any confusion). The price to pay for weakening the condition of identically distributed random variables to the weakly dominated on the average is to only be able to prove sufficient conditions. In some cases we will also consider a narrower class of summability methods; that is, for a given set W, apart from the conditions (A1) and (A2'), we will assume that

(A3) there exists a constant $a > 0$ such that, for each $k > 1$,

$$\sum_{n=1}^{k} \frac{\tau_{W,\delta}(n)}{\phi(n)} \le a \frac{T_{W,\delta}(k)}{\phi(k)}. \tag{2.7}$$

With such preparations we can formulate our first main result.

Theorem 2.2. *Let* $(X_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ *be a field of independent random variables weakly dominated on the average in the set* W *by the random variable* Y*. Moreover, let* (A1), (A2'), *and* (2.1) *be satisfied. If*

$$E\left(T_{W,\delta}\left(\phi^{-1}(|Y|)\right)\right) < \infty, \tag{2.8}$$

then

$$\lim_{W} \frac{1}{g(\delta(\underline{n}))} \sum_{W\ni \underline{k}\leq \underline{n}} \frac{X_{\underline{k}} - m_{\underline{k}}}{h(\delta(\underline{k}))} = 0, \quad \textit{almost surely}. \tag{2.9}$$

If additionally $\phi(n)/n \to 0$ *as* $n \to \infty$ *and the function* ϕ *satisfies the condition* $(A3)$*, then*

$$\lim_{W} \frac{1}{g(\delta(\underline{n}))} \sum_{W\ni \underline{k}\leq \underline{n}} \frac{X_{\underline{k}} - E(X_{\underline{k}})}{h(\delta(\underline{k}))} = 0, \quad \textit{almost surely}. \tag{2.10}$$

Proof. From the Kolmogorov-type maximal inequality and the strong law of large numbers which is due to Christofides and Serfling (see [11, Corollary 2.5 and Theorem 2.8]), it follows that the sufficient condition for the convergence of series of independent random fields is analogous to the one-dimensional case. Therefore, it suffices to prove that the series

$$\sum_{\underline{n}\in W} E\left(\frac{Y_{\underline{n}}}{\phi(\delta(\underline{n}))}\right)^2, \tag{2.11}$$

is convergent. We have

$$\begin{aligned}
\sum_{\underline{n}\in W} \frac{E(Y_{\underline{n}})^2}{\phi^2(\delta(\underline{n}))} &= \sum_{\underline{n}\in W} \frac{1}{\phi^2(\delta(\underline{n}))} \int_0^{\infty} P\left(Y_{\underline{n}}^2 \geq t\right) dt \\
&\leq \sum_{\underline{n}\in W} \frac{1}{\phi^2(\delta(\underline{n}))} \int_0^{\phi^2(\delta(\underline{n}))} P\left(X_{\underline{n}}^2 \geq t\right) dt \\
&= \sum_{k=1}^{\infty} \frac{1}{\phi^2(k)} \int_0^{\phi^2(k)} \sum_{\{\underline{n}\in W:\delta(\underline{n})=k\}} P\left(X_{\underline{n}}^2 \geq t\right) dt \\
&\leq C \sum_{k=1}^{\infty} \frac{1}{\phi^2(k)} \tau_{W,\delta}(k) \int_0^{\phi^2(k)} P\left(t \leq Y^2 < \phi^2(k)\right) dt \\
&\quad + C \sum_{k=1}^{\infty} \frac{1}{\phi^2(k)} \tau_{W,\delta}(k) \int_0^{\phi^2(k)} P\left(\phi^2(k) \leq Y^2\right) dt
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{k=1}^{\infty}\frac{1}{\phi^2(k)}\tau_{W,\delta}(k)\int_0^{\phi^2(k)} P\left(t \le Y^2 < \phi^2(k)\right)dt \\
&\quad + C\sum_{k=1}^{\infty}\tau_{W,\delta}(k)P\left(\phi^2(k) \le Y^2\right)dt.
\end{aligned} \tag{2.12}
$$

Now

$$
\begin{aligned}
\sum_{k=1}^{\infty}\tau_{W,\delta}(k)P\left(\phi^2(k) \le Y^2\right) &\le \sum_{m=1}^{\infty}P\left(\phi^2(m) \le Y^2 \le \phi^2(m+1)\right)\sum_{k=1}^{m}\tau_{W,\delta}(k) \\
&= \sum_{m=1}^{\infty}T_{W,\delta}(m)P\left(\phi^2(m) \le Y^2 \le \phi^2(m+1)\right),
\end{aligned} \tag{2.13}
$$

which is equivalent to $ET_{W,\delta}(\phi^{-1}(|Y|)) < \infty$ since the function $T_{W,\delta}$ satisfies (2.1). It is clear that $\int_0^{\phi^2(k)} P(t \le Y^2 < \phi^2(k))dt = EY^2 I_{\{|Y| \le \phi(k)\}}$, and from the relation (A2′) we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{\phi^2(k)}\tau_{W,\delta}(k)\int_0^{\phi^2(k)} P\left(t \le Y^2 < \phi^2(k)\right)dt \\
&\quad = \sum_{k=1}^{\infty}\frac{1}{\phi^2(k)}\tau_{W,\delta}(k)EY^2 I_{\{|Y|<\phi(k)\}} \\
&\quad = \sum_{m=1}^{\infty}EY^2 I_{\{\phi(m-1)\le|Y|<\phi(m)\}}\sum_{k=m}^{\infty}\frac{\tau_{W,\delta}(k)}{\phi^2(k)} \\
&\quad \le a\sum_{m=1}^{\infty}EY^2 I_{\{\phi(m-1)\le|Y|<\phi(m)\}}\frac{T_{W,\delta}(m)}{\phi^2(m)} \\
&\quad \le a\sum_{m=1}^{\infty}T_{W,\delta}(m)P(\phi(m-1) \le |Y| < \phi(m)),
\end{aligned} \tag{2.14}
$$

which again is equivalent to $ET_{W,\delta}(\phi^{-1}(|Y|)) < \infty$ by (2.1). Therefore, the first part of the theorem is proved. In order to prove the second part of the theorem, let us observe that

$$
\begin{aligned}
\frac{1}{g(\delta(\underline{n}))}\sum_{W\ni\underline{k}\le\underline{n}}\frac{X_{\underline{k}} - EX_{\underline{k}}}{h(\delta(\underline{k}))} &= \frac{1}{g(\delta(\underline{n}))}\sum_{W\ni\underline{k}\le\underline{n}}\frac{Y_{\underline{k}} - EY_{\underline{k}}}{h(\delta(\underline{k}))} - \frac{1}{g(\delta(\underline{n}))}\sum_{W\ni\underline{k}\le\underline{n}}\frac{EX_{\underline{k}} - EY_{\underline{k}}}{h(\delta(\underline{k}))} \\
&\quad + \frac{1}{g(\delta(\underline{n}))}\sum_{W\ni\underline{k}\le\underline{n}}\frac{X_{\underline{k}} - Y_{\underline{k}}}{h(\delta(\underline{k}))}.
\end{aligned} \tag{2.15}
$$

The first summand converges to 0 by the first part of the theorem applied to the random variables $Y_{\underline{k}}$ defined by (2.4). The convergence of the third one follows from the Borel-Cantelli lemma since

$$\sum_{\underline{n}\in W} P\left(X_{\underline{n}} \neq Y_{\underline{n}}\right) \leq \sum_{k=1}^{\infty} T_{W,\delta}(k) P\left(\phi(k-1) \leq |Y| < \phi(k)\right) < \infty. \tag{2.16}$$

It remains to prove that the second summand converges to 0. In order to prove this let us put $Z_{\underline{n}} = X_{\underline{n}} - Y_{\underline{n}}$, and we will prove that the series

$$\sum_{\underline{n}\in W} E\left(\frac{Z_{\underline{n}}}{\phi(\delta(\underline{n}))}\right) \tag{2.17}$$

is absolutely convergent. We have

$$\begin{aligned}
\sum_{\underline{n}\in W}\left|\frac{EZ_{\underline{n}}}{\phi(\delta(\underline{n}))}\right| &\leq \sum_{\underline{n}\in W}\frac{1}{\phi(\delta(\underline{n}))}\int_0^{\infty} P\left(\left|Z_{\underline{n}}\right| \geq t\right)dt \\
&= \sum_{\underline{n}\in W}\frac{1}{\phi(\delta(\underline{n}))}\int_{\phi(\delta(\underline{n}))}^{\infty} P\left(\left|X_{\underline{n}}\right| \geq t\right)dt \\
&\quad + \sum_{\underline{n}\in W}\frac{1}{\phi(\delta(\underline{n}))}\phi(\delta(\underline{n}))P\left(\left|X_{\underline{n}}\right| \geq \phi(\delta(\underline{n}))\right) \\
&= \sum_{m=1}^{\infty}\frac{1}{\phi(m)}\sum_{\{\underline{n}\in W:\delta(\underline{n})=m\}}\int_{\phi(m)}^{\infty} P\left(\left|X_{\underline{n}}\right| \geq t\right)dt \\
&\quad + \sum_{m=1}^{\infty}\frac{1}{\phi(m)}\sum_{\{\underline{n}\in W:\delta(\underline{n})=m\}}\phi(\delta(\underline{n}))P\left(\left|X_{\underline{n}}\right| \geq \phi(\delta(\underline{n}))\right) \\
&\leq C\sum_{m=1}^{\infty}\frac{1}{\phi(m)}\tau_{W,\delta}(m)\int_{\phi(m)}^{\infty} P(|Y| \geq t)dt \\
&\quad + C\sum_{m=1}^{\infty}\tau_{W,\delta}(m)P\left(|Y| \geq \phi(m)\right).
\end{aligned} \tag{2.18}$$

Now, since $ET_{W,\delta}(\phi^{-1}(|Y|)) < \infty$, the second series on the r.h.s. of the above inequality is convergent by the same argument as in the proof of the first part, whereas for the second series it is true that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{1}{\phi(m)}\tau_{W,\delta}(m)\int_{\phi(m)}^{\infty} P(|Y| \geq t)dt \\
&\quad = \sum_{m=1}^{\infty}\frac{\tau_{W,\delta}(m)}{\phi(m)}\sum_{k=m}^{\infty}\int_{\phi(k)}^{\phi(k+1)} P(|Y| \geq t)dt
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} \int_{\phi(k)}^{\phi(k+1)} P(|Y| \geq t)dt \sum_{m=1}^{k} \frac{\tau_{W,\delta}(m)}{\phi(m)} \\
&\leq a \sum_{k=1}^{\infty} \frac{T_{W,\delta}(k)}{\phi(k)} \int_{\phi(k)}^{\phi(k+1)} P(|Y| \geq t)dt \\
&\leq a \sum_{k=1}^{\infty} T_{W,\delta}(k) P\big(|Y| \geq \phi(k)\big),
\end{aligned} \tag{2.19}$$

which is convergent again by $ET_{W,\delta}(\phi^{-1}(|Y|)) < \infty$. □

Under the assumption that the random variables are identically distributed, one may obtain a necessary and sufficient condition for the WSLLN.

Theorem 2.3. *Let* $(X_{\underline{n}})_{\underline{n} \in \mathbb{N}^d}$ *be a field of i.i.d. random variables with the same distribution as the random variable* Y. *Moreover, let* (A1), (A2′), *and* (2.1) *be satisfied. Then* (2.8) *is equivalent to* (2.9).

Proof. In view of Theorem 2.2, it suffices to prove the necessity of (2.8).

Similarly as in [4] we have

$$\lim_{W} \frac{m_{\underline{n}}}{\phi(\delta(\underline{n}))} = 0, \quad \text{almost surely,} \tag{2.20}$$

from the Lebesgue dominated convergence criterion. Now—again analogously to [4]—for $\underline{n} \in W$ write

$$\frac{X_{\underline{n}} - m_{\underline{n}}}{\phi(\delta(\underline{n}))} = \frac{1}{g(\delta(\underline{n}))} \left(\sum_{\underline{a} \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^{d} a_i} S_{\underline{n}-\underline{a}} \right), \tag{2.21}$$

where $S_{\underline{n}} = \sum_{\underline{k} \leq \underline{n}, \underline{k} \in W} ((X_{\underline{k}} - m_{\underline{k}})/h(\delta(\underline{k})))$ and $\underline{a} = (a_1, \ldots, a_d)$ with $a_i = 0$ or $a_i = 1$ for $i = 1, \ldots, d$. (To see this one should simply put $X_{\underline{n}} = 0$ for $\underline{n} \notin W$ and apply a well-known summation technique to the whole $\mathbb{N}^d$.) From the relations (2.21) and (2.9), we easily obtain that

$$\lim_{W} \frac{X_{\underline{n}} - m_{\underline{n}}}{\phi(\delta(\underline{n}))} = 0, \quad \text{almost surely,} \tag{2.22}$$

for details see [4, page 20]. Thus, from the above and (2.20), we obtain the necessity of condition (2.9) via the Borel-Cantelli lemma. □

Remark 2.4. In both of the above theorems, we do not need to assume that the function $T_{W,\delta}$ is regularly varying in the sense of (2.1) nor that the function ϕ is invertible (we may omit (A1)). If we omit both of these assumptions, then, instead of the condition $E(T_{W,\delta}(\phi^{-1}(|Y|))) < \infty$, one ought to write

$$\sum_{k=1}^{\infty} T_{W,\delta}(k) P\big(\phi(k-1) \leq |Y| < \phi(k)\big) < \infty, \tag{2.23}$$

and, instead of $E(\phi^{-1}(|Y|)) < \infty$,

$$\sum_{k=1}^{\infty} P\big(\phi(k-1) \le |Y| < \phi(k)\big) < \infty, \tag{2.24}$$

with some minor technical changes in the proofs.

3. WSLLN in the Case $d = 2$

In this section we aim to extend the results of the papers [1, 2], which are "sectorial SLLN" for fields of i.i.d. random variables in the case $d = 2$. In this section we will study the weighted strong law of large numbers. Here we will use a less general definition of a sector than in the previous section. In this case it is possible to obtain the necessary and sufficient conditions for a stronger form of the WSLLN; in other words, we will set

$$S(\underline{n}) = \sum_{\underline{k} \le \underline{n}} \frac{X_{\underline{k}} - E(X_{\underline{k}})}{h(|\underline{k}|)} \tag{3.1}$$

and consider the almost sure convergence

$$\lim_{A_{f,F}} \frac{S(\underline{n})}{g(|\underline{n}|)}. \tag{3.2}$$

Let us note that the main difference lies in the set over which we sum up the random variables. In the former section we were performing the summation only over the indices which belong to the sector, and now we sum up over all indices in $\mathbb{N}^d$ and the difference between the present setting and the classical SLLN for random fields is in the very definition of convergence. In order to be able to prove our main results, we have to adopt some techniques form [2] to the setting of the WSLLN, but, for the sake of brevity, we will not include all the justifications here, and instead we refer to proper lines of the proofs in [2] or in [1]. In what follows, we will assume that the sector $A_{f,F}$ satisfies the conditions:

$$f \text{ is increasing,} \tag{3.3}$$

$$f(x) \le x \le F(x), \quad f(1) \le x < F(1), \tag{3.4}$$

$$\frac{f(x)}{x} \text{ is nonincreasing,} \quad \frac{F(x)}{x} \text{ is nondecreasing.} \tag{3.5}$$

These conditions were originally introduced in [1]. Let us observe that from (3.5) it follows that $f(x)/x \le f(1)$ and $F(x)/x \ge F(1)$, for $x \ge 1$. Thus, according to (3.4), $f(x) \le xf(1) < xF(1) \le F(x)$. From this inequality it follows that $A_\theta \subset A_{f,F}$, that is, that the nonlinear sector $A_{f,F}$ contains a standard sector A_θ, where $\theta = \max\{f(1); 1/F(1)\}$. Moreover, we assume that the functions g, h satisfy the assumption (A1) and (A2'). Since we are dealing with a different problem now, then we have to use a modification of the definition of weakly mean domination on the average.

Let $A_{f,F}$ be an arbitrary, nonempty sector, defined by the functions f and F. Let us divide the set $\mathbb{N}^2$ into three parts:

$$\begin{gathered} A_{f,F}, \\ A_f := \left\{ \underline{n} \in \mathbb{N}^2 : 1 \le n_i < f(n_2) \right\}, \\ A_F := \left\{ \underline{n} \in \mathbb{N}^2 : n_2 > F(n_1) \right\}. \end{gathered} \tag{3.6}$$

A random field $(X_{\underline{n}})_{\underline{n} \in \mathbb{N}^d}$ is said to be weakly dominated on the average by the random variable Y if (2.6) holds for $A = A_{f,F}, A_f$ and A_F.

Let us now state the main result of this section.

Theorem 3.1. *Let the functions f, F, the sector $A_{f,F}$, and the function $T_{f,F}$ be as above and satisfy* (2.1)*,* (3.3)–(3.5)*. Moreover, let the functions g, h satisfy the conditions* (A1) *and* (A2')*. If a field $(X_{\underline{n}})_{\underline{n} \in \mathbb{N}^2}$ of independent random variables is weakly dominated on the average by the random variable Y and $E(T_{f,F}(\phi^{-1}(|Y|))) < \infty$, then*

$$\lim_{A_{f,F}} \frac{S_{\underline{n}}}{g(|\underline{n}|)} = 0, \quad \textit{almost surely}. \tag{3.7}$$

Assume additionally that $\phi(n)/n \to 0$ as $n \to \infty$ and the function ϕ satisfies the condition $(A3)$. If $E(T_{f,F}(\phi^{-1}(|Y|))) < \infty$, then

$$\lim_{A_{f,F}} \frac{1}{g(|\underline{n}|)} \sum_{\underline{k} \le \underline{n}} \frac{X_{\underline{k}} - E(X_{\underline{k}})}{h(|\underline{k}|)} = 0, \quad \textit{almost surely}. \tag{3.8}$$

Proof. Let us begin with the justification of the first assertion. Assume that $ET_{f,F}(\phi^{-1}(|Y|)) < \infty$, then $E\phi^{-1}(|Y|) < \infty$. Let us consider the truncated random variables $(Y_{\underline{n}})_{\underline{n} \in \mathbb{N}^2}$ as defined in (2.3). From the Borel-Cantelli lemma, we have that, since $E\phi^{-1}(|Y|) < \infty$, then

$$P\left(|X_{\underline{n}} - Y_{\underline{n}}| \neq 0 \text{ i.o. } \underline{n} \in \mathbb{N}^2 \right) = 0, \tag{3.9}$$

and this of course means that it suffices to prove that

$$\lim_{A_{f,F}} \frac{1}{g(|\underline{n}|)} \sum_{\{\underline{n} \in \mathbb{N}^d : \underline{n} \le \underline{k}\}} \frac{Y_{\underline{n}} - m_{\underline{n}}}{h(|\underline{n}|)} = 0, \quad \text{almost surely}. \tag{3.10}$$

Now let us divide the partial sums into three terms as in [2, 12]; in other, words let us consider

$$\begin{gathered} A_f \cap \left\{ \underline{n} \in \mathbb{N}^d : \underline{n} \le \underline{k} \right\}, \qquad A_F \cap \left\{ \underline{n} \in \mathbb{N}^d : \underline{n} \le \underline{k} \right\}, \\ A_{f,F} \cap \left\{ \underline{n} \in \mathbb{N}^d : \underline{n} \le \underline{k} \right\}, \end{gathered} \tag{3.11}$$

where A_f and A_F are defined in (3.6). It is clear (see [2, 12] for details) that the families

$$\left(A_f \cap \left\{\underline{n} \in \mathbb{N}^d : \underline{n} \leq \underline{k}\right\}\right)_{\underline{k} \in \mathbb{N}^d}, \quad \left(A_F \cap \left\{\underline{n} \in \mathbb{N}^d : \underline{n} \leq \underline{k}\right\}\right)_{\underline{k} \in \mathbb{N}^d} \tag{3.12}$$

are linearly ordered by inclusion, thus may be enumerated in an ascending order. We also write

$$S_B(\underline{k}) := \sum_{B \cap \left\{\underline{n} \in \mathbb{N}^d : \underline{n} \leq \underline{k}\right\}} \frac{Y_{\underline{k}} - m_{\underline{n}}}{h(|\underline{n}|)}, \tag{3.13}$$

for $B \subset \mathbb{N}^2$. By the above remark the partial sums $S_{A_f}(\underline{k}),\ S_{A_F}(\underline{k})$ may be seen as subsequences of cumulative sums of weakly dominated random variables. Thus, we may use the sufficient condition for the the Feller-Jajte WSLLN for weakly dominated random variables (see Section 2) to conclude that the condition $E\phi^{-1}(|Y|) < \infty$ is sufficient for $S_{A_f}(\underline{k}) \rightarrow 0$ *a.s.* and $S_{A_F}(\underline{k}) \rightarrow 0$ *a.s.* Therefore, it remains to prove that $S_{A_{f,F}}(\underline{k}) \rightarrow 0$ *a.s.*, which by the results of Klesov [12, 13] is implied by the fact that

$$\sum_{A_{f,F} \cap \left\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\right\}} P\left(\left|Y_{\underline{n}}\right| > \phi(|\underline{n}|)\right) < \infty, \qquad \sum_{A_{f,F} \cap \left\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\right\}} E\left(\frac{Y_{\underline{n}} - m_{\underline{n}}}{\phi(|\underline{n}|)}\right) < \infty,$$
$$\sum_{A_{f,F} \cap \left\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\right\}} \operatorname{Var}\left(\frac{Y_{\underline{n}} - m_{\underline{n}}}{\phi(|\underline{n}|)}\right) < \infty. \tag{3.14}$$

The first summand is bounded by $ET_{f,F}(\phi^{-1}(|Y|)) < \infty$; the second is bounded since $EY_{\underline{n}} = m_{\underline{n}}$. The fact that, under the above assumptions on the functions g and h, the convergence of the last summand follows from the assumption $ET_{f,F}(\phi^{-1}(|Y|)) < \infty$ follows by the same lines as in the proofs from the preceding section.

The proof of the second assertion of the theorem is much more the same as in the proof of Theorem 2.2 from the former section. Let us first consider the division of the partial sums of the random field $(X_{\underline{n}})_{\underline{n} \in \mathbb{N}^d}$ (weakly dominated on the average by the random variable Y) into three terms as given in (3.21). Let us notice that by applying the beforehand defined truncation technique we obtain again that

$$\begin{aligned} S_B(\underline{k}) &= \sum_{B \cap \left\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\right\}} \frac{X_{\underline{n}} - EX_{\underline{n}}}{h(|\underline{n}|)} \\ &= \sum_{B \cap \left\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\right\}} \frac{Y_{\underline{n}} - EY_{\underline{n}}}{h(|\underline{n}|)} - \sum_{B \cap \left\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\right\}} \frac{EX_{\underline{n}} - EY_{\underline{n}}}{h(|\underline{n}|)} + \sum_{B \cap \left\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\right\}} \frac{X_{\underline{n}} - Y_{\underline{n}}}{h(|\underline{n}|)}, \end{aligned} \tag{3.15}$$

where $B = A_{f,F}, A_f, A_F$. We will only give the proof for $B = A_{f,F}$, remembering that for $B = A_f, A_F S_B(\underline{k})$ may be seen as a subsequence of a sequence of weakly mean dominated (see [10]) random variables (indexed by natural numbers), and; therefore, the same techniques may be used to prove the convergence of $S_{A_f}(\underline{k})$ and $S_{A_F}(\underline{k})$ to zero (the same as for the convergence of $S_{A_{f,F}}(\underline{k})$) with the change that $d = 1$. Now from the Borel-Cantelli lemma we infer that, since

$$\sum_{\underline{n}\in A_{f,F}} P(X_{\underline{n}} \neq Y_{\underline{n}}) \leq \sum_{k=1}^{\infty} T_{f,F}(k) P(\phi(k-1) \leq |Y| < \phi(k)) < \infty, \tag{3.16}$$

then

$$\frac{1}{g(|\underline{n}|)} \sum_{A_{f,F}\cap\{\underline{n}\in\mathbb{N}^2 : \underline{n}\leq\underline{k}\}} \frac{X_{\underline{n}} - Y_{\underline{n}}}{h(|\underline{n}|)} \longrightarrow 0, \quad \text{almost surely.} \tag{3.17}$$

Now, the almost sure convergence to zero of the first term on the r.h.s. of (3.15) follows by the same lines as in the first part of the proof. Therefore, it remains to prove that

$$\lim_{\underline{k}\to\infty} \frac{1}{g(|\underline{k}|)} \sum_{A_{f,F}\cap\{\underline{n}\in\mathbb{N}^2 : \underline{n}\leq\underline{k}\}} \frac{EX_{\underline{n}} - EY_{\underline{n}}}{h(|\underline{n}|)} = 0, \tag{3.18}$$

which follows from the proof of the Theorem 2.2 in the preceding section. □

As before, below we show that under the assumption that the random variables are i.i.d. the sufficient conditions in the above proofs become necessary.

Theorem 3.2. *For f, F, the sector $A_{f,F}$, the function $T_{f,F}$, the functions g, h as above, and the field $(X_{\underline{n}})_{\underline{n}\in\mathbb{N}^2}$ of i.i.d. random variables, the following conditions are equivalent:*

(1) $\lim_{A_{f,F}}(S_{\underline{n}}/g(|\underline{n}|)) = 0$, *almost surely,*

(2) $E(T_{f,F}(\phi^{-1}(|X_{\underline{1}}|))) < \infty$.

Proof. As obviously i.i.d. random variables are weakly dominated on the average for each sector in $\mathbb{N}^2$, then only the necessity needs justification. Let us, therefore, assume that $\lim_{A_{f,F}}(S_{\underline{n}}/g(|\underline{n}|)) = 0$, a.s. From this it immediately follows that

$$\lim_{k\to\infty} \frac{1}{g(k^2)} \sum_{\underline{n}\leq\overline{k}} \frac{X_{\underline{n}} - m_{\underline{n}}}{h(|\underline{n}|)} = 0 \quad \text{a.s.,} \tag{3.19}$$

where

$$\mathbb{N}^2 \ni \overline{k} = (k, k). \tag{3.20}$$

Therefore, from the Feller-Jajte WSLLN (see [5, 6]), we infer that $E\phi^{-1}(|X_{\underline{1}}|) < \infty$. As in (3.6) divide the set $\{\underline{n} \in \mathbb{N}^2 : \underline{n} \leq \underline{k}\}$ into three subsets $A_{f,F}$, A_f, A_F and—as before—write

$$S(\underline{k}) = \sum_{\{\underline{n}\in\mathbb{N}^2:\underline{n}\leq\underline{k}\}} \frac{X_{\underline{n}} - m_{\underline{n}}}{h(|\underline{n}|)} = S_{A_{f,F}}(\underline{n}) + S_{A_f}(\underline{k}) + S_{A_F}(\underline{k}). \tag{3.21}$$

Since, as we have noted, the last two sums may be seen as subsequences of a sequence of i.i.d. random variables with the same distribution as $X_{\underline{1}}$, then from the fact that $E\phi^{-1}(|X_{\underline{1}}|) < \infty$ we obtain $S_{A_f}(\underline{k}) \to 0$ and $S_{A_F}(\underline{k}) \to 0$ as $\underline{k} \to \infty$; thus, $S_{A_{f,F}}(\underline{k}) \to 0$ as $\underline{k} \to \infty$. As in the proof of Theorem 2.3 in the preceding section we have $m_{\underline{n}}/\phi(|\underline{n}|) \to 0$ as $\underline{k} \to \infty$, and in turn this implies that

$$P\left(\frac{|X_{\underline{n}}|}{h(|\underline{n}|)} > g(|\underline{n}|) \text{ i.o. for } \underline{n} \in A_{f,F}\right) = 0, \tag{3.22}$$

(for details see [4]). Since the $X_{\underline{n}}'s$ are independent, then the Borel-Cantelli lemma implies that $ET_{f,F}(\phi^{-1}(|X_{\underline{1}}|)) < \infty$ and the necessary part of the Theorem is proved. □

4. Examples

The aim of the present section is to show some examples of functions Φ for which the condition (A2′) is satisfied. In the case $d = 2$ and for $\Phi(x) = x$, Klesov and Rychlik proved that (see [1, Lemma 2])

$$\sum_{k=n}^{\infty} \frac{\tau_{f,F}(k)}{k^2} \leq C\frac{T_{f,F}(n)}{n^2}, \tag{4.1}$$

where the sector $A_{f,F}$ is defined by the functions f and F such that

$$f(x) \leq x \leq F(x), \quad \frac{f(x)}{x} \text{ is nonincreasing}, \quad \frac{F(x)}{x} \text{ is nondecreasing}. \tag{4.2}$$

We will extend this result to more general classes of functions Φ satisfying some additional technical conditions. We will apply the theory of regularly varying functions (we refer the reader to [14] for details), and in order to use integrals and sums interchangeably we impose the following conditions:

$$\Phi \text{ is positive, nondecreasing on } [1,\infty], \tag{4.3}$$

$$\frac{\Phi(k+1)}{\Phi(k)} \leq C_1, \quad \text{for some } C_1 > 0 \text{ and every } k \in N. \tag{4.4}$$

Furthermore, let Φ be differentiable with the derivative such that

$$\frac{\Phi'(y)}{\Phi'(x)} \le C_2, \quad \text{for some } C_2 > 0 \text{ and every } 1 \le x \le y \le x+1,$$
$$\frac{k\Phi'(k)}{\Phi(k)} \le C_3, \quad \text{for some } C_3 > 0 \text{ and every } k \in N. \tag{4.5}$$

Proposition 4.1. *Let $d = 2$ and the sector $A_{f,F}$ be defined by the functions f and F satisfying* (4.2). *Assume that the function Φ is regularly varying with index $\delta > 1/2$ and the conditions* (4.3)–(4.5) *are satisfied. Then*

$$\sum_{k=n}^{\infty} \frac{\tau_{f,F}(k)}{\Phi^2(k)} \le C \frac{T_{f,F}(n)}{\Phi^2(n)}, \quad \textit{for some } C > 0 \textit{ and every } n \in \mathbb{N}. \tag{4.6}$$

Proof. We will make use of the Abel transform:

$$\sum_{k=n}^{\infty} \frac{\tau_{f,F}(k)}{\Phi^2(k)} = \lim_{N\to\infty} \left(\frac{T_{f,F}(N)}{\Phi^2(N+1)} - \frac{T_{f,F}(n-1)}{\Phi^2(n-1)} + \sum_{k=n-1}^{N} T_{f,F}(k) \left(\frac{1}{\Phi^2(k)} - \frac{1}{\Phi^2(k+1)} \right) \right). \tag{4.7}$$

From Lemma 4 of [1], we have

$$c_1 m l_*(m) \le T_{f,F}(m) \le c_2 m l_*(m), \tag{4.8}$$

where $l_*(m) = \log \sqrt{m} - \log x_m$ and $l_*(t) = l_*([t])$ is a slowly varying function (even belonging to the Zygmund class). Since Φ is regularly varying with index δ, we have

$$\Phi(N) = N^{\delta} l(N), \tag{4.9}$$

for some slowly varying function l. Thus, from (4.8) and (4.9) we get

$$\frac{T_{f,F}(N)}{\Phi^2(N+1)} \le \frac{T_{f,F}(N)}{\Phi^2(N)} \le \frac{c_2 N l_*(N)}{N^{2\delta} l^2(N)} \longrightarrow 0, \tag{4.10}$$

since $2\delta > 1$ and $l_*(N)/l(N)$ is slowly varying. By mean value theorem and (4.4)–(4.5), we get

$$\sum_{k=n-1}^{N} T_{f,F}(k) \left(\frac{1}{\Phi^2(k)} - \frac{1}{\Phi^2(k+1)} \right) \le C \sum_{k=n-1}^{\infty} T_{f,F}(k) \frac{\Phi'(\theta_k)}{\Phi^3(\theta_k)}$$
$$\le C \sum_{k=n-1}^{\infty} \frac{l_*(k)}{\Phi^2(k)}, \tag{4.11}$$

where $\theta_k \in (k, k+1)$. Let us note that the function $f(t) = l_*(t)/\Phi^2(t)$ is regularly varying with index $-2\delta < -1$; therefore, by Theorem 1.5.11 in [14] with $\sigma = 0$, we get

$$\frac{yf(y)}{\int_y^\infty f(y)} \longrightarrow 2\delta - 1, \quad \text{as } y \longrightarrow \infty. \tag{4.12}$$

In consequence, for some $C > 0$,

$$\int_y^\infty \frac{l_*(t)}{\Phi^2(t)} dt \le Cy \frac{l_*(y)}{\Phi^2(y)}, \tag{4.13}$$

from this it follows that

$$\begin{aligned} \sum_{k=n-1}^{\infty} \frac{l_*(k)}{\Phi^2(k)} &\le C \frac{(n-1)l_*(n-1)}{\Phi^2(n-1)} \le \frac{C}{c_1} \frac{T_{f,F}(n-1)}{\Phi^2(n-1)} \\ &\le \frac{C}{c_1 C_1} \frac{T_{f,F}(n)}{\Phi^2(n)}. \end{aligned} \tag{4.14}$$

Now, from (4.7) and (4.10), the conclusion follows. □

The above proof was essentially based on the inequality (4.8) and may be repeated in higher dimensions for any sector with such asymptotics for $T_{f,F}(n)$. This is the case for nonlinear sectors such that $A_{\theta_1} \subset A_{f,F} \subset A_{\theta_2}$, for some θ_1 and θ_2. Therefore, we may state the following proposition (in view of the remark following (3.5), the first inclusion holds automatically).

Proposition 4.2. *Let $A_{f,F}$ be any sector in $\mathbb{N}^d$, $d \ge 2$ such that $A_{\theta_1} \subset A_{f,F} \subset A_{\theta_2}$, for some θ_1 and θ_2. Assume that Φ is a regularly varying function with index $\delta > 1/2$ satisfying the conditions* (4.3)–(4.5). *Then* (4.6) *holds.*

Remark 4.3. It is well known (see [14]) that a function $\Phi(x)$ on $[1, \infty)$, regularly varying with index δ, may be represented in the form $\Phi(x) = x^\delta l(x)$, where $l(x)$ is slowly varying and admits the representation:

$$l(x) = c(x) \exp\left(\int_1^x \frac{\varepsilon(u)}{u} du \right), \tag{4.15}$$

where $c(x) \to c > 0$ and $\varepsilon(u) \to 0$ as $u \to \infty$. To prove (4.6) it suffices to consider the case $c(x) \equiv c$, and it is easy to see that (4.3)–(4.5) are satisfied if $\varepsilon(u)$ is positive, continuous, and nonincreasing function which tends to 0 as $u \to \infty$.

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