Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 179724, 11 pages
doi:10.1155/2009/179724

Research Article

PD Control for Vibration Attenuation in a Physical Pendulum with Moving Mass

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Received 8 December 2008; Accepted 22 May 2009

Recommended by John Burns

This paper proposes a Proportional Derivative controller plus gravity compensation to damp out the oscillations of a frictionless physical pendulum with moving mass. A mass slides along the pendulum main axis and operates as an active vibration-damping element. The Lyapunov method together with the LaSalle's theorem allows concluding closed-loop asymptotic stability. The proposed approach only uses measurements of the moving mass position and velocity and it does not require synchronization of the pendulum and moving mass movements. Numerical simulations assess the performance of the closed-loop system.

1. Introduction

Vibrating mechanical systems are an important class of dynamic systems including buildings, bridges, car suspensions, pacemakers, wind generators, and hi-fi speakers [1]. In physical terms, the behavior of a vibrating system is describe by the interplay between an energy-storing component and an energy-carrying component. Thus, the system dynamics are described in terms of energy changes, that is, the motion of the system results from an energy exchange. The control of vibrating mechanical systems is an important area of research, which has provided technological solutions to several problems concerning oscillatory behaviors of some important classes of dynamic systems. For instance, active control of vibrations allows attenuating undesired oscillations in buildings affected by external forces such as strong winds and earthquakes (see, e.g., [2–9] and the references therein), and computer-based active suspension systems are now common in cars as a mean to improve road handling.

As far as mathematical tools are concerned, the control of vibrations has mainly been tackled via frequency-domain techniques, which are essentially restricted to linear systems

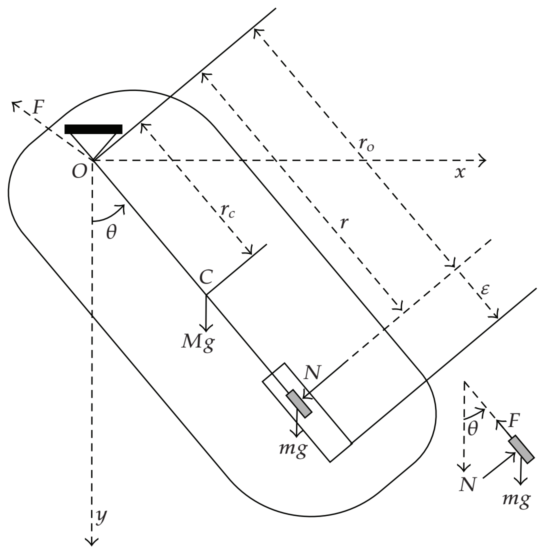


Figure 1: Physical pendulum with moving mass.

(see, e.g., [10, 11]). If the vibrating systems are nonlinear and if they oscillate too far away from their equilibrium points, then, frequency-domain techniques are not suitable. In the case of nonlinear systems characterized by small domains of attraction around their equilibrium points, the linear approach is not very effective. Hence, modern approaches, employing time-domain nonlinear control strategies, would yield better performance.

This paper focuses on active control of a class of underdamped lumped nonlinear under-actuated vibrating mechanical systems following an energy-based approach, that is, the control of vibrations is tackled via the shaping of the energy flow that characterizes the system. The control of vibrations is considered in terms of the solution of a particular asymptotic stabilizing feedback control problem around a selected equilibrium point. A stabilizing controller is then obtained following an energy-based Lyapunov approach, which exploits the physical properties of the mechanical system. Intuitively speaking, the energy-based Lyapunov control shapes, using a feedback loop, the potential, and kinetic energies of the controlled system to ensure a motion guaranteeing the control objective (see, e.g., [8, 12, 13]). Moreover, this approach requires the total energy of the system to be a non-increasing function. The total energy function is also required to be at least locally positive definite around the selected equilibrium point (see, e.g., [6, 7, 9, 13]). In this way, the proposed approach avoids conservative control strategies based either on high gains or on canceling nonlinear terms [14, 15].

The problem tackled in this work is the stabilization of a frictionless under-actuated physical pendulum with a radially moving mass. This system was studied in [16], where several control strategies solved the aforementioned stabilization problem. The proposed stabilizing strategies include a modified nonlinear Proportional Derivative (PD) controller and a neural network approach. Further works also studied this physical system [17, 18]. The proposed approach in these two references synchronizes the movement of the moving mass with the pendulum oscillation to damp out the pendulum oscillation. It is interesting to point out that the aforementioned approaches are mainly heuristic and they do not provide rigorous stability proofs. Reference [19] provides a control algorithm based on a switching strategy. The model of the pendulum includes damping friction, and the proposed control law needs measurement of the pendulum angle and the moving mass position. It is also worth remarking that, if not properly tuned, the switching strategy could introduce unbounded control signal chattering.

This paper proposes a simple control law for damping out the oscillation of a frictionless pendulum through the movement of a mass sliding along the pendulum main axis. The control law is composed of two parts, a linear PD controller corresponds to the first part, and the second part is a constant term, that is, equal to the gravity force term associated to the moving mass. Compared with previous approaches, the proposed control law only needs measurements of the moving mass position and velocity and does not relies on the synchronization of the pendulum oscillation. Moreover, it is simple and it could be implemented using processors with limited computing capabilities.

The contribution is organized as follows: Section 2 presents the model of the physical pendulum with moving mass, as well as its main physical properties. Section 3 describes the proposed control law and the stability analysis of the closed loop system. Section 4 depicts some computer simulations. The paper ends with some final comments.

2. Physical Pendulum with Moving Mass

2.1. Lagrangian Modeling

Consider a mechanical system consisting of a physical frictionless pendulum of mass M with its pivot at O and an auxiliary mass m able to slide to and from the pivot as depicted in Figure 1. The moment of inertia of the pendulum about the pivot is given by I_0, and its center of mass C is located at a distance r_c from the pivot. The forces acting on the mass m are the gravitational force mg and a force F parallel to the guide $\overline{OC}$ and supplied by an actuator, that is, an electric motor, attached to the auxiliary mass. The pivot O is the origin of the reference frame $x - y$. The x-axis is set in the horizontal direction and the y-axis is set in the vertical direction. The set of generalized coordinates are the angle θ between $\overline{OC}$ and the y-axis, and the radial displacement r of the mass m from the pivot O. It is easy to show that the total kinetic energy K_c and the total potential energy K_p for this system are given by

$$\begin{aligned} K_c &= \frac{1}{2}I_0\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2, \\ K_p &= -Mgr_c\cos\theta - mgr\cos\theta, \end{aligned} \tag{2.1}$$

respectively. Note that the total kinetic energy comprises the rotational energy of the pendulum as well as the translational and rotational energy of the sliding mass. The above equations allow writing the Lagrangian function

$$L(q,\dot{q}) = K_c - K_{p,} \tag{2.2}$$

where $q := [r,\theta]^T$. From the above, the corresponding Euler-Lagrange equations are given by

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta &= F, \\ \left(mr^2 + I_0\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + g(Mr_c + mr)\sin\theta &= 0. \end{aligned} \tag{2.3}$$

2.2. Model Properties

Define the force F as

$$F = v - mg, \tag{2.4}$$

where variable v is a new input, and mg is a gravity compensation term. Substituting (2.4) into (2.3) leads to the following Euler-Lagrange system

$$M(q)\ddot{q} + C(q,q)\dot{q} + \nabla_q K_i(q) = Gv, \tag{2.5}$$

where $G := [1,0]^T$ and $K_i(q) := -Mgr_c\cos\theta + mgr(1-\cos\theta)$ and

$$M(q) = \begin{bmatrix} m & 0 \\ 0 & mr^2 + I_0 \end{bmatrix}, \qquad C(q,\dot{q}) = \begin{bmatrix} 0 & -mr\dot{\theta} \\ mr\dot{\theta} & mr\dot{r}. \end{bmatrix}. \tag{2.6}$$

System (2.5) satisfies the following properties:

(**P1**) $M(q)$ is positive definite.

(**P2**) $H := \dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric with

$$H = \begin{bmatrix} 0 & -mr\dot{\theta} \\ mr\dot{\theta} & 0 \end{bmatrix}. \tag{2.7}$$

(**P3**) The operator $v \longrightarrow \dot{r}$ is passive.

Properties (**P1**) and (**P2**) are shared by any Euler-Lagrange mechanical system. In order to prove property (**P3**), define the following storage function:

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + K_i(q). \tag{2.8}$$

Taking the time derivative of (2.8) and using properties (**P1**) and (**P2**) yields

$$\dot{E} = v\dot{r}. \tag{2.9}$$

According to standard results [20, page 236] the operator $v \longrightarrow \dot{r}$ is passive. Finally, the following remark concerns the local controllability of (2.5).

Remark 2.1. Define $x = [q,\dot{q}]^T = [r,\theta,\dot{r},\dot{\theta}]^T$. Then, linearization of system (2.5) around $\overline{x} = [\overline{r},0,0,0]^T$, $\overline{r} > 0$ produces

$$\begin{gathered} m\ddot{r} = v, \\ \left(m\overline{r}^2 + I_0\right)\ddot{\theta} + g(Mr_c + m\overline{r})\theta = 0. \end{gathered} \tag{2.10}$$

From the above, it is clear that (2.10) is not locally controllable since there is no way to affect the dynamics of θ. Further, $\overline{x}$ is a stable equilibrium point of (2.5) if $v = 0$ and $-\pi/2 < \theta < \pi/2$.

3. The Control Law

Before establishing the control objective of this work, we define the admissible set $Q \subset R^2$ as

$$Q = \left\{ q = [r, \theta]^T : 0 < \overline{r} - \varepsilon < r < \overline{r} + \varepsilon,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ \varepsilon > 0 \right\}. \tag{3.1}$$

The control objective is defined as follows.

Problem 3.1. Consider the physical pendulum with moving mass described in (2.5), under the assumption that the initial conditions satisfy $q(0) \in Q - \overline{q}$. Then, the control objective is to bring asymptotically the rotating pendulum with moving mass to the equilibrium point $\overline{\mathrm{x}} = [\overline{r}, 0, 0, 0]^T$, while $q = [r, \theta]^T \in Q$.

To solve the aforementioned control problem, define the following Lyapunov function candidate:

$$E_T(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + K_m(q), \tag{3.2}$$

where $K_m(q)$ is the modified potential energy stated as

$$K_m(q) = \frac{k_p}{2}(r - \overline{r})^2 + K_i(q) + Mgr_c \tag{3.3}$$

with $k_p > 0$.

Remark 3.2. The selected potential energy $K_m(q)$ has a minimum at $\overline{q} = [\overline{r}, 0]^T$ since

$$K_m(\overline{q}) = 0, \qquad \nabla_q K_m(q)_{q=\overline{q}} = 0, \qquad \nabla_q^2 K_m(q)_{q=\overline{q}} = \begin{bmatrix} k_p & 0 \\ 0 & Mgr_c + mg\overline{r} \end{bmatrix} > 0. \tag{3.4}$$

As a matter of fact, the above condition implies that $K_m(q)$ is a convex function around $\overline{q}$. In geometrical terms, the level curves of $K_m(q)$ consist of a set of closed curves around $\overline{q}$. On the other hand, function $K_m(q)$ is positive definite as long as $-\pi/2 < \theta < \pi/2$.

Taking into account property (**P3**), the first time derivative of E_T along a trajectory of (2.5) is given by

$$\dot{E}_T(q, \dot{q}) = v\dot{r} + k_p(r - \overline{r})\dot{r}. \tag{3.5}$$

Define the control input v as a Proportional Derivative control law

$$v = -k_p(r - \overline{r}) - k_d\dot{r} \tag{3.6}$$

with $k_p > 0$, $k_d > 0$. Therefore, substituting control law (3.6) into (3.5) yields

$$\dot{E}_T(q, \dot{q}) = -k_d\dot{r}^2. \tag{3.7}$$

As a consequence, $\dot{E}_T \leq 0$. Thus, this condition establishes stability of the equilibrium point $\overline{x}$ in the Lyapunov sense. Moreover, it also shows that function $E_T(q, \dot{q})$ is not increasing so does q. Therefore, if $q(0) = [r(0), \theta(0)]^T \in Q$, then, $q = [r, \theta]^T \in Q$ as $t \to \infty$. On the other hand, since $E_T(q, \dot{q})$ is not increasing, then, $E_T(q, \dot{q}) \leq E_T(q(0), \dot{q}(0))$. The above result allows defining a compact invariant set Ω as follows:

$$\Omega = \left\{ x = [q, \dot{q}]^T : E_T(q, \dot{q}) \leq \overline{C} \right\}, \tag{3.8}$$

where $\overline{C} = E_T(q(0), \dot{q}(0))$. Therefore, if $x(0) \in \Omega$, then, $x \in \Omega$ as $t \to \infty$.

To end the stability proof, La Salle's Theorem [21] will allow concluding asymptotic stability. To this end, define the invariant set S as follows:

$$S = \left\{ [q, \dot{q}]^T \in \Omega : \dot{E}_T(q, \dot{q}) = 0 \right\}. \tag{3.9}$$

Clearly, in the set S, we have that $\dot{r} = 0$ and as a consequence $\ddot{r} = 0$ and $r = \underline{r}$, where $\underline{r} > 0$ is a constant. Thus, substituting these quantities into (2.5) leads to

$$\begin{aligned} -m\underline{r}\dot{\theta}^2 - mg(\cos\theta - 1) + k_p(\underline{r} - \overline{r}) &= 0, \\ \left(m\underline{r}^2 + I_0\right)\ddot{\theta} + g(Mr_c + m\underline{r})\sin\theta &= 0. \end{aligned} \tag{3.10}$$

The time derivative of the first equation in (3.10) yields

$$(2\underline{r}\ddot{\theta} - g\sin\theta)\dot{\theta} = 0. \tag{3.11}$$

Two cases must then be analyzed.

Case 1. If $\dot{\theta} = 0$ in the set S, it also follows that $\ddot{\theta} = 0$. From the second differential equation of (3.10), it is clear that $\sin\theta = 0$ since $Mr_c + m\underline{r}$ is strictly positive. Hence, it follows that $\theta = 0$ in the set S. As a consequence, from the first equation of (3.10) $\underline{r} - \overline{r} = 0$. Therefore, $r = \overline{r}$ in the set S.

Case 2. If $\dot{\theta} \neq 0$ in the set S, then (3.11) implies that

$$\ddot{\theta} = \frac{g\sin\theta}{2\underline{r}}. \tag{3.12}$$

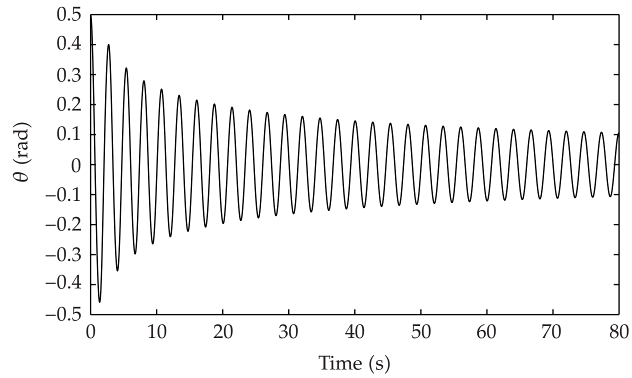


Figure 2: Angular displacement θ.

Since $\underline{r} > 0$, thus, $\ddot{\theta}$ is well defined. Taking into account the second differential equation in (3.10) and (3.12) yields to the following algebraic equation for the variable θ:

$$0 = \left(2Mr_c\underline{r} + 3m\underline{r}^2 + I_0\right)\sin\theta. \tag{3.13}$$

This last equation implies that $\theta = 0$ on the set S because $2Mr_c\underline{r} + 3m\underline{r}^2 + I_0 > 0$. This means that Case 2 is not possible since it assumes that $\dot{\theta} \neq 0$. Therefore, $\dot{\theta} = 0$ and Case 1 is the only possibility.

The above analysis allows concluding that the largest invariant set contained in S is given by $\overline{x}$. According to the LaSalle's invariance theorem, all the trajectories starting in Ω asymptotically converge towards $\overline{x} = [\overline{r}, 0, 0, 0]^T$. The following proposition resumes the stability result previously presented.

Proposition 3.3. *Consider the closed-loop dynamic system given by* (2.5) *in closed-loop with the control law*

$$F = -k_p(r - \overline{r}) - k_d\dot{r} - mg. \tag{3.14}$$

Then, all the trajectories starting in Ω asymptotically converge towards the equilibrium point $\overline{x} = [\overline{r}, 0, 0, 0]^T$.

It is worth noting that the three terms defining (3.14) are the proportional part $k_p(r - \overline{r})$, the derivative part $k_d\dot{r}$, and a gravity compensation term mg. Moreover, (3.7) indicates that damping introduced by the derivative part $k_d\dot{r}$ provides energy dissipation.

4. Numerical Simulations

To illustrate the performance of the proposed control law, a numerical simulation was carried out using the MATLAB program. The system physical parameters were set as follows:

$$m = 1[\text{Kg}], \qquad M = 2.5[\text{Kg}], \qquad r_c = 0.7[\text{m}], \qquad I_0 = 1.22\left[\text{Kg}\cdot\text{m}^2\right]. \tag{4.1}$$

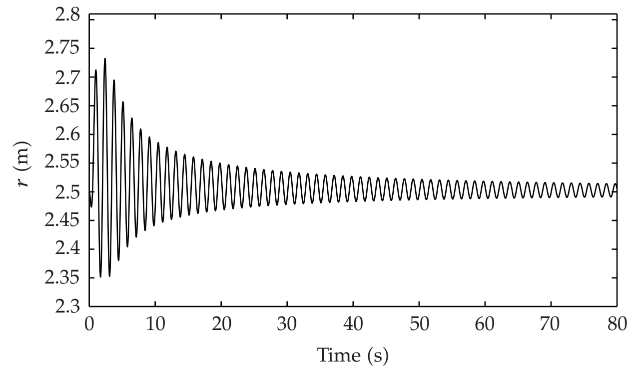
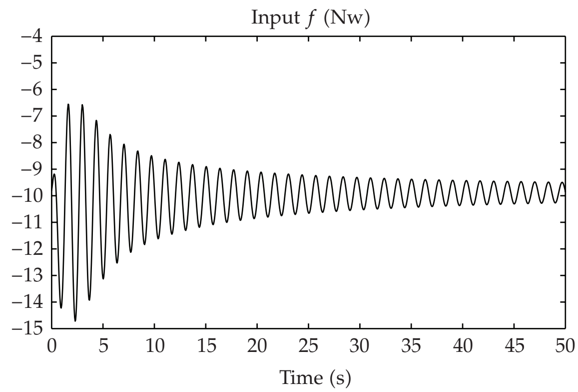
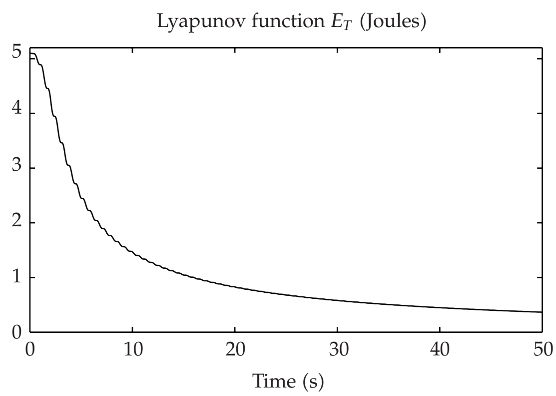


Figure 3: Moving mass displacement r.



(a)



(b)

Figure 4: Force F applied to the moving mass and time evolution of the Lyapunov function $E_T(q, \dot{q})$.

The radial displacement of m was given by $\overline{r} = 2.5[\text{m}]$ and $\varepsilon = 0.75[\text{m}]$. The initial conditions were chosen to be $\theta(0) = 0.5[\text{rad}]$, $r(0) = 2.5[\text{m}]$, $\dot{\theta}(0) = 0[\text{rad/s}]$ and $\dot{r}(0) = 0[\text{m/s}]$. The control gains, empirically proposed to increase the convergence rate of the closed-loop system, were set as $k_p = 19.6$ and $k_d = 1.9$. Figure 2 shows the angular displacement, Figure 3 depicts the displacement of the moving mass. Figure 4 displays the

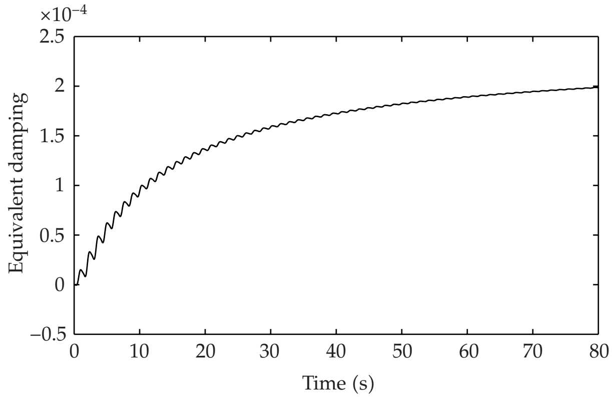


Figure 5: Time evolution of the equivalent damping ξ_{EQ}.

time evolution of both, the force applied to the moving mass and the Lyapunov function $E_T(q,\dot{q})$. From the above results, it is evident that the proposed control law attenuates the pendulum oscillations by a factor of five; the radial displacement remain bounded and converges to the value $\overline{r} = 2.5$[m]. On the other hand, the applied force stays also bounded and converges to −9.8[Nw], that is, the value given by the gravity compensation mg. Note also that the Lyapunov function $E_T(q,\dot{q})$, which accounts for the kinetic and potential energy of the closed-loop system, also decreases. Figure 5 displays the equivalent damping. This concept is introduced in [17] and is given by the following equation:

$$\xi_{\mathrm{EQ}} = \frac{1}{2\pi}\frac{\int_0^{\tau}(\dot{r}/r)\left(\dot{\theta}^2 + (g/4r)\theta^2\right)dt}{\left[(1/2)\dot{\theta}^2(0) + (g/2r(0))\theta^2(0)\right]}. \quad (4.2)$$

Even if this measure was originally intended for evaluating the damping in one cycle of the pendulum oscillation, equation allows computing ξ_{EQ} for any time τ provided that $\theta \ll 1$. Note that the time evolution of r and $\dot{r}$ determines the behaviour of ξ_{EQ}. The equivalent damping has final value of 2×10^{-4}, which is indeed very small.

It must be pointed out that the control strategy is effective in reducing the pendulum oscillations with reasonable control input effort for a large deviation from the equilibrium point. However, the damping injection capability of the proposed strategy is somewhat limited, that is, the system is brought to the desired equilibrium very slowly. This last observation is in agreement with the small value of the equivalent damping. Nevertheless, in real systems there always exists viscous friction, which helps to accelerate convergence to the equilibrium point. Also, Figure 6 shows the level curves associated with the modified potential energy $K_m(q)$, for the two different values of the parameter $k_p = 19.6$ and $k_p = 40$ and, as we can see, the region of attraction of the specified equilibrium point can be increased by just augmenting the value of k_p. However, it is not convenient to consider high values for the proportional gain k_p, because it may generate high frequencies oscillations in the closed-loop system. Thus, it is better to consider small values for parameter k_p in order to guarantee that $|r - \overline{r}| < \varepsilon$ holds.

5. Conclusions

This paper proposes a Proportional Derivative control law plus gravity compensation for active vibration damping in a frictionless physical pendulum with moving mass. The control

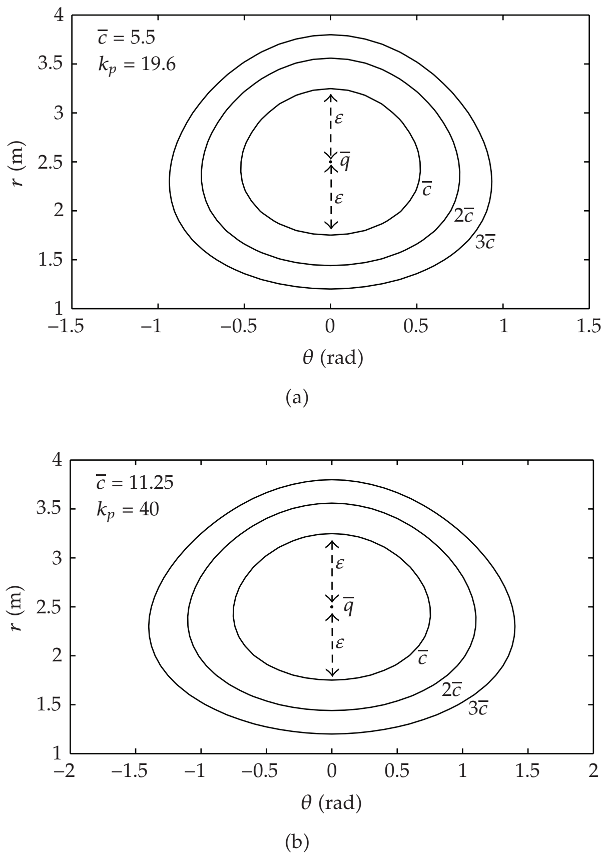


Figure 6: Level curves of $K_m(q)$ around the origin for (a) $k_p = 19.6$ and (b) $k_p = 40$.

law is able to damp out the oscillations of the pendulum by using the moving mass as active damper, and its design exploits the underlying physical properties of this system to shape a Lyapunov function candidate. LaSalle's theorem allows concluding asymptotic stability of the closed-loop system. Moreover, the control law only needs measurements of the position and velocity of the moving mass. Compared with previous approaches [16, 18], the proposed methodology does not need to synchronize the motion of the moving mass with the swings of the pendulum, then avoiding measurement of the pendulum position and velocity. The proposed strategy could be considered as a first step towards the reduction of undesirable oscillation in civil structures.

Acknowledgments

This research was supported by the Centro de Investigación en Computación of the Instituto Politecnico Nacional, by the Secretara de Investigación y Posgrado of the Instituto Politecnico Nacional (SIP-IPN), under Research Grant 20082694, and by the Centro de Investigación y Estudios Avanzados del Instituto Politecnico Nacional (Cinvestav-IPN) by CONACYT-México under Research Grant 32681-A. Octavio Gutiérrez is a scholarship holder of the Consejo Nacional de Ciencia y Tecnologia (CONACYT-México).

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