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Research Article

The Pullback Attractors for the Nonautonomous Camassa-Holm Equations

Delin Wu

College of Science, China Jiliang University, Hangzhou 310018, China

Correspondence should be addressed to Delin Wu, wudelin@gmail.com

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We consider the pullback attractors for the three-dimensional nonautonomous Camassa-Holm equations in the periodic box $\Omega = [0, L]^3$. Assuming $f \in L^2_{\text{loc}}((0, T); D(A^{-1/2}))$, which is translation bounded, the existence of the pullback attractor for the three-dimensional nonautonomous Camassa-Holm system is proved in $D(A^{1/2})$ and $D(A)$.

1. Introduction

We consider the following viscous version of the three-dimensional Camassa-Holm equations in the periodic box $\Omega = [0, L]^3$:

$$\begin{gathered}\frac{\partial}{\partial t}\left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) - \nu\Delta\left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) - u \times \left(\nabla \times \left(\alpha_0^2 u - \alpha_1^2 \Delta u\right)\right) + \frac{1}{\rho_0}\nabla p = f(x,t), \\ \nabla \cdot u = 0, \\ u(x,\tau) = u_\tau(x),\end{gathered} \tag{1.1}$$

where $p/\rho_0 = \pi/\rho_0 + \alpha_0^2|u|^2 - \alpha_1^2(u \cdot \Delta u)$ is the modified pressure, while π is the pressure, $\nu > 0$ is the constant viscosity and $\rho_0 > 0$ is a constant density. The function f is a given body forcing $\alpha_0 > 0$ and $\alpha_1 \geq 0$ are scale parameters. Notice α_0 is dimensionless while α_1 has units of length. Also observe that at the limit $\alpha_0 = 1$, $\alpha_1 = 0$ we obtain the three dimensional Navier-Stokes equations with periodic boundary conditions.

We consider this equaton in an appropriate Hilbert space and show that there is an attractor $\mathcal{A}$ which all solutions approach as $\tau \to -\infty$. The basic idea of our construction, which is motivated by the works of [1].

In addition, we assume that the function $f(\cdot,t) =: f(t) \in L^2_{\text{loc}}(R;E)$ is translation bounded, where $E = D(A^{-1/2})$. This property implies that

$$\|f\|^2_{L^2_b} = \|f\|^2_{L^2_b(R;E)} = \sup_{t\in R}\int_t^{t+1} \|f(s)\|^2_E \, ds < \infty. \tag{1.2}$$

In [1] the authors established the global regularity of solutions of the autonomous Camassa-Holm, or Navier-Stokes-alpha (NS-α) equations, subject to periodic boundary conditions. The inviscid NS-α equations (Euler-α) were introduced in [2] as a natural mathematical generalization of the integrable inviscid one-dimensional Camassa-Holm equation discovered in [3] through a variational formulation. An alternative more physical derivation for the inviscid NS-α equations (Euler-α), was introduced in [4, 5] (see also [6]). For more information and a brief guide to the previous literature specifically about the NS-α model, see [7].

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a system having some dissipativity properties is to analyse the existence and structure of its global attractor, which in the autonomous case, is an invariant compact set which attracts all the trajectories of the system, uniformly on bounded sets. However, nonautonomous systems are also of great importance and interest as they appear in many applications to natural sciences. On some occasions, some phenomena are modeled by nonlinear evolutionary equations which do not take into account all the relevant information of the real systems. Instead some neglected quantities can be modeled as an external force which in general becomes time dependent. In this situation, there are various options to deal with the problem of attractors for nonautonomous systems (kernel sections [8], skew-product formalism [9, 10], etc.), for our particular situation we have preferred to choose that of pullback attractor (see [11–15]) which has also proved extremely fruitful, particularly in the case of random dynamical systems (see [14, 16]).

In this paper, we study the existence of compact pullback attractor for the nonautonomous three-dimensional-Camassa-Holm equations in bounded domain Ω with periodic boundary condition. We apply the concept of measure of noncompactness to nonautonomous Camassa-Holm equation with external forces $f(x,t)$ in $L^2_{\text{loc}}(R;E)$ which is translation bounded.

From (1.1) one can easily see, after integration by parts, that

$$\frac{d}{dt}\int_\Omega \left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) dx = \int_\Omega f \, dx, \tag{1.3}$$

on the other hand, because of the spatial periodicity of the solution, we have $\int_\Omega \Delta u \, dx = 0$. As a result, we have $(d/dt)\int_\Omega \alpha_0^2 u \, dx = \int_\Omega f \, dx$; that is, the mean of the solution is invariant provided the mean of the forcing term is zero. In this paper we will consider forcing terms and initial values with spatial means that are zero; that is, we will assume $\int_\Omega u_\tau(x) dx = \int f \, dx = 0$ and hence $\int_\Omega u \, dx = 0$.

Next, let us introduce some notation and background.

(i) Let E be a linear subspace of integrable functions defined on the domain Ω, we denote

$$\dot{E} := \left\{ \varphi \in E : \int_{\Omega} \varphi(x)dx = 0 \right\}. \tag{1.4}$$

(ii) We denote $\mathcal{V} = \{\varphi : \varphi$ is a vector valued trigonometric polynomial defined on Ω, such that $\nabla \cdot \varphi = 0$ and $\int_{\Omega}\varphi(x)dx = 0\}$, and let H and V be the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and in $H^1(\Omega)^3$, respectively; observe that $H^{\perp}$, the orthogonal complement of H in $L^2(\Omega)^3$ is $\{\nabla p : p \in H^1(\Omega)\}$ (cf. [17] or [18]).

(iii) We denote $P : L^2(\Omega)^3 \rightarrow H$ the L^2 orthogonal projection, usually referred as Helmholtz-Leray projector, and by $A = -P\Delta$ the Stokes operator with domain $D(A) = (H^2(\Omega))^3 \cap V$. Notice that in the case of periodic boundary condition $A = -\Delta|_{D(A)}$ is a selfadjoint positive operator with compact inverse. Hence the space H has an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ of eigenfunctions of A, that is, $Aw_j = \lambda_j w_j$, with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \longrightarrow +\infty, \quad \text{as} \quad j \longrightarrow \infty, \tag{1.5}$$

in fact these eigenvalues have the form $|k|^2 4\pi/L^2$ with $k \in Z^3 \setminus \{0\}$.

(iv) The scalar product on H is denoted by $(\cdot,\cdot)$, the one on V is denoted by $((\cdot,\cdot))$, and the associated norms are denoted by $|\cdot|$ and $\|\cdot\| = |A^{1/2}\cdot|$, respectively. Notice that the inner product $((\cdot,\cdot))$ is equivalent to the H^1 inner product

$$[u, v] = \alpha_0^2(u, v) + \alpha_1^2((u, v)) \quad \text{for} \quad u, v \in V, \tag{1.6}$$

provided $\alpha_1 > 0$.

2. Abstract Results

We now discuss the theory of pullback attractors, as developed in [11, 12, 15]. As it is well known, in the case of nonautonomous differential equations the initial time is just as important as the final time, and the classical semigroup property of autonomous dynamical systems is no longer available.

Instead of a family of one time-dependent maps $S(t)$ we need to use a two parameter process $U(t,\tau)$ on the complete metric space E, $U(t,\tau)\varrho$ uses to denote the value of the solution at time t which was equal to the initial value ϱ at time τ.

The semigroup property is replaced by the process composition property

$$U(t,\tau)U(\tau,s) = U(t,s) \quad \forall t \geq \tau \geq s, \tag{2.1}$$

and, obviously, the initial condition implies $U(\tau,\tau) = Id$. As with the semigroup composition $S(t)S(\tau) = S(t+\tau)$, this just expresses the uniqueness of solutions.

It is also possible to present the theory within the more general framework of cocycle dynamical systems. In this case the second component of U is viewed as an element of some parameter space J, so that the solution can be written as $U(t,p)\varrho$, and a shift map $\theta_t : J \to J$ is defined so that the process composition becomes the cocycle property,

$$U(t+\tau,p) = U(t,\theta_\tau p)U(\tau,p). \tag{2.2}$$

However, when one tries to develop a theory under a unified abstract formulation, the context of cocycle (or skew-product flows) may not be the most appropriate to deal with the problem, since it is not known how to construct the set J (the same happens with the construction of the symbols set if one wishes to apply the theory of kernel sections as developed by [8]). For this reason, we do not pursue this approach here, but note that it has proved extremely fruitful, particularly in the case of random dynamical systems. For various examples using this general setting, see [13]. For this reason, pullback attractors are often referred to as cocycle attractors.

As in the standard theory of attractors, we seek an invariant attracting set. However, since the equation is nonautonomous this set also depends on time. By $\mathcal{B}(E)$ we denote the collection of the *bounded* sets of E.

Definition 2.1. Let U be a process on a complete metric space E. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a pullback attractor for U if, for all $\tau \in \mathbb{R}$, it satisfies

(i) $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau$,

(ii) $\lim_{s\to\infty}\operatorname{dist}(U(t,t-s)D,\mathcal{A}(t)) = 0$, for $D \in \mathcal{B}(E)$.

The pullback attractor is said to be uniform if the attraction property is uniform in time, that is,

$$\lim_{s\to\infty}\sup_{t\in\mathbb{R}}\operatorname{dist}(U(t,t-s)D,\mathcal{A}(t)) = 0, \quad \text{for } D \in \mathcal{B}(E). \tag{2.3}$$

Definition 2.2. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a forward attractor for U if, for all $\tau \in \mathbb{R}$, it satisfies

(i) $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau$,

(ii) $\lim_{t\to\infty}\operatorname{dist}(U(t,\tau)D,\mathcal{A}(t)) = 0$, for $D \in \mathcal{B}(E)$.

The forward attractor is said to be uniform if the attraction property is uniform in time, that is,

$$\lim_{t\to\infty}\sup_{\tau\in\mathbb{R}}\operatorname{dist}(U(t+\tau,\tau)D,\mathcal{A}(t+\tau)) = 0, \quad \text{for } D \in \mathcal{B}(E). \tag{2.4}$$

In the definition, $\operatorname{dist}(A,B)$ is the Hausdorff semidistance between A and B, defined as

$$\operatorname{dist}(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b), \quad \text{for } A, B \subseteq E. \tag{2.5}$$

Property (i) is a generalization of the invariance property for autonomous dynamical systems. The pullback attracting property (ii) considers the state of the system at time t when the initial time $t-s$ goes to $-\infty$ (see also [19]).

The notion of an attractor is closely related to that of an absorbing set.

Definition 2.3. The family $\{B(t)\}_{t\in\mathbb{R}}$ is said to be (pullback) absorbing with respect to the process U if, for all $t \in \mathbb{R}$ and $D \in \mathcal{B}(E)$, there exists $S(D,t) > 0$ such that for all $s \geq S(D,t)$

$$U(t, t-s)D \subset B(t). \tag{2.6}$$

The absorption is said to be uniform if $S(D,t)$ does not depend on the time variable t.

Indeed, just as in the autonomous case, the existence of compact absorbing sets is the crucial property in order to obtain pullback attractors. For the following result see [11].

Theorem 2.4. *Let $U(t,\tau)$ be a two-parameter process, and suppose $U(t,\tau) : E \to E$ is continuous for all $t \geq \tau$. If there exists a family of compact (pullback) absorbing sets $\{B(t)\}_{t\in\mathbb{R}}$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ and $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$.*

$$\mathcal{A}(t\ ) = \overline{\bigcup_{D\in\mathcal{B}(E)} \omega(D,t)}, \tag{2.7}$$

where

$$\omega(D,t) = \bigcap_{n\in N} \overline{\bigcup_{s\geq n} U(t,t-s)D}. \tag{2.8}$$

Now we recall the abstract results in [15].

Definition 2.5. The family of processes $\{U(t,t-s)\}$ is said to be satisfying pullback Condition (C) if for any fixed $B \in \mathcal{B}(E)$ and $\varepsilon > 0$, there exist $s_0 = s(B,t,\varepsilon) \geq 0$ and a finite dimensional subspace E_1 of E such that

(i) $\{\|P(\bigcup_{s\geq s_0} U(t,t-s)B)\|_E\}$ is bounded,

(ii) $\|(I-P)(\bigcup_{s\geq s_0} U(t,t-s)B)\|_E \leq \varepsilon$,

where $P : E \to E_1$ is a bounded projector.

Theorem 2.6. *Let the family of processes $\{U(t,\tau)\}$ acting in E be continuous and possesses compact pullback attractor $\mathcal{A}(t)$ satisfying*

$$\mathcal{A}(t) = \overline{\bigcup_{B\in\mathcal{B}} \omega(B,t)}, \quad \textit{for } t \in \mathbb{R}, \tag{2.9}$$

if it

(i) *has a bounded (pullback) absorbing set B,*

(ii) *satisfies pullback Condition (C).*

Moreover if E is a uniformly convex Banach space then the converse is true.

3. Pullback Attractor of Nonautonomous Camassa-Holm Equations

This section deals with the existence of the attractor for the three-dimensional nonautonomous Camassa-Holm equations in a bounded domain Ω with periodic boundary condition (see [1]).

We use the notation in [1] to obtain the equivalent system of equations

$$\begin{gathered}\frac{dv}{dt}+\nu Av+B(v)u+B^{*}(v)u=Pf,\\ v(x,\tau)=v_{\tau}(x)\in V.\end{gathered}\tag{3.1}$$

Here $v=\alpha_0^2u+\alpha_1^2Au$ and $B^{*}(v)$ denotes the adjoint operator of the linear operator $B(v)$ defined above.

It is similar to autonomous case that we can establish the existence of solution of (3.1) by the standard Faedo-Galerkin method.

In [1], the authors have shown that the semigroup $S(t):V\to V\ \ (t\geq 0)$ associated with the autonomous systems (3.1) possesses a global attractor in V and $D(A)$. The main objective of this section is to prove that the nonautonomous system (3.1) has pullback attractors in V and $D(A)$.

To this end, we first state some the following results.

Proposition 3.1. *Let $f\in V'$ and let $u_\tau\in V$. Then the problem* (1.1) *has a unique solution $u(t)$ such that for any $T>0$,*

$$u\in C([0,T);V)\cap L^2([0,T);D(A)),\qquad \frac{du}{dt}\in L^2([0,T),H),\tag{3.2}$$

and such that for almost all $t\in[0,T)$ and for any $w\in D(A)$,

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}\left(\alpha_0^2u-\alpha_1^2\Delta u\right),w\right\rangle_{D(A)'}+\nu\left\langle A\left(\alpha_0^2u-\alpha_1^2\Delta u\right),w\right\rangle_{D(A)'}+\left\langle\widetilde{B}\left(u,\alpha_0^2u-\alpha_1^2\Delta u\right),w\right\rangle_{D(A)'}\\ &\quad=(f,w),\end{aligned}\tag{3.3}$$

here

$$\left(\widetilde{B}(u,v),w\right)=(B(u,v),w)-(B(w,v),u)=(B(v)u-B^{*}(v)u,w)\tag{3.4}$$

for every $u,v,w\in V$.

Proof. The Proof of Proposition 3.1 is similar to autonomous Camassa-Holm in [1]. □

Proposition 3.2. *The process $\{U(t, t-s)\} : V \to V$ associated with the system* (3.1) *possesses (pullback) absorbing sets, that is, there exists a family $\{B(t)\}_{t\in R}$ of bounded (pullback) absorbing sets in V and $D(A)$ for the process U, which is given by*

$$\mathcal{B}_0 = B(t) = \{v \in V \mid \|v\| \le r_0\},$$
$$\mathcal{B}_1 = B(t) = \{v \in D(A) \mid |Av| \le r_1\}, \tag{3.5}$$

which absorb all bounded sets of V. Moreover $\mathcal{B}_0$ and $\mathcal{B}_1$ absorb all bounded sets of V and $D(A)$ in the norms of V and $D(A)$, respectively.

Proof. The proof of Proposition 3.2 is similar to autonomous Camassa-Holm equation. We can obtain absorbing sets in V and $D(A)$ the following from [1]. □

The main results in this section are as follows.

Now we prove the existence of compact pullback attractors in V and $D(A)$ by applying Theorem 2.6.

Theorem 3.3. *If $f(x,t) \in L_b^2(R; V')$ and $u_\tau \in V$, then the processes $\{U(t, t-s)\}$ corresponding to problem* (3.1) *possesses compact pullback attractor $\mathcal{A}_0(t)$ in V which coincides with the pullback attractor*

$$\mathcal{A}_0(t) = \overline{\bigcup_{\mathcal{B}_0 \in \mathcal{B}} \omega(\mathcal{B}_0, t)}, \tag{3.6}$$

where $\mathcal{B}_0$ is the (pullback) absorbing set in V.

Proof. As in the previous section, for fixed N, let H_1 be the subspace spanned by $w_1; \cdots ; w_N$, and H_2 the orthogonal complement of H_1 in H. We write

$$u = u_1 + u_2; \quad u_1 \in H_1,\ u_2 \in H_2 \text{ for any } u \in H. \tag{3.7}$$

Now we testify that the family of processes $\{U(t, \tau)\}$ corresponding to (3.1) satisfies pullback Condition (C). Namely, we need to estimate $|u_2(t)|$, where $u(t) = u_1(t) + u_2(t)$ is a solution of (3.1) given in Proposition 3.1

Multiplying (3.1) by u_2, we have

$$\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2|u_2|^2 + \alpha_1^2\|u_2\|^2\right) + \nu\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) + \left(\widetilde{B}\left(u, \alpha_0^2 u - \alpha_1^2 Au\right), u_2\right) = (Pf, u_2). \tag{3.8}$$

Notice that

$$\left|(Pf, u_2)\right| \le \left|f\right|_{V'}\|u_2\| \le \frac{\left|f\right|_{V'}^2}{\nu\alpha_0^2} + \frac{\nu}{4}\alpha_0^2\|u_2\|^2. \tag{3.9}$$

From the above inequalities we get

$$\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2|u_2|^2+\alpha_1^2\|u_2\|^2\right)+\frac{3\nu}{4}\left(\alpha_0^2\|u_2\|^2+\alpha_1^2|Au_2|^2\right)+\left(\widetilde{B}\left(u,\alpha_0^2u-\alpha_1^2Au\right),u_2\right)\leq\frac{|f|_{V'}^2}{\nu\alpha_0^2}. \tag{3.10}$$

We use part (iii) of Lemma 1 in [1] to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\alpha_0^2|u_2|^2+\alpha_1^2\|u_2\|^2\right)+\frac{3\nu}{4}\left(\alpha_0^2\|u_2\|^2+\alpha_1^2|Au_2|^2\right)\\&\leq c\left(\alpha_0^2\|u_2\|^2+\alpha_1^2\|u_2\|\|Au_2\|\right)|u_2|^{1/2}\|u_2\|^{1/2}+\frac{|f|_{V'}^2}{\nu\alpha_0^2}.\end{aligned} \tag{3.11}$$

By Young's inequality, together with H^1—Estimates in [1], we have

$$\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2|u_2|^2+\alpha_1^2\|u_2\|^2\right)+\frac{\nu}{4}\left(\alpha_0^2|u_2|^2+\alpha_1^2\|u_2\|^2\right)\leq M_1+\frac{|f|_{V'}^2}{\nu\alpha_0^2}. \tag{3.12}$$

Here $M_1=M_1(\alpha_0,\alpha_1,r_0)$ depends on λ_{m+1} is not increasing as λ_{m+1} is increasing.

Therefore, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2|u_2|^2+\alpha_1^2\|u_2\|^2\right)+\frac{\nu}{4}\left(\alpha_0^2|u_2|^2+\alpha_1^2\|u_2\|^2\right)\leq M_1+\frac{c}{\nu}|f|_{V'}^2. \tag{3.13}$$

By the Gronwall inequality, the above inequality implies

$$\begin{aligned}\alpha_0^2|u_2|^2+\alpha_1^2\|u_2\|^2\leq&\left(\alpha_0^2|u_2(\tau)|^2+\alpha_1^2\|u_2(\tau)\|^2\right)e^{-\nu\lambda_{m+1}(t-\tau)/2}\\&+\frac{2M_1}{\nu\lambda_{m+1}}+\frac{2c}{\nu}\int_\tau^t e^{-\nu\lambda_{m+1}(t-s)/2}|f|_{V'}^2ds.\end{aligned} \tag{3.14}$$

If we consider the time $t-s$ instead of τ (so that we can use more easily the definition of pullback attractors) we have $\frac{3}{\nu}\int_\tau^t e^{-\nu\lambda_{m+1}(t-\sigma)/2}|f(\sigma)|_{V'}^2d\sigma=\frac{3}{\nu}\int_{t-s}^t e^{-\nu\lambda_{m+1}(t-\sigma)/2}|f(\sigma)|_{V'}^2d\sigma$.

Applying continuous integral and [8, Lemma II 1.3] for any ε, there exists $\eta=\eta(\varepsilon)>0$ such that

$$\int_{t-\eta}^t|f(\sigma)|_{V'}^2d\sigma<\frac{\nu\varepsilon}{18}, \tag{3.15}$$

thus, we have

$$\frac{3}{\nu}\int_{t-\eta}^{t} e^{-\nu\lambda_{m+1}(t-\sigma)/2}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma \leq \frac{\varepsilon}{6}, \tag{3.16}$$

$$\begin{aligned}
&\frac{3}{\nu}\int_{t-s}^{t-\eta} e^{-\nu\lambda_{m+1}\,(t-\sigma)/2}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma\\
&\quad\leq \frac{3}{\nu}\int_{t-\eta-1}^{t-\eta} e^{-\nu\lambda_{m+1}\,(t-\sigma)/2}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma\\
&\qquad+\frac{3}{\nu}\int_{t-\eta-2}^{t-\eta-1} e^{-\nu\lambda_{m+1}\,(t-\sigma)/2}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma+\cdots\\
&\quad\leq \frac{3}{\nu}\,e^{-\nu\lambda_{m+1}\eta/2}\left(\int_{t-\eta-1}^{t-\eta}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma+e^{-\nu\lambda_{m+1}/2}\int_{t-\eta-2}^{t-\eta-1}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma+\cdots\right)\\
&\quad\leq \frac{3}{\nu}\,e^{-\nu\lambda_{m+1}\eta/2}\left(1+e^{-\nu\lambda_{m+1}/2}+\cdots\right)\sup_{s\in R}\int_{s-1}^{s}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma\\
&\quad\leq \frac{(3/\nu)e^{-\nu\lambda_{m+1}\eta/2}}{1-e^{-\nu\lambda_{m+1}/2}}\left\|f\right\|_{L_b^2}^{2}.
\end{aligned} \tag{3.17}$$

Using (1.5) and let $s_1 = 2/\nu\lambda_{m+1}\ln(3r_0^2/\varepsilon)$, then $s \geq s_1$ implies

$$\begin{gathered}
\frac{3}{\nu}\int_{t-s}^{t-\eta} e^{-\nu\lambda_{m+1}(t-\sigma)}\left|f(\sigma)\right|_{V'}^{2}\,d\sigma \leq \frac{(3/\nu)e^{-\nu\lambda_{m+1}\eta/2}}{1-e^{-\nu\lambda_{m+1}/2}}\left\|f\right\|_{L_b^2(R;V')}^{2} \leq \frac{\varepsilon}{6},\\
\frac{2M_1}{\nu\lambda_{m+1}} \leq \frac{\varepsilon}{3};\\
k_1 e^{-\nu\lambda_{m+1}(t-\tau)/2} \leq r_0^2 e^{-\nu\lambda_{m+1}s_1/2} \leq \frac{\varepsilon}{3}.
\end{gathered} \tag{3.18}$$

Therefore, we deduce from (3.14) that

$$\left\|u_2\right\|^2 \leq \varepsilon, \quad \forall s \geq s_1, \tag{3.19}$$

which indicates $\{U(t,\tau)\}$ satisfying pullback Condition (C) in V. Applying Theorem 2.6 the proof is complete. □

According to Propositions 3.1-3.2, we can now regard that the families of processes $\{U(t,\tau)\}$ are defined in $D(A)$ and $\mathcal{B}_1$ is a pullback absorbing set in $D(A)$.

Theorem 3.4. *If $f(x,t) \in L_b^2(R;V')$, then the processes $\{U(t,\tau)\}$ corresponding to problem* (3.1) *possesses compact pullback attractor $\mathcal{A}_1(t)$ in V or $D(A)$:*

$$\mathcal{A}_1(t) = \overline{\bigcup_{\mathcal{B}_1 \in \mathcal{B}} \omega(\mathcal{B}_1, t)}, \tag{3.20}$$

where $\mathcal{B}_1$ is the absorbing set in $D(A)$.

Proof. Using Proposition 3.2, we have the family of processes $\{U(t,\tau)\}$ corresponding to (3.1) possesses the pullback absorbing set in $D(A)$.

Similarly, we only have to verify pullback Condition (C).

Multiplying the equation (3.1) by Au_2, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) + \nu\left(\alpha_0^2|Au_2|^2 + \alpha_1^2\left|A^{3/2}u_2\right|^2\right) + \left(\widetilde{B}\left(u, \alpha_0^2 u - \alpha_1^2 Au\right), Au_2\right) \\ &\quad = (Pf, Au_2).\end{aligned} \tag{3.21}$$

Notice that

$$\left|(Pf, Au_2)\right| \le |f|_{V'}\left|A^{3/2}u_2\right| \le \frac{|f|_{V'}^2}{\nu\alpha_1^2} + \frac{\nu}{4}\alpha_1^2\left|A^{3/2}u_2\right|^2. \tag{3.22}$$

Therefore we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2\|Au_2\|^2\right) + \frac{3\nu}{4}\left(\alpha_0^2\|Au_2\|^2 + \alpha_1^2\left|A^{3/2}u_2\right|^2\right) + \left(\widetilde{B}\left(u, \alpha_0^2 u - \alpha_1^2 Au\right), u_2\right) \\ &\quad \le \frac{|f|_{V'}^2}{\nu\alpha_1^2}.\end{aligned} \tag{3.23}$$

We use part (iii) of Lemma 1 in [1] to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) + \frac{3\nu}{4}\left(\alpha_0^2|u_2|^2 + \alpha_1^2\left|A^{3/2}u_2\right|^2\right) \\ &\quad \le c\|u_2\|\left(\alpha_0^2\|u_2\| + \alpha_1^2\left|A^{3/2}u_2\right|\right)|Au_2|\left|A^{3/2}u_2\right|^{1/2} + \frac{|f|_{V'}^2}{\nu\alpha_1^2}.\end{aligned} \tag{3.24}$$

By Young's inequality, together with H^1—Estimates in [1], we have

$$\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) + \frac{\nu}{4}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) \le M_2 + \frac{|f|_{V'}^2}{\nu\alpha_1^2}. \tag{3.25}$$

Here $M_2 = M_2(\alpha_0, \alpha_1, r_1)$ depends on λ_{m+1} is not increasing as λ_{m+1} is increasing.

By the Gronwall inequality, the above inequality implies

$$\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2 \le \left(\alpha_0^2\|u_2(\tau)\|^2 + \alpha_1^2|Au_2(\tau)|^2\right)e^{-\nu\lambda_{m+1}(t-\tau)/2} + \frac{2M_2}{\nu\lambda_{m+1}} + \frac{2c}{\nu}\int_\tau^t e^{-\nu\lambda_{m+1}(t-\sigma)/2}|f|_{V'}^2 d\sigma. \tag{3.26}$$

We consider the time $t-s$ instead of τ. The following result is similar to (3.16)-(3.18), for any ε

$$\frac{2c}{\nu}\int_\tau^t e^{-\nu\lambda_{m+1}(t-\sigma)/2}|f|_{V'}^2\, d\sigma \le \frac{\varepsilon}{3}. \tag{3.27}$$

Using (1.5) and let $s_2 = (2/\nu\lambda_{m+1})\ln(3r_1^2/\varepsilon)$, then $s \ge s_2$ implies

$$\begin{gathered}\frac{2M_2}{\nu\lambda_{m+1}} \le \frac{\varepsilon}{3},\\ \left(\alpha_0^2\|u_2(\tau)\|^2 + \alpha_1^2|Au_2(\tau)|^2\right)e^{-\nu\lambda_{m+1}(t-\tau)/2} \le k_2 e^{-\nu\lambda_{m+1}s/2} < \frac{\varepsilon}{3}.\end{gathered} \tag{3.28}$$

Therefore, we deduce from (3.26) that

$$|Au_2|^2 \le \varepsilon, \quad \forall s \ge s_1, \tag{3.29}$$

which indicates $\{U(t,\tau)\}$ satisfying pullback Condition (C) in $D(A)$. □

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