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Research Article

Interior Controllability of a Broad Class of Reaction Diffusion Equations

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We prove the interior approximate controllability of the following broad class of reaction diffusion equation in the Hilbert spaces $Z = L^2(\Omega)$ given by $z' = -Az + 1_\omega u(t)$, $t \in [0, \tau]$, where Ω is a domain in $\mathbb{R}^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L^2(0, t_1; L^2(\Omega))$ and $A : D(A) \subset Z \to Z$ is an unbounded linear operator with the following spectral decomposition: $Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}$. The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors of A. The operator $-A$ generates a strongly continuous semigroup $\{T(t)\}$ given by $T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}$. Our result can be applied to the nD heat equation, the Ornstein-Uhlenbeck equation, the Laguerre equation, and the Jacobi equation.

1. Introduction

In this paper we prove the interior approximate controllability of the following broad class of reaction diffusion equation in the Hilbert space $Z = L^2(\Omega)$ given by

$$\begin{aligned} z' &= -Az + 1_\omega u(t), \quad t \in [0, \tau], \\ z(0) &= z_0, \end{aligned} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, and the distributed control $u \in L^2(0, t_1; L^2(\Omega))$ and $A : D(A) \subset Z \to Z$

is an unbounded linear operator. Here we assume the following spectral decomposition for A:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{1.2}$$

with $\langle \cdot, \cdot \rangle$ denoting an inner product in Z, and

$$E_j z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \tag{1.3}$$

The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \lambda_n \to \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors of A. So, $\{E_j\}$ is a complete family of orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z,\ z \in Z$. The operator $-A$ generates a strongly continuous semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z. \tag{1.4}$$

Systems of the form (1.1) are thoroughly studied in [1, 2], but the interior controllability is not considered there.

Examples of this class of equations are the following well-known partial differential equations.

Example 1.1. The interior controllability of the heat equation,

$$\begin{aligned} z_t &= \Delta z + 1_\omega u(t,x), \quad \text{in } (0,\tau) \times \Omega, \\ z &= 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ z(0,x) &= z_0(x), \quad \text{in } \Omega, \end{aligned} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$ of class C^2, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, $z_0 \in L^2(\Omega)$, and the distributed control $u \in L^2(0,\tau; L^2(\Omega))$.

Example 1.2 (see [3, 4]). (1) The interior controllability of the Ornstein-Uhlenbeck equation is

$$z_t = \sum_{i=1}^{d} \left[x_i \frac{\partial^2 z}{\partial x_i^2} - x_i \frac{\partial z}{\partial x_i} \right] + 1_\omega u(t,x), \quad t > 0,\ x \in \mathbb{R}^d, \tag{1.6}$$

where $u \in L^2(0,\tau; L^2(\mathbb{R}^d, \mu))$, $\mu(x) = (1/\pi^{d/2}) \prod_{i=1}^{d} e^{-|x_i|^2} dx$ is the Gaussian measure in $\mathbb{R}^d$, and ω is an open nonempty subset of $\mathbb{R}^d$.

(2) The interior controllability of the Laguerre equation is

$$z_t = \sum_{i=1}^{d} \left[x_i \frac{\partial^2 z}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial z}{\partial x_i} \right] + 1_\omega u(t,x), \quad t > 0, \ x \in \mathbb{R}_+^d, \tag{1.7}$$

where $u \in L^2(0,\tau; L^2(\mathbb{R}_+^d, \mu_\alpha))$, $\mu_\alpha(x) = \prod_{i=1}^{d} (x_i^{\alpha_i} e^{-x_i} / \Gamma(\alpha_i + 1)) dx$ is the Gamma measure in $\mathbb{R}_+^d$ and ω is an open nonempty subset of $\mathbb{R}_+^d$.

(3) The interior controllability of the Jacobi equation is

$$z_t = \sum_{i=1}^{d} \left[\left(1 - x_i^2\right) \frac{\partial^2 z}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial z}{\partial x_i} \right] + 1_\omega u(t,x), \tag{1.8}$$

where $t > 0$, $x \in (-1,1)^d$, $u \in L^2(0,\tau; L^2([-1,1]^d, \mu_{\alpha,\beta}))$, $\mu_{\alpha,\beta}(x) = \prod_{i=1}^{d} (1 - x_i)^{\alpha_i} (1 + x_i)^{\beta_i} dx$ is the Jacobi measure in $[-1,1]^d$ and ω is an open nonempty subset of $[-1,1]^d$.

To complete the exposure of this introduction, we mention some works done by other authors showing the difference between our results and those of them: the interior approximate controllability is very well-known fascinate and important subject in systems theory; there are some works done by [5–9].

Particularly, Zuazua in [9] proves the interior approximate controllability of the heat equation (1.5) in two different ways. In the first one, he uses the Hahn-Banach theorem, integrating by parts the adjoint equation, the Carleman estimates and the Holmgren Uniqueness theorem [10]. But, the Carleman estimates depend on the Laplacian operator Δ, so it may not be applied to those equations that do not involve the Laplacian operator, like the Ornstein-Uhlenbeck equation, the Laguerre equation, and the Jacobi equation.

The second method is constructive and uses a variational technique: let us fix the control time $\tau > 0$, the initial and final state, $z_0 = 0$, $z_1 \in L^2(\Omega)$, respectively, and $\epsilon > 0$. The control steering the initial state z_0 to a ball of radius $\epsilon > 0$, and center z_1 is given by the point in which the following functional achieves its minimum value:

$$J_\epsilon(\varphi_\tau) = \frac{1}{2} \int_0^\tau \int_\omega \varphi^2 dx\, dt + \epsilon \|\varphi_\tau\|_{L^2(\Omega)} - \int_\Omega z_1 \varphi_\tau, \tag{1.9}$$

where φ is the solution of the corresponding adjoint equation with initial data φ_τ.

The technique given here is motivated by the following results.

Theorem 1.3 (see [11, Theorem 1.23, page 20]). *Suppose $\Omega \subset \mathbb{R}^n$ is open, nonempty and connected set, and f is real analytic function in Ω with $f = 0$ on a nonempty open subset ω of Ω. Then, $f = 0$ in Ω.*

Lemma 1.4 (see [1, Lemma 3.14, page 62]). *Let $\{\alpha_j\}_{j\geq 1}$ and $\{\beta_{i,j} : i = 1,2,\dots,m\}_{j\geq 1}$ be two sequences of real numbers such that $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0,t_1], \ i = 1,2,\dots,m, \tag{1.10}$$

if and only if

$$\beta_{i,j} = 0, \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, \infty. \tag{1.11}$$

2. Main Theorem

In this section we will prove the main result of this paper on the controllability of the linear system (1.1). But before that, we will give the definition of approximate controllability for this system. To this end, the system (1.1) can be written as follows:

$$\begin{gathered} z' = -Az + B_\omega u(t), \quad z \in Z, \\ z(0) = z_0, \end{gathered} \tag{2.1}$$

where the operator $B_\omega : Z \to Z$ is defined by $B_\omega f = 1_\omega f$. For all $z_0 \in Z$ and $u \in L^2(0, \tau; Z)$ the initial value problem (2.1) admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega u(s)ds, \quad t \in [0, \tau]. \tag{2.2}$$

Definition 2.1 (exact controllability). The system (2.1) is said to be exactly controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$ there exists $u \in L^2(0, \tau; Z)$ such that the solution z of (2.2) corresponding to u satisfies $z(\tau) = z_1$.

Definition 2.2 (approximate controllability). The system (2.1) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; Z)$ such that the solution z of (2.2) corresponding to u satisfies

$$\|z(\tau) - z_1\| < \varepsilon. \tag{2.3}$$

Remark 2.3. The following result was proved in [12]. If the semigroup $\{T(t)\}$ is compact, then the system $z' = -Az + B_\omega u(t)$ can never be exactly controllable on time $\tau > 0$, which is the case of the heat equations, the Ornstein-Uhlenbeck equation, the Laguerre equation, the Jacobi equation, and many others partial differential equations.

The following theorem can be found in a general form for evolution equation in [2].

Theorem 2.4. *The system* (2.1) *is approximately controllable on* $[0, \tau]$ *if, and only if,*

$$B_\omega^* T^*(t)z = 0, \quad \forall t \in [0, \tau] \Longrightarrow z = 0. \tag{2.4}$$

Now, we are ready to formulate and prove the main theorem of this paper.

Theorem 2.5. *If for an open nonempty set* $\omega \subset \Omega$ *the restrictions* $\phi_{j,k}^\omega = \phi_{j,k}|_\omega$ *to* ω *are linearly independent functions on* ω*, then for all* $\tau > 0$ *the system* (2.1) *is approximately controllable on* $[0, \tau]$.

Proof. We will apply Theorem 2.4 to prove the approximate controllability of system (2.1). To this end, we observe that $B_\omega = B_\omega^*$ and $T^*(t) = T(t)$. Suppose that $B_\omega^* T^*(t)z = 0,\ \forall t \in [0,\tau]$. Then,

$$B_\omega^* T^*(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} B_\omega^* E_j z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle 1_\omega \phi_{j,k} = 0$$
$$\Longleftrightarrow \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle 1_\omega(x) \phi_{j,k}(x) = 0, \quad \forall x \in \Omega \tag{2.5}$$
$$\Longleftrightarrow \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \forall x \in \omega.$$

Hence, from Lemma 1.4, we obtain that

$$\sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \forall x \in \omega,\ j = 1,2,3,\ldots. \tag{2.6}$$

Since $\phi_{j,k}$ are linearly independent on ω, we obtain that $\langle z, \phi_{j,k} \rangle = 0,\ j = 1,2,\ldots$. Therefore, $E_j z = 0,\ j = 1,2,3,\ldots$, which implies that $z = 0$. So, the system (2.1) is approximately controllable on $[0,\tau]$. □

Corollary 2.6. *If $\phi_{j,k}$ are analytic functions on Ω, then for all open nonempty set $\omega \subset \Omega$ and $\tau > 0$ the system* (2.1) *is approximately controllable on* $[0,\tau]$.

Proof. It is enough to prove that, for all open nonempty set $\omega \subset \Omega$ the restrictions $\phi_{j,k}^\omega = \phi_{j,k}|_\omega$ to ω are linearly independent functions on ω, which follows directly from Theorem 1.3. □

3. Applications

As an application of our result we will prove the controllability of the nD heat equation, the Ornstein-Uhlenbeck equation, the Laguerre equation and the Jacobi equation.

3.1. The Interior Controllability of the Heat Equation (1.5)

In this subsection we will prove the controllability of system (1.5), but before that, we will prove the following theorem.

Theorem 3.1. *The eigenfunctions of the operator $-\Delta$ with Dirichlet boundary conditions on Ω are real analytic functions in Ω.*

To this end, first, we will consider the following definition and results from [13].

Definition 3.2. A differential operator L is say to be hypoelliptic analytic if for each open subset Ω of $\mathbb{R}^n$ and each distribution $u \in D'(\Omega)$, we have that: if $L(u)$ is an analytic function in Ω, then u is an analytic function in Ω.

Corollary 3.3 (see [13, page 15]). *Every second-order elliptic operator with constant coefficients is hypoelliptic analytic.*

Proof of Theorem 3.1. Let ϕ be an eigenfuction of $-\Delta$ with corresponding eigenvalue $\lambda > 0$. Then, the second-order differential operator $L = \Delta + \lambda$ is an elliptic operator according to [13, Definiton 7.2, page 97]. Therefore, applying the foregoing corollary we get that $L = \Delta + \lambda$ hypoelliptic analytic.

On the other hand, we know that $L\phi = \Delta\phi + \lambda\phi = 0$, which is trivially an analytic function, then ϕ is an analytic function in Ω. □

Now, we will make the abstract formulation of the problem, and to this end, let us consider $Z = L^2(\Omega)$ and the linear unbounded operator $A : D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{3.1}$$

It is well-known that this operator A has spectral decomposition given by (1.2) and the system (1.5) can be written as an abstract equation in the space $Z = L^2(\Omega)$

$$\begin{gathered} z' = -Az + B_\omega u(t), \quad z \in Z, \\ z(0) = z_0, \end{gathered} \tag{3.2}$$

where the control function u belongs to $L^2(0, \tau; Z)$, and the operator $B_\omega : Z \to Z$ is defined by $B_\omega f = 1_\omega f$.

Theorem 3.4. *For all open nonempty set $\omega \subset \Omega$ and $\tau > 0$ the system* (3.2) *is approximately controllable on* $[0, \tau]$.

3.2. *The Interior Controllability of* (1.6), (1.7), *and* (1.8)

Theorem 3.5. *The systems* (1.6), (1.7), *and* (1.8) *are approximately controllable.*

Proof. It is enough to prove that the operators

(i) Ornstein-Uhlenbeck operator: $-A = (1/2)\nabla - \langle x, \Delta_x \rangle$, defined on $\Omega = \mathbb{R}^d$,with $\Delta_x = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$ in the space $Z = L^2(\mathbb{R}^d, \mu)$;

(ii) Laguerre operator: $A = -\sum_{i=1}^d [x_i(\partial^2 z/\partial x_i^2) + (\alpha_i + 1 - x_i)(\partial z/\partial x_i)]$, defined on $\Omega = (0, \infty)^d$, with $\alpha_i > -1$, $i = 1, \ldots, d$ in the space $Z = L^2(\mathbb{R}_+^d, \mu_\alpha)$;

(iii) Jacobi operator: $A = -\sum_{i=1}^d [(1 - x_i^2)(\partial^2 z/\partial x_i^2) + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2)x_i)(\partial z/\partial x_i)]$, $\Omega = (-1, 1)^d$, with $\alpha_i, \beta_i > -1$, $i = 1, \ldots, d$ in the space $Z = L^2([-1, 1]^d, \mu_{\alpha,\beta})$

can be represented in the form of (1.2). This was done in [3, 4], where they prove that the eigenfunctions in these cases are polynomial functions in multiple variables, which are trivially analytic functions. □

4. Final Remark

The result presented in this paper can be formulated in a more general setting. Indeed, we can consider the following evolution equation in a general Hilbert space Z:

$$z' = -Az + Bu, \quad z, u \in Z, \ t \in [0, \tau],$$
$$z(0) = z_0, \tag{4.1}$$

where $A : D(A) \subset Z \to Z$ is an unbounded linear operator in Z with the spectral decomposition given by (1.2), the control $u \in L^2(0, \tau; Z)$ and $B : Z \to Z$ is a linear and bounded operator (linear and continuous).

In this case the characteristic function set is a particular operator B, and the following theorem is a generalization of Theorem 2.5.

Theorem 4.1. *If the vectors $B^*\phi_{j,k}$ are linearly independent in Z, then the system* (4.1) *is approximately controllable on* $[0, \tau]$.

Proof. From [2, Theorem 4.1.7, part (b)-(iii)], it is enough to prove that

$$B^*T^*(t)z = 0, \quad \forall t \in [0, \tau] \Longrightarrow z = 0. \tag{4.2}$$

To this end, we observe that

$$B^*T^*(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} B^* E_j z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle B^* \phi_{j,k} = 0. \tag{4.3}$$

Hence, from Lemma 1.4, we obtain that $\sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle B^* \phi_{j,k} = 0, \ j = 1, 2, \ldots$. Since $B^*\phi_{j,k}$ are linearly independent on Z, we obtain that $\langle z, \phi_{j,k} \rangle = 0, \ j = 1, 2, \ldots$. Therefore, $E_j z = 0, \ j = 1, 2, 3, \ldots$, which implies that $z = 0$. So, the system (4.1) is approximately controllable on $[0, \tau]$. □

Remark 4.2. As future researches, we will try to use this technique to study the controllability of other partial differential equations such as the thermoelastic plate equation, the equation modelling the damped flexible beam, and the strongly damped wave equation.

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