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Research Article

An Inverse Eigenvalue Problem for Damped Gyroscopic Second-Order Systems

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The inverse eigenvalue problem of constructing symmetric positive semidefinite matrix D (written as $D \geq 0$) and real-valued skew-symmetric matrix G (i.e., $G^T = -G$) of order n for the quadratic pencil $Q(\lambda) := \lambda^2 M_a + \lambda(D+G) + K_a$, where $M_a > 0$, $K_a \geq 0$ are given analytical mass and stiffness matrices, so that $Q(\lambda)$ has a prescribed subset of eigenvalues and eigenvectors, is considered. Necessary and sufficient conditions under which this quadratic inverse eigenvalue problem is solvable are specified.

1. Introduction

Vibrating structures such as beams, buildings, bridges, highways, and large space structures, are distributed parameter systems. While it is desirable to obtain a solution of a vibration problem in its own natural setting of distributed parameter systems; due to the lack of appropriate computational methods, in practice, very often a distributed parameter system is first discretized to a matrix second-order model (referred to as an analytical model) using techniques of finite elements or finite differences and then an approximate solution is obtained from the solution of the problem in the analytical model. A matrix second-order model of the free motion of a vibrating system is a system of differential equations of the form

$$M_a \ddot{x}(t) + (D_a + G_a)\dot{x}(t) + K_a x(t) = 0, \tag{1.1}$$

where M_a, D_a, G_a, and K_a are, respectively, analytical mass, damping, gyroscopic and stiffness matrices.

The system represented by (1.1) is called damped gyroscopic system. The gyroscopic matrix G_a is always skew symmetric and, in general, the mass matrix M_a is symmetric and positive definite and D_a, K_a are symmetric positive semidefinite; the system is called symmetric definite system. If the gyroscopic force is not present, then the system is called nongyroscopic.

It is well known that all solutions of the differential equation of (1.1) can be obtained via the algebraic equation

$$\left(\lambda^2 M_a + \lambda(D_a + G_a) + K_a\right)x = 0. \tag{1.2}$$

Complex numbers λ and nonzero vectors x for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. The "forward" problem is, of course, to find the eigenvalues and eigenvectors when the coefficient matrices are given. Many authors have devoted to this kind of problem and a series of good results have been made (see, e.g., [1–7]). Generally speaking, very often natural frequencies and mode shapes (eigenvalues and eigenvectors) of an analytical model described by (1.2) do not match very well with experimentally measured frequencies and mode shapes obtained from a real-life vibrating structure. Thus, a vibration engineer needs to update the theoretical analytical model to ensure its validity for future use. In view of in analytical model (1.1) for structure dynamics, the mass and stiffness are, in general, clearly defined by physical parameters. However, the effect of damping and Coriolis forces on structural dynamic systems is not well understood because it is purely dynamics property that cannot be measured statically. Our main interest in this paper is the corresponding inverse problem, given partially measured information about eigenvalues and eigenvectors, we reconstruct the damping and gyroscopic matrices to produce an adjusted analytical model with modal properties that closely match the experimental modal data. Recently, the quadratic inverse eigenvalue problems over the complex field have been well studied and there now exists a wealth of information. Many papers have been written (see, e.g., [8–15]), and a complete book [16] has been devoted to the subject. In the present paper we will consider an inverse problem related to damped gyroscopic second-order systems.

Problem P

Given a pair of matrices (Λ, X) in the form

$$\Lambda = \operatorname{diag}\left\{\lambda_1, \lambda_2, \ldots, \lambda_{2l-1}, \lambda_{2l}, \lambda_{2l+1}, \ldots, \lambda_p\right\} \in \mathbf{C}^{p\times p}, \tag{1.3}$$

$$X = \left[x_1, x_2, \ldots, x_{2l-1}, x_{2l}, x_{2l+1}, \ldots, x_p\right] \in \mathbf{C}^{n\times p}, \tag{1.4}$$

where Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \overline{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \overline{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \ldots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l+1, \ldots, p$, we find symmetric positive semidefinite matrix D and real-valued skew-symmetric matrix G that satisfy the following equation:

$$M_a X\Lambda^2 + (D + G)X\Lambda + K_a X = 0. \tag{1.5}$$

In other words, each pair $(\lambda_t, x_t), t = 1, \ldots, p$, is an eigenpair of the quadratic pencil

$$Q(\lambda) := \lambda^2 M_a + \lambda(D + G) + K_a, \tag{1.6}$$

where $M_a > 0$ and $K_a \geq 0$ are given analytical mass and stiffness matrices.

The goal of this paper is to derive the necessary and sufficient conditions on the spectral information under which the inverse problem is solvable. Our proof is constructive. As a byproduct, numerical algorithm can also be developed thence. A numerical example will be discussed in Section 3.

In this paper we will adopt the following notation. $\mathbf{C}^{m\times n}$, $\mathbf{R}^{m\times n}$ denote the set of all $m \times n$ complex and real matrices, respectively. $\mathbf{OR}^{n\times n}$ denotes the set of all orthogonal matrices in $\mathbf{R}^{n\times n}$. Capital letters $A, B, C, \ldots$ denote matrices, lower case letters denote column vectors, Greek letters denote scalars, $\overline{\alpha}$ denotes the conjugate of the complex number α, A^T denotes the transpose of the matrix A, I_n denotes the $n \times n$ identity matrix, and A^+ denotes the Moore-Penrose generalized inverse of A. We write $A > 0$ $(A \geq 0)$ if A is real symmetric positive definite (positive semi-definite).

2. Solvability Conditions for Problem P

Let $\alpha_i = \mathrm{Re}(\lambda_i)$ (the real part of the complex number λ_i), $\beta_i = \mathrm{Im}(\lambda_i)$ (the imaginary part of the complex number λ_i), $y_i = \mathrm{Re}(x_i)$, $z_i = \mathrm{Im}(x_i)$ for $i = 1, 3, \ldots, 2l-1$. Define

$$\widetilde{\Lambda} = \mathrm{diag}\left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{2l-1} & \beta_{2l-1} \\ -\beta_{2l-1} & \alpha_{2l-1} \end{bmatrix}, \lambda_{2l+1}, \ldots, \lambda_p \right\} \in \mathbf{R}^{p\times p}, \tag{2.1}$$

$$\widetilde{X} = \left[y_1, z_1, \ldots, y_{2l-1}, z_{2l-1}, x_{2l+1}, \ldots, x_p \right] \in \mathbf{R}^{n\times p}, \tag{2.2}$$

$$C = D + G. \tag{2.3}$$

Then the equation of (1.5) can be written equivalently as

$$M_a \widetilde{X} \widetilde{\Lambda}^2 + C \widetilde{X} \widetilde{\Lambda} + K_a \widetilde{X} = 0, \tag{2.4}$$

and the relations of C, D, and G are

$$D = \frac{1}{2}\left(C + C^T\right), \qquad G = \frac{1}{2}\left(C - C^T\right). \tag{2.5}$$

In order to solve the equation of (2.4), we shall introduce some lemmas.

Lemma 2.1 (see [17]). *If $A \in \mathbf{R}^{m\times l}$, $F \in \mathbf{R}^{q\times l}$, then $ZA = F$ has a solution $Z \in \mathbf{R}^{q\times m}$ if and only if $FA^+A = F$. In this case, the general solution of the equation can be described as $Z = FA^+ + L(I_m - AA^+)$, where $L \in \mathbf{R}^{q\times m}$ is an arbitrary matrix.*

Lemma 2.2 (see [18, 19]). *Let $A \in \mathbf{R}^{m\times m}$, $B \in \mathbf{R}^{m\times l}$, then*

$$ZB^T + BZ^T = A \tag{2.6}$$

has a solution $Z \in \mathbf{R}^{m\times l}$ if and only if

$$A = A^T, \quad (I_m - BB^+)A(I_m - BB^+) = 0. \tag{2.7}$$

When condition (2.7) *is satisfied, a particular solution of* (2.6) *is*

$$Z_0 = \frac{1}{2}A(B^+)^T + \frac{1}{2}(I_m - BB^+)A(B^+)^T, \tag{2.8}$$

and the general solution of (2.6) *can be expressed as*

$$Z = Z_0 + 2V - VB^+B - BV^T(B^+)^T - (I_m - BB^+)VB^+B, \tag{2.9}$$

where $V \in \mathbf{R}^{m\times l}$ is an arbitrary matrix.

Lemma 2.3 (see [20]). *Let $\widetilde{H} = [\widetilde{H}_{ij}]$ be a real symmetric matrix partitioned into 2×2 blocks, where $\widetilde{H}_{11}$ and $\widetilde{H}_{22}$ are square submatrices. Then $\widetilde{H}$ is a symmetric positive semi-definite matrix if and only if*

$$\widetilde{H}_{11} \geq 0, \qquad \widetilde{H}_{22} - \widetilde{H}_{21}\widetilde{H}_{11}^{+}\widetilde{H}_{12} \geq 0, \qquad \operatorname{rank}\left(\widetilde{H}_{11}\right) = \operatorname{rank}\left(\left[\widetilde{H}_{11}, \widetilde{H}_{12}\right]\right). \tag{2.10}$$

Lemma 2.3 directly results in the following lemma.

Lemma 2.4. *Let $\widetilde{H} = [\widetilde{H}_{ij}] \in \mathbf{R}^{n\times n}$ be a real symmetric matrix partitioned into 2×2 blocks, where $\widetilde{H}_{11} \in \mathbf{R}^{r\times r}$ is the known symmetric submatrix, and $\widetilde{H}_{12}, \widetilde{H}_{22}$ are two unknown submatrices. Then there exist matrices $\widetilde{H}_{12}, \widetilde{H}_{22}$ such that $\widetilde{H}$ is a symmetric positive semi-definite matrix if and only if $\widetilde{H}_{11} \geq 0$. Furthermore, all submatrices $\widetilde{H}_{12}$, $\widetilde{H}_{22}$ can be expressed as*

$$\widetilde{H}_{12} = \widetilde{H}_{11}Y, \qquad \widetilde{H}_{22} = Y^T\widetilde{H}_{11}Y + H, \tag{2.11}$$

where $Y \in \mathbf{R}^{r\times(n-r)}$ is an arbitrary matrix and $H \in \mathbf{R}^{(n-r)\times(n-r)}$ is an arbitrary symmetric positive semi-definite matrix.

By Lemma 2.1, the equation of (2.4) with respect to unknown matrix $C \in \mathbf{R}^{n\times n}$ has a solution if and only if

$$\left(M_a\widetilde{X}\widetilde{\Lambda}^2 + K_a\widetilde{X}\right)(\widetilde{X}\widetilde{\Lambda})^{+}\widetilde{X}\widetilde{\Lambda} = M_a\widetilde{X}\widetilde{\Lambda}^2 + K_a\widetilde{X}. \tag{2.12}$$

In this case, the general solution of (2.4) can be written as

$$C = C_0 + W\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right), \tag{2.13}$$

where $W \in \mathbf{R}^{n\times n}$ is an arbitrary matrix and

$$C_0 = -\left(M_a\widetilde{X}\widetilde{\Lambda}^2 + K_a\widetilde{X}\right)(\widetilde{X}\widetilde{\Lambda})^+. \tag{2.14}$$

From (2.5) and (2.13) we have

$$W\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) + \left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right)W^T = 2D - C_0 - C_0^T. \tag{2.15}$$

For a fixed symmetric positive semi-definite matrix D, we know, from the lemma (2.2), that the equation of (2.15) has a solution $W \in \mathbf{R}^{n\times n}$ if and only if

$$(\widetilde{X}\widetilde{\Lambda})^T D\widetilde{X}\widetilde{\Lambda} = \frac{1}{2}\left(\widetilde{X}\widetilde{\Lambda}\right)^T\left(C_0 + C_0^T\right)\widetilde{X}\widetilde{\Lambda}. \tag{2.16}$$

Let the singular value decomposition (SVD) of $\widetilde{X}\widetilde{\Lambda}$ be

$$\widetilde{X}\widetilde{\Lambda} = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}P^T = U_1\Sigma P_1^T, \tag{2.17}$$

where $U = [U_1, U_2] \in \mathbf{OR}^{n\times n}$, $P = [P_1, P_2] \in \mathbf{OR}^{p\times p}$, $\Sigma = \operatorname{diag}\{\sigma_1, \ldots, \sigma_r\} > 0$, and define

$$U^T DU = \begin{bmatrix}D_{11} & D_{12}\\ D_{12}^T & D_{22}\end{bmatrix} \text{ with } D_{11} \in \mathbf{R}^{r\times r}. \tag{2.18}$$

Then (2.16) becomes

$$\Sigma D_{11}\Sigma = \frac{1}{2}\Sigma U_1^T\left(C_0 + C_0^T\right)U_1\Sigma. \tag{2.19}$$

Clearly, $D_{11} \geq 0$ if and only if

$$U_1^T\left(C_0 + C_0^T\right)U_1 \geq 0 \tag{2.20}$$

or equivalently,

$$\left(\widetilde{X}\widetilde{\Lambda}\right)^T\left(C_0 + C_0^T\right)\widetilde{X}\widetilde{\Lambda} \geq 0. \tag{2.21}$$

According to Lemma 2.4, we know if condition (2.21) holds, then there are a family of symmetric positive semi-definite matrices

$$D = U\begin{bmatrix} D_{11} & D_{11}Y \\ Y^T D_{11} & Y^T D_{11} Y + H \end{bmatrix} U^T, \tag{2.22}$$

where $D_{11} = (1/2)U_1^T(C_0 + C_0^T)U_1$, $Y \in \mathbf{R}^{r\times(n-r)}$ is an arbitrary matrix, and $H \in \mathbf{R}^{(n-r)\times(n-r)}$ is an arbitrary symmetric positive semi-definite matrix, satisfying the equation of (2.16).

Applying Lemma 2.2 again to the equation of (2.15) yields

$$\begin{aligned} W = W_0 &+ 2V - V\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) \\ &- \left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right)V^T\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) \\ &- \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+ V\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} W_0 = &\frac{1}{2}\left(2D - C_0 - C_0^T\right)\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) \\ &+ \frac{1}{2}\widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\left(2D - C_0 - C_0^T\right)\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) \end{aligned} \tag{2.24}$$

is a particular solution of (2.15) with D the same as in (2.22), and $V \in \mathbf{R}^{n\times n}$ is an arbitrary matrix.

Since $C_0(I_n - \widetilde{X}\widetilde{\Lambda}(\widetilde{X}\widetilde{\Lambda})^+) = 0$, it follows from (2.13) and (2.23) that

$$\begin{aligned} G &= \frac{1}{2}\left(C - C^T\right) \\ &= \frac{1}{2}\left(C_0 - C_0^T\right) + \frac{1}{2}\left(W_0\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) - \left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right)W_0^T\right) \\ &\quad + \left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right)\left(V - V^T\right)\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) \\ &= \frac{1}{2}\left(C_0 - C_0^T\right) + \frac{1}{2}\left(\left(2D - C_0^T\right)\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) - \left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right)(2D - C_0)\right) \\ &\quad + \left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right)\left(V - V^T\right)\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right) \\ &:= G_0 + \left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right)J\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^+\right), \end{aligned} \tag{2.25}$$

where

$$G_0 = \frac{1}{2}\left(C_0 - C_0^T\right) + \frac{1}{2}\left(2D - C_0^T\right)\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^{+}\right) - \frac{1}{2}\left(I_n - \widetilde{X}\widetilde{\Lambda}\left(\widetilde{X}\widetilde{\Lambda}\right)^{+}\right)(2D - C_0), \tag{2.26}$$

and J is an arbitrary skew-symmetric matrix.

By now, we have proved the following result.

Theorem 2.5. *Let $M_a > 0$, $K_a \geq 0$, and let the matrix pair $(X, \Lambda) \in \mathbf{C}^{n\times p} \times \mathbf{C}^{p\times p}$ be given as in* (1.3) *and* (1.4)*. Separate matrices Λ and X into real parts and imaginary parts resulting $\widetilde{\Lambda}$ and $\widetilde{X}$ expressed as in* (2.1) *and* (2.2)*. Let the SVD of $\widetilde{X}\widetilde{\Lambda}$ be* (2.17)*. Then Problem P is solvable if and only if conditions* (2.12) *and* (2.21) *are satisfied, in which case, D and G are given, respectively, by* (2.22) *and* (2.25).

Note that when $\operatorname{rank}(\widetilde{X}\widetilde{\Lambda}) = n$, that is, $\widetilde{X}\widetilde{\Lambda}$ is full row rank, then the arbitrary matrices Y and H in the equation of (2.22) disappear, in this case, D is uniquely determined, and so is G. Thus, we have the following corollary.

Corollary 2.6. *Under the same assumptions as in Theorem 2.5, suppose that rank $(\widetilde{X}\widetilde{\Lambda}) = n$, if condition* (2.12) *and $C_0 + C_0^T \geq 0$ are satisfied. Then there exist unique matrices D and G such that* (1.5) *holds. Furthermore, D and G can be expressed as*

$$D = \frac{1}{2}\left(C_0 + C_0^T\right), \qquad G = \frac{1}{2}\left(C_0 - C_0^T\right). \tag{2.27}$$

3. A Numerical Example

Based on Theorem 2.5 we can state the following algorithm.

Algorithm 3.1. An algorithm for solving Problem P.

(1) Input M_a, K_a, Λ, X.

(2) Separate matrices Λ and X into real parts and imaginary parts resulting $\widetilde{\Lambda}$ and $\widetilde{X}$ given as in (2.1) and (2.2).

(3) Compute the SVD of $\widetilde{X}\widetilde{\Lambda}$ according to (2.17).

(4) If (2.12) and (2.21) hold, then continue, otherwise, go to (1).

(5) Choose matrices $Y \in \mathbf{R}^{r\times(n-r)}$, $H \in \mathbf{R}^{(n-r)\times(n-r)}$ with $H \geq 0$, and $J \in \mathbf{R}^{n\times n}$ with $J^T = -J$.

(6) According to (2.22) and (2.25) calculate D and G.

Example 3.2. Consider a five-DOF system modelled analytically with mass and stiffness matrices given by

$$M_a = \operatorname{diag}\{1,\ 2,\ 5,\ 4,\ 3\},$$
$$K_a = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ -20 & 120 & -35 & 0 & 0 \\ 0 & -35 & 80 & -12 & 0 \\ 0 & 0 & -12 & 95 & -40 \\ 0 & 0 & 0 & -40 & 124 \end{bmatrix}. \tag{3.1}$$

The measured eigenvalue and eigenvector matrices Λ and X are given by

$$\Lambda = \operatorname{diag}\{-1.7894 + 7.6421i \quad -1.7894 - 7.6421i \quad -1.6521 + 3.9178i \quad -1.6521 - 3.9178i\},$$
$$X = \begin{bmatrix} 0.1696 + 0.6869i & 0.1696 - 0.6869i & 0.0245 - 0.0615i & 0.0245 + 0.0615i \\ 0.3906 + 0.5733i & 0.3906 - 0.5733i & -0.0820 - 0.2578i & -0.0820 + 0.2578i \\ 0.0210 - 0.1166i & 0.0210 + 0.1166i & -0.3025 - 0.5705i & -0.3025 + 0.5705i \\ -0.0389 + 0.0079i & -0.0389 - 0.0079i & 0.5205 + 0.2681i & 0.5205 - 0.2681i \\ -0.0486 + 0.0108i & -0.0486 - 0.0108i & 0.1806 + 0.3605i & 0.1806 - 0.3605i \end{bmatrix}. \tag{3.2}$$

According to Algorithm 3.1, it is calculated that conditions (2.12) and (2.21) hold. Thus, by choosing

$$Y = \begin{bmatrix} 0.3742 & 0.3062 & 0.3707 & 0.7067 \end{bmatrix}^T,$$
$$H = 10,$$
$$J = \begin{bmatrix} 0 & -0.4512 & 0.1879 & 0.0747 & -0.4468 \\ 0.4512 & 0 & 0.2956 & -0.0395 & 0.0506 \\ -0.1879 & -0.2956 & 0 & -0.6044 & 0.5844 \\ -0.0747 & 0.0395 & 0.6044 & 0 & 0.1974 \\ 0.4468 & -0.0506 & -0.5844 & -0.1974 & 0 \end{bmatrix}, \tag{3.3}$$

Table 1

(λ_i, x_i)	(λ_1, x_1)	(λ_2, x_2)	(λ_3, x_3)	(λ_4, x_4)
$\mathrm{res}(\lambda_i, x_i)$	8.6580e-014	8.6580e-014	8.2462e-014	8.2462e-014

we can figure out

$$D = \begin{bmatrix} 10.8255 & -8.5715 & -4.6840 & 0.0327 & -7.7270 \\ -8.5715 & 15.9097 & 2.6332 & 1.2234 & 11.2417 \\ -4.6840 & 2.6332 & 9.2185 & -0.5837 & 0.1449 \\ 0.0327 & 1.2234 & -0.5837 & 13.8235 & 3.2361 \\ -7.7270 & 11.2417 & 0.1449 & 3.2361 & 26.5027 \end{bmatrix},$$
$$G = \begin{bmatrix} 0.0000 & -1.0438 & -3.7921 & -0.5470 & -8.6740 \\ 1.0438 & -0.0000 & 6.6747 & -0.7391 & 10.3262 \\ 3.7921 & -6.6747 & -0.0000 & -7.0774 & -6.2101 \\ 0.5470 & 0.7391 & 7.0774 & -0.0000 & 12.6496 \\ 8.6740 & -10.3262 & 6.2101 & -12.6496 & -0.0000 \end{bmatrix}. \tag{3.4}$$

We define the residual as

$$\mathrm{res}(\lambda_i, x_i) = \left\| \left(\lambda_i^2 M_a + \lambda_i (D + G) + K_a \right) x_i \right\|, \tag{3.5}$$

where $\| \cdot \|$ is the Frobenius norm, and the numerical results shown in Table 1.

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