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Research Article

Existence of Weak Solutions to a Class of Degenerate Semiconductor Equations Modeling Avalanche Generation

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We consider the drift-diffusion model with avalanche generation for evolution in time of electron and hole densities n, p coupled with the electrostatic potential ψ in a semiconductor device. We also assume that the diffusion term is degenerate. The existence of local weak solutions to this Dirichlet-Neumann mixed boundary value problem is obtained.

1. Introduction

In this paper, we consider the following degenerate semiconductor equations modeling avalanche generation:

$$-\nabla\cdot(\nabla\psi) = p - n + C(x), \tag{1.1}$$

$$n_t - \nabla\cdot J_n = R(n,p) + g, \qquad J_n = \nabla(n^\gamma) - \mu_1 n\nabla\psi, \tag{1.2}$$

$$p_t + \nabla\cdot J_p = R(n,p) + g, \qquad -J_p = \nabla(p^\gamma) + \mu_2 p\nabla\psi \tag{1.3}$$

with initial and boundary conditions

$$(\psi,n,p) = (\overline{\psi},\overline{n},\overline{p}), \quad (x,t)\in\Sigma_D\equiv\Gamma_D\times(0,T), \tag{1.4}$$

$$\left(\frac{\partial\psi}{\partial\eta},\frac{\partial n}{\partial\eta},\frac{\partial p}{\partial\eta}\right) = (0,0,0), \quad (x,t)\in\Sigma_N\equiv\Gamma_N\times(0,T), \tag{1.5}$$

$$(n,p) = (n_0,p_0), \quad x\in\Omega,\ t=0. \tag{1.6}$$

Here the unknowns ψ, n, and p denote the electrostatic potential, the electron density, and the hole density, respectively. The boundary $\partial\Omega$ consists of two disjoint subsets Γ_D and Γ_N. The carrier densities and the potential are fixed at Γ_D (Ohmic contacts), whereas Γ_N models the union of insulating boundary segments. J_n represents the electron current, and J_p is the analogously defined physical quantity of the positively charged holes. Function $C(x)$ denotes the doping profile (fixed charged background ions) characterizing the semiconductor under consideration, while the term $g = \alpha_1(\nabla\psi)|J_n| + \alpha_2(\nabla\psi)|J_p|$ models the effect of impact ionization (avalanche generation of charged particles) (cf. [1, 2] for details). $R(n,p) = r(n,p)(1-np)$ is the net recombination-generation rate, where r characterizes the mechanism of particle transition. The constant γ is the adiabatic or isothermal (if $\gamma = 1$) exponent. The regime $0 < \gamma < 1$ describes a fast diffusion process in the electron (hole) density, whereas $1 < \gamma \leq 5$ is related to slow diffusion.

The standard drift-diffusion model corresponding to $\gamma = 1$ has been mathematically and numerically investigated in many papers (see [3–6]). Existence and uniqueness of weak solutions have been shown. The standard model can be derived from Boltzmann's equation once assumed that the semiconductor device is in the low injection regime, that is, for small absolute values of the applied voltage. In [7] Jüngel showed that in the high-injection regime diffusion terms are no longer linear. A useful choice for γ is $\gamma = 5/3$. In this case, the parabolic equations (1.2) and (1.3) become of degenerate type, and existence of solutions does not follow from standard theory. Recently, many authors [8–10] have studied the existence and uniqueness of weak solutions of this type of degenerate semiconductor equations without avalanche generation term. In [9], the degenerate semiconductor equations based on Fermi-Dirac statistics were introduced by Jüngel for the first time. The existence and uniqueness results are shown under the assumption that the solution ψ of Poisson equation with Dirichlet-Neumann mixed boundary conditions had the regularity $\psi \in W^{2,r}(\Omega)\,(r > N)$, this amounts to a geometric condition on Ω, for example $\Omega \in C^{1,1}$ and $\Gamma_D \cap \Gamma_N = \emptyset$. ([11, Theorem 3.29]). Then Guan and Wu [8] obtain similar results without the assumption above.

There are some papers concerning the semiconductor equations modeling avalanche generation. For instance, the existence of weak solutions of nondegenerate stationary problem has been investigated in [12, 13]. When $\gamma = 1$, that is, the diffusion term is not degenerate, the authors [14] obtained the existence of local weak solutions of problem (1.1)–(1.6).

Our main goal in this paper is to study the existence of weak solutions of problem (1.1)–(1.6). In contrast to the above works, degeneration of diffusion term we are going to study introduces significant new technical difficulties to estimate the avalanche term.

We make the following assumptions:

(H1) $\Omega \subset R^N\,(N = 1,2,3)$ is bounded and $\partial\Omega \in C^{0,1}$, whose outward normal vector is η and $\overline{\partial\Omega} = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \Gamma_D \cap \Gamma_N = \emptyset$, $\text{meas}_{N-1}(\Gamma_D) > 0$;

(H2) $C(x) \in L^\infty(\Omega)$;

(H3) $r(n,p)$ is a locally Lipschitz continuous function defined for (n,p) and $0 \leq r(n,p) \leq \overline{r} < \infty$;

(H4) $\alpha_i \in C(R^N)$, $0 \leq \alpha_i(\xi) \leq \alpha_{i0} = \text{const.} < +\infty$, for all $\xi \in R^N\,(i = 1,2)$;

(H5) $\overline{n}, \overline{p} \in W^{1,\infty}(\Omega) \cap L^\infty(\Omega)$, $\overline{\psi} \in H^1(\Omega) \cap L^\infty(\Omega)$, and $\overline{n}, \overline{p} \geq 0$ in Ω;

(H6) $n_0, p_0 \in L^\infty(\Omega)$ and $n_0, p_0 \geq 0$ a.e. in Ω.

Let

$$Y = \left\{ \omega \in H^1(\Omega) | \omega|_{\Gamma_D} = 0 \right\}. \tag{1.7}$$

Definition 1.1. (ψ, n, p) is called the weak solution to the problem (1.1)–(1.6) if $n^\gamma \in \overline{n}^\gamma + L^2(0,T;Y)$, $p^\gamma \in \overline{p}^\gamma + L^2(0,T;Y)$, $\psi \in \overline{\psi} + L^2(0,T;Y)$, $n_t, p_t \in L^2(0,T;Y^*)$, $n|_{t=0} = n_0$, $p|_{t=0} = p_0$, and there hold

$$\begin{aligned}
&\int_\Omega \nabla\psi \cdot \nabla\zeta \mathrm{d}x = \int_\Omega (p - n + C(x))\zeta \mathrm{d}x, \quad \forall t \in (0,T),\ \forall \zeta \in Y \\
&\int_0^T \langle n_t, \zeta\rangle_{Y^*,Y} + \int_0^T \int_\Omega (\nabla(n^\gamma) - \mu_1 n \nabla\psi) \cdot \nabla\zeta \mathrm{d}x\, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega [r(n,p)(1 - np) + g]\zeta \mathrm{d}x\, \mathrm{d}t, \quad \forall \zeta \in L^2(0,T;Y), \\
&\int_0^T \langle p_t, \zeta\rangle_{Y^*,Y} + \int_0^T \int_\Omega (\nabla(p^\gamma) + \mu_2 p \nabla\psi) \cdot \nabla\zeta \mathrm{d}x\, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega [r(n,p)(1 - np) + g]\zeta \mathrm{d}x\, \mathrm{d}t, \quad \forall \zeta \in L^2(0,T;Y).
\end{aligned} \tag{1.8}$$

Our main result in this paper is as follows.

Theorem 1.2. *Under hypotheses (H1)–(H6), there exists at least one local weak solution to the problem* (1.1)–(1.6).

2. Approximate Problem

For simplicity, we assume that $\mu_i = 1, i = 1, 2$. As [14], we first construct the following bound approximate sequence g_τ of avalanche generation term g:

$$g_\tau(n, p, \nabla n, \nabla p, \nabla\psi) = (\alpha_1(\nabla\psi)|J_n| + \alpha_2(\nabla\psi)|J_p|) \cdot \left[1 + \tau(\alpha_1(\nabla\psi)|J_n| + \alpha_2(\nabla\psi)|J_p|)\right]^{-1}, \tag{2.1}$$

here $0 < \tau < 1$. Obviously, $0 < g_\tau < 1/\tau$.

Now we introduce the following approximate problem with the initial and boundary conditions (1.4)–(1.6):

$$-\nabla \cdot (\nabla\psi) = p - n + C(x), \tag{2.2}$$

$$n_t - \nabla \cdot (\nabla n^\gamma - n\nabla\psi) = r(n,p)(1 - np) + g_\tau(n, p, \nabla n, \nabla p, \nabla\psi), \tag{2.3}$$

$$p_t - \nabla \cdot (\nabla p^\gamma + p\nabla\psi) = r(n,p)(1 - np) + g_\tau(n, p, \nabla n, \nabla p, \nabla\psi). \tag{2.4}$$

This section is devoted to the proof of global existence of weak solutions to the above approximate problem (2.2)–(2.4), (1.4)–(1.6). We will prove the following existence theorem.

Theorem 2.1. *Under hypotheses (H1)–(H6), there exists at least one global weak solution to the problem* (2.2)–(2.4)*,* (1.4)–(1.6)*.*

The proof is based on Schauder's fixed pointed theorem. The main difficulty in the proof is that problem (2.2)–(2.4), (1.4)–(1.6) is degenerated at points where $n, p = 0$. This difficulty leads us to consider the following auxiliary regularized problem with the initial and boundary conditions (1.4)–(1.6):

$$-\nabla \cdot (\nabla \psi) = p_k - n_k + C(x), \tag{2.5}$$

$$n_t - \nabla \cdot \left(\left(\gamma n_k^{\gamma-1} + \varepsilon \right) \nabla n - n_k \nabla \psi \right) = r(n,p)(1 - n p_k) + h(n, p, \nabla n, \nabla p, \nabla \psi), \tag{2.6}$$

$$p_t - \nabla \cdot \left(\left(\gamma p_k^{\gamma-1} + \varepsilon \right) \nabla p + p_k \nabla \psi \right) = r(n,p)(1 - n_k p) + h(n, p, \nabla n, \nabla p, \nabla \psi), \tag{2.7}$$

where $s_k = \min\{\max\{0, s\}, k\}$ and

$$\begin{aligned} h(\xi, \eta, \xi', \eta', \zeta) = {} & \left(\alpha_1(\zeta) \left| \gamma \xi_k^{\gamma-1} \xi' - \xi_k \zeta \right| + \alpha_2(\zeta) \left| \gamma \eta_k^{\gamma-1} \eta' + \eta_k \zeta \right| \right) \\ & \times \left[1 + \tau \left(\alpha_1(\zeta) \left| \gamma \xi_k^{\gamma-1} \xi' - \xi_k \zeta \right| + \alpha_2(\zeta) \left| \gamma \eta_k^{\gamma-1} \eta' + \eta_k \zeta \right| \right) \right]^{-1} \end{aligned} \tag{2.8}$$

for any $\xi, \eta \in R$ and $\xi', \eta', \zeta \in R^N$.

Let

$$\mathcal{K} = \left\{ x \in L^2(Q_T) \mid \|x\|_{L^2(Q_T)} \leq R \right\}, \tag{2.9}$$

and $\widetilde{n}, \widetilde{p} \in \mathcal{K}$. It is obvious that $\mathcal{K}$ is a closed convex set and weakly compact in $L^2(Q_T)$.

The theory of linear elliptic boundary value problems [15] gives a unique ψ such that

$$-\nabla \cdot (\nabla \psi) = \widetilde{p}_k - \widetilde{n}_k + C(x), \tag{2.10}$$

$$\psi \mid_{\Sigma_D} = \overline{\psi}, \qquad \left. \frac{\partial \psi}{\partial \eta} \right|_{\Sigma_N} = 0, \tag{2.11}$$

$$\|\nabla \psi\|_{L^2(Q_T)} + \|\psi\|_{L^\infty(Q_T)} \leq C, \tag{2.12}$$

where C is dependent on k, Q_T and the L^∞ norms for $C(x)$ and $\overline{\psi}$, but not on R.

Next, for the unique weak solution ψ to problem (2.10)–(2.11), we consider the following problem:

$$n_t - \nabla \cdot \left(\left(\gamma \tilde{n}_k^{\gamma-1} + \varepsilon \right) \nabla n - \tilde{n}_k \nabla \psi \right) = r(\tilde{n}, \tilde{p})(1 - n\tilde{p}_k) + h(\tilde{n}, \tilde{p}, \nabla n, \nabla p, \nabla \psi), \tag{2.13}$$

$$p_t - \nabla \cdot \left(\left(\gamma \tilde{p}_k^{\gamma-1} + \varepsilon \right) \nabla p + \tilde{p}_k \nabla \psi \right) = r(\tilde{n}, \tilde{p})(1 - \tilde{n}_k p) + h(\tilde{n}, \tilde{p}, \nabla n, \nabla p, \nabla \psi), \tag{2.14}$$

$$(n, p)|_{\Sigma_D} = (\overline{n}, \overline{p}), \qquad \left(\frac{\partial n}{\partial \eta}, \frac{\partial p}{\partial \eta} \right) \Big|_{\Sigma_N} = (0, 0), \tag{2.15}$$

$$(n, p) = (n_0, p_0), \quad x \in \Omega, \; t = 0. \tag{2.16}$$

Lemma 2.2. *Under hypotheses (H1)–(H6), there exists one global and unique weak solution to the problem* (2.13)–(2.16).

Further there are bounds on $\|n\|_{L^2(0,T;H^1(\Omega))}$ and $\|n_t\|_{L^2(0,T;Y^*)}$ which depend on $\varepsilon, k, \tau, Q_T$, and the known data, but not on R. Similar estimates also hold for p.

Proof. We begin by choosing a constant ρ such that

$$\rho \geq \frac{\max\{\alpha_{10}, \alpha_{20}\}^2 \gamma^2 k^{2\gamma-2}}{2\min\{\alpha_{11}, \alpha_{21}\}} \tag{2.17}$$

and observe that (n, p) satisfies (2.13)–(2.16) if and only if $(U, V) = (e^{-\rho t} n, e^{-\rho t} p)$ satisfies

$$U_t - \nabla \cdot (\alpha_{11} \nabla U - e^{-\rho t} \tilde{n}_k \nabla \psi) + \alpha_{12} U = e^{-\rho t} H(e^{\rho t} \nabla U, e^{\rho t} \nabla V) + e^{-\rho t} r(\tilde{n}, \tilde{p}), \tag{2.18}$$

$$V_t - \nabla \cdot (\alpha_{21} \nabla V + e^{-\rho t} \tilde{p}_k \nabla \psi) + \alpha_{22} V = e^{-\rho t} H(e^{\rho t} \nabla U, e^{\rho t} \nabla V) + e^{-\rho t} r(\tilde{n}, \tilde{p}), \tag{2.19}$$

$$(U, V)|_{\Sigma_D} = (e^{-\rho t} \overline{n}, e^{-\rho t} \overline{p}), \qquad \left(\frac{\partial U}{\partial \eta}, \frac{\partial V}{\partial \eta} \right) \Big|_{\Sigma_N} = (0, 0), \tag{2.20}$$

$$(U, V) = (n_0, p_0), \quad x \in \Omega, \; t = 0, \tag{2.21}$$

or if and only if $(u, v) = (U - e^{-\rho t} \overline{n}, V - e^{-\rho t} \overline{p})$ satisfies

$$u_t - \nabla \cdot (\alpha_{11} \nabla u) + \alpha_{12} u = e^{-\rho t} H(e^{\rho t} \nabla u + \nabla \overline{n}, e^{\rho t} \nabla v + \nabla \overline{p}) + F_1, \tag{2.22}$$

$$v_t - \nabla \cdot (\alpha_{21} \nabla v) + \alpha_{22} v = e^{-\rho t} H(e^{\rho t} \nabla u + \nabla \overline{n}, e^{\rho t} \nabla v + \nabla \overline{p}) + F_2, \tag{2.23}$$

$$(u, v)|_{\Sigma_D} = (0, 0), \qquad \left(\frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} \right) \Big|_{\Sigma_N} = \left(-e^{-\rho t} \frac{\partial \overline{n}}{\partial \eta}, -e^{-\rho t} \frac{\partial \overline{p}}{\partial \eta} \right), \tag{2.24}$$

$$(u, v) = (n_0 - \overline{n}, p_0 - \overline{p}), \quad x \in \Omega, \; t = 0. \tag{2.25}$$

where

$$
\begin{aligned}
(\alpha_{ij})_{2\times 2} &= \begin{pmatrix} \gamma\tilde{n}_k^{\gamma-1}+\varepsilon & \rho+r(\tilde{n},\tilde{p})\tilde{p}_k \\ \gamma\tilde{p}_k^{\gamma-1}+\varepsilon & \rho+r(\tilde{n},\tilde{p})\tilde{n}_k \end{pmatrix}, \\
H(\nabla u,\nabla v) &= h(\tilde{n},\tilde{p},\nabla u,\nabla v,\nabla\psi), \\
\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} &= \begin{pmatrix} e^{-\rho t}\left(\nabla\cdot(\alpha_{11}\nabla\overline{n}-\tilde{n}_k\nabla\psi)+(\rho-\alpha_{12})\overline{n}+r(\tilde{n},\tilde{p})\right) \\ e^{-\rho t}\left(\nabla\cdot(\alpha_{21}\nabla\overline{p}+\tilde{p}_k\nabla\psi)+(\rho-\alpha_{22})\overline{p}+r(\tilde{n},\tilde{p})\right) \end{pmatrix}.
\end{aligned}
\tag{2.26}
$$

Clearly, $(F_1,F_2)\in(L^2(0,T;Y^*))^2$.

First, we prove the uniqueness of weak solution to the problem (2.13)–(2.16) which is equivalent to (2.18)–(2.21). Let $(U_i,V_i), i=1,2$ be two weak solutions to the problem (2.18)–(2.21), then $(N,P)=(U_1-U_2,V_1-V_2)$ satisfies

$$N_t-\nabla\cdot(\alpha_{11}\nabla N)+\alpha_{12}N=e^{-\rho t}G(\nabla U_1,\nabla U_2,\nabla V_1,\nabla V_2), \tag{2.27}$$

$$P_t-\nabla\cdot(\alpha_{21}\nabla P)+\alpha_{22}P=e^{-\rho t}G(\nabla U_1,\nabla U_2,\nabla V_1,\nabla V_2), \tag{2.28}$$

$$(N,P)|_{\Sigma_D}=(0,0),\qquad \left(\frac{\partial N}{\partial\eta},\frac{\partial P}{\partial\eta}\right)\bigg|_{\Sigma_N}=(0,0), \tag{2.29}$$

$$(U,V)=(0,0),\quad x\in\Omega,\ t=0, \tag{2.30}$$

here

$$G(\nabla U_1,\nabla U_2,\nabla V_1,\nabla V_2)=H\left(e^{\rho t}\nabla U_1,e^{\rho t}\nabla V_1\right)-H\left(e^{\rho t}\nabla U_2,e^{\rho t}\nabla V_2\right) \tag{2.31}$$

such that

$$
\begin{aligned}
G(\nabla U_1,\nabla U_2,\nabla V_1,\nabla V_2) &\le \alpha_1(\nabla\psi)\left|\left|\gamma\tilde{n}_k^{\gamma-1}e^{\rho t}\nabla U_1-\tilde{n}_k\nabla\psi\right|-\left|\gamma\tilde{n}_k^{\gamma-1}e^{\rho t}\nabla U_2-\tilde{n}_k\nabla\psi\right|\right| \\
&\quad+\alpha_2(\nabla\psi)\left|\left|\gamma\tilde{p}_k^{\gamma-1}e^{\rho t}\nabla V_1+\tilde{p}_k\nabla\psi\right|-\left|\gamma\tilde{p}_k^{\gamma-1}e^{\rho t}\nabla V_2+\tilde{p}_k\nabla\psi\right|\right| \\
&\le \max\{\alpha_{10},\alpha_{20}\}\gamma k^{\gamma-1}e^{\rho t}(|\nabla N|+|\nabla P|).
\end{aligned}
\tag{2.32}
$$

Take N,P as test functions in (2.27), (2.28), respectively. By (2.32) and Hölder inequality,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(|N(t)|^2+|P(t)|^2\right)+\min\{\alpha_{11},\alpha_{21}\}\int_\Omega\left(|\nabla N|^2+|\nabla P|^2\right)+\rho\int_\Omega\left(N^2+P^2\right) \\
&\quad\le \max\{\alpha_{10},\alpha_{20}\}\gamma k^{\gamma-1}\int_\Omega(|\nabla N|+|\nabla P|)(|N|+|P|) \\
&\quad\le \frac{1}{2}\min\{\alpha_{11},\alpha_{21}\}\int_\Omega\left(|\nabla N|^2+|\nabla P|^2\right)+\frac{\max\{\alpha_{10},\alpha_{20}\}^2\gamma^2k^{2\gamma-2}}{2\min\{\alpha_{11},\alpha_{21}\}}\int_\Omega\left(N^2+P^2\right).
\end{aligned}
\tag{2.33}
$$

Thus the uniqueness is established by Gronwall's inequality.

We are now in a position to prove the existence result. Define $W = L^2(0,T;Y)$, $\mathcal{V} = W \times W$. Clearly, $\mathcal{V}$ is a Hilbert space with respect to the scalar product

$$((u,v),(\xi,\eta)) = \int_0^T \int_\Omega (\nabla u \cdot \nabla \xi + \nabla v \cdot \nabla \eta). \tag{2.34}$$

Set, for $(u,v),(\xi,\eta) \in \mathcal{V}$,

$$\begin{aligned}\langle \mathcal{A}(u,v),(\xi,\eta)\rangle &= \int_0^T \int_\Omega (\alpha_{11}\nabla u \cdot \nabla \xi + \alpha_{21}\nabla v \cdot \nabla \eta) + \int_0^T \int_\Omega (\alpha_{12} u\xi + \alpha_{22} v\eta) \\ &\quad + \int_0^T \int_\Omega e^{-\rho t} H\left(e^{\rho t}\nabla u + \nabla \overline{n}, e^{\rho t}\nabla v + \nabla \overline{p}\right)(\xi + \eta).\end{aligned} \tag{2.35}$$

The operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ is well defined and bounded (because $0 \le H \le 1/\tau$). To prove the existence result by using [16, Theorem 30.A], it suffices to verify that the operator $\mathcal{A}$ is hemicontinuous, monotone, and coercive.

Note that

$$\begin{aligned}&e^{-\rho t}\left|H\left(e^{\rho t}\nabla(u+\lambda\varphi) + \nabla\overline{n}, e^{\rho t}\nabla(v+\lambda\phi) + \nabla\overline{p}\right) - H\left(e^{\rho t}\nabla u + \nabla\overline{n}, e^{\rho t}\nabla v + \nabla\overline{p}\right)\right| \\ &\quad \le \max\{\alpha_{10},\alpha_{20}\}\lambda\gamma k^{\gamma-1}\left(|\nabla\varphi| + |\nabla\phi|\right)\end{aligned} \tag{2.36}$$

for any $(u,v),(\varphi,\phi) \in \mathcal{V}$. The hemicontinuity of $\mathcal{A}$ is easily obtained by the standard method.

For the monotone, we first notice that

$$\begin{aligned}&\int_0^T \int_\Omega \left[H\left(e^{\rho t}\nabla u_1 + \nabla\overline{n}, e^{\rho t}\nabla v_1 + \nabla\overline{p}\right) - H\left(e^{\rho t}\nabla u_2 + \nabla\overline{n}, e^{\rho t}\nabla v_2 + \nabla\overline{p}\right)\right]\left[(u_1-u_2) + (v_1-v_2)\right] \\ &\quad \ge -\max\{\alpha_{10},\alpha_{20}\}\gamma k^{\gamma-1}\int_0^T \int_\Omega (|\nabla(u_1-u_2)| + |\nabla(v_1-v_2)|)(|u_1-u_2| + |v_1-v_2|) \\ &\quad \ge -\frac{\min\{\alpha_{11},\alpha_{21}\}}{2}\int_0^T \int_\Omega \left(|\nabla(u_1-u_2)|^2 + |\nabla(v_1-v_2)|^2\right) \\ &\qquad - \frac{\max\{\alpha_{10},\alpha_{20}\}^2\gamma^2 k^{2\gamma-2}}{2\min\{\alpha_{11},\alpha_{21}\}}\int_0^T \int_\Omega \left[(u_1-u_2)^2 + (v_1-v_2)^2\right].\end{aligned} \tag{2.37}$$

Hence

$$\begin{aligned}&\langle \mathcal{A}(u_1,v_1) - \mathcal{A}(u_2,v_2),(u_1-u_2,v_1-v_2)\rangle \\ &\quad \ge \frac{\min\{\alpha_{11},\alpha_{21}\}}{2}\int_0^T \int_\Omega \left(|\nabla(u_1-u_2)|^2 + |\nabla(v_1-v_2)|^2\right) \\ &\qquad + \left(\rho - \frac{\max\{\alpha_{10},\alpha_{20}\}^2\gamma^2 k^{2\gamma-2}}{2\min\{\alpha_{11},\alpha_{21}\}}\right)\int_0^T \int_\Omega \left[(u_1-u_2)^2 + (v_1-v_2)^2\right].\end{aligned} \tag{2.38}$$

By the choice of ρ, we can easily obtain the monotone of $\mathscr{A}$. Moreover, from (2.38) we also know that the operator $\mathscr{A}$ is coercive.

Therefore, there exists a unique $(u,v) \in \mathcal{V}$ with $(u_t, v_t) \in \mathcal{V}^*$ such that $(u_0, v_0) = (n_0 - \overline{n}, p_0 - \overline{p})$ in Ω and

$$\left\langle (u_t, v_t) + \mathscr{A}(u,v), (\xi,\eta) \right\rangle = \left\langle (F_1, F_2), (\xi,\eta) \right\rangle, \quad \forall (\xi,\eta) \in \mathcal{V}. \tag{2.39}$$

Especially,

$$\begin{gathered}\langle (u_t, v_t) + \mathscr{A}(u,v), (\xi, 0) \rangle = \langle (F_1, F_2), (\xi, 0) \rangle, \\ \langle (u_t, v_t) + \mathscr{A}(u,v), (0,\eta) \rangle = \langle (F_1, F_2), (0,\eta) \rangle.\end{gathered} \tag{2.40}$$

That is, (u,v) is a weak solution to the problem (2.22)–(2.25).

Finally, noting that $0 \le h \le 1/\tau$ we can easily establish the bounds on $\|\nabla u\|_{L^2(0,T;H^1(\Omega))}$ and $\|u_t\|_{L^2(0,T;Y^*)}$ by the standard energy estimate. □

Lemma 2.3. *Under hypotheses (H1)–(H6), there exists at least one global weak solution to the problem* (2.5)–(2.7), (1.4)–(1.6).

Proof. We define the mapping S as

$$S : \mathscr{K}^2 \longrightarrow \left(L^2(Q_T)\right)^2, \qquad (\tilde{n}, \tilde{p}) \longmapsto (n,p) \tag{2.41}$$

with (n,p) solution of (2.13)–(2.16). From Lemma 2.2, we know that S is well defined and compact. Indeed, (n,p) lies in a bounded sunsets of $(L^2(0,T;H^1(\Omega)))^2$, and (n_t, p_t) lies in a bounded subset of $(L^2(0,T;Y^*))^2$. Since the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we conclude from Aubin's lemma that S is relatively compact in $(L^2(Q_T))^2$. And for given τ and k, $S(\mathscr{K}^2) \hookrightarrow \mathscr{K}^2$ holds if we choose R large enough.

To apply Schauder's fixed point theorem, we still need to prove that the mapping S is continuous. Consider any sequence $(\tilde{n}_j, \tilde{p}_j) \subset \mathscr{K}^2 \to (\tilde{n}, \tilde{p})$ strongly in $(L^2(Q_T))^2$ and let $S(\tilde{n}_j, \tilde{p}_j) = (n_j, p_j)$. Since $S(\mathscr{K}^2)$ is relatively compact in $(L^2(Q_T))^2$ and bounded in $(L^2(0,T;H^1(\Omega)))^2$, we can extract subsequences such that

$$\begin{aligned}&\left((n_j)_t, (p_j)_t\right) \longrightarrow (n_t, p_t) \quad \text{weakly in } L^2(0,T;Y^*), \\ &n_j \longrightarrow n, \quad p_j \longrightarrow p \quad \text{strongly in } L^2(Q_T), \\ &n_j \longrightarrow n, \quad p_j \longrightarrow p \quad \text{weakly in } L^2\left(0,T;H^1(\Omega)\right), \\ &n_j \longrightarrow n, \quad p_j \longrightarrow p \quad \text{a.e. in } Q_T.\end{aligned} \tag{2.42}$$

We only have to show $S(\tilde{n},\tilde{p}) = (n,p)$. To do this, we only need to prove

$$\int_0^T\int_\Omega h(\tilde{n}_j,\tilde{p}_j,\nabla n_j,\nabla p_j,\nabla\psi_j)\cdot\zeta \longrightarrow \int_0^T\int_\Omega h(\tilde{n},\tilde{p},\nabla n,\nabla p,\nabla\psi)\cdot\zeta, \tag{2.43}$$

where $\zeta \in L^2(0,T;Y)$ is a test function, and ψ_j,ψ are solutions of (2.10)–(2.11) corresponding to $(\tilde{n}_j,\tilde{p}_j),(\tilde{n},\tilde{p})$, respectively. The reminder of convergence proof is standard (details see [8] or [9]). Use $n_j - n \in L^2(0,T;Y)$ as test function in a modification of (2.13) in which the functions $\tilde{n},\tilde{p}$ have been replaced by $\tilde{n}_j,\tilde{p}_j$, respectively. Then we have

$$\begin{aligned}
\varepsilon\int_0^T\int_\Omega|\nabla(n_j-n)|^2 \le{}& -\int_0^T\langle n_t,n_j-n\rangle_{Y^*,Y} - \varepsilon\int_0^T\int_\Omega \nabla n\cdot\nabla(n_j-n)\\
&+\int_0^T\int_\Omega(\tilde{n}_j)_k\nabla\psi_j\cdot\nabla(n_j-n) + \int_0^T\int_\Omega r(\tilde{n}_j,\tilde{p}_j)\Big(1-n_j(\tilde{p}_j)_k\Big)(n_j-n)\\
&+\max\{\alpha_{01},\alpha_{02}\}\max\Big\{\gamma k^{\gamma-1},2k\Big\}\int_0^T\int_\Omega\left(|\nabla n_j|+|\nabla p_j|+|\nabla\psi_j|\right)(n_j-n)\\
\le{}& -\int_0^T\langle n_t,n_j-n\rangle_{Y^*,Y} - \varepsilon\int_0^T\int_\Omega \nabla n\cdot\nabla(n_j-n)\\
&+\frac{\varepsilon}{4}\Big(\|\nabla(n_j-n)\|^2_{L^2(Q_T)}+\|\nabla(p_j-p)\|^2_{L^2(Q_T)}\Big)+C\int_0^T\int_\Omega(1+|n_j|)|n_j-n|\\
&+C(\varepsilon)\left(\|\nabla(\psi_j-\psi)\|^2_{L^2(Q_T)}+\left|\int_0^T\int_\Omega\nabla\psi\cdot\nabla(n_j-n)\right|+\|n_j-n\|^2_{L^2(Q_T)}\right)\\
&+C(\varepsilon)\int_0^T\int_\Omega(|\nabla n|+|\nabla p|+|\nabla\psi|)|n_j-n|.
\end{aligned} \tag{2.44}$$

A similar estimate holds for $\varepsilon\int_0^T\int_\Omega|\nabla(p_j-p)|^2$. Then adding the two inequalities and using $\nabla\psi_j \to \nabla\psi$ strongly in $L^2(Q_T)$ (details see [9]) and (2.42), we conclude that $\|\nabla(n_j-n)\|_{L^2(Q_T)} + \|\nabla(p_j-p)\|_{L^2(Q_T)} \to 0$. This implies that $(\nabla n_j,\nabla p_j) \to (\nabla n,\nabla p)$ a.e. in Q_T. Then we can easily prove (2.43) by using Vitali's theorem.

Now existence of a fixed point of S follows which is a solution of (2.5)–(2.7), (1.4)–(1.6). □

To obtain the existence result of problem (2.2)–(2.4), (1.4)–(1.6), the following L^∞ estimates on n,p,ψ uniformly in ε are necessary.

Lemma 2.4. *The solutions of problem* (2.5)–(2.7), (1.4)–(1.6) *satisfy the estimates*

$$0 \le n(x,t),\quad p(x,t) \le C,\quad \text{a.e. } (x,t)\in Q_T, \tag{2.45}$$

where C is dependent on τ, Q_T and the known data, but not on ε.

Proof. By taking $n^- = \min\{n, 0\} \in L^2(0, T; Y)$ as test function in (2.6), we have

$$\frac{1}{2}\int_\Omega n^-(t)^2 + \varepsilon\int_0^T\int_\Omega |\nabla n^-|^2 \le \int_0^T\int_\Omega n_k \nabla\psi\cdot\nabla n^- - \int_0^T\int_\Omega r(n,p)p_k {n^-}^2 + \int_0^T\int_\Omega (r(n,p)+h)n^-. \tag{2.46}$$

By taking into account $n_k = 0$ in $\{n \le 0\}$ and the nonnegativity of r and h we obtain

$$\frac{1}{2}\int_\Omega n^-(t)^2 \le 0, \tag{2.47}$$

and thus $n(x,t) \ge 0$ a.e. in Q_T. Similarly, we have $p(x,t) \ge 0$ a.e. in Q_T.

To obtain the upper bound set

$$M = \max\left\{\|\overline{n}\|_{L^\infty(\Omega)}, \|n_0\|_{L^\infty(\Omega)}, \|\overline{p}\|_{L^\infty(\Omega)}, \|p_0\|_{L^\infty(\Omega)}\right\}, \quad k \ge M \tag{2.48}$$

and use $(n_k - M)^{+^q}$ as test function in (2.6), then

$$\begin{aligned}
&\frac{1}{q+1}\int_\Omega (n_k - M)^{+^{(q+1)}} + \varepsilon\int_0^T\int_\Omega q(n_k - M)^{+^{(q-1)}}|\nabla(n_k - M)^+|^2 \\
&\le \int_0^T\int_\Omega \nabla\psi\cdot\nabla\left(\frac{q}{q+1}(n_k - M)^{+^{(q+1)}} + M(n_k - M)^{+^q}\right) \\
&\quad + \int_0^T\int_\Omega (r(n,p)(1 - np_k) + h)(n_k - M)^{+^q} \\
&\le \int_0^T\int_\Omega (p_k - n_k + C(x))\left(\frac{q}{q+1}(n_k - M)^{+^{(q+1)}} + M(n_k - M)^{+^q}\right) + \int_0^T\int_\Omega \left(\overline{r} + \frac{1}{\tau}\right)(n_k - M)^{+^q}.
\end{aligned} \tag{2.49}$$

Adding the equality (2.49) for n and a similar inequality for p, we get

$$\begin{aligned}
&\frac{1}{q+1}\int_\Omega \left((n_k - M)^{+^{(q+1)}} + (p_k - M)^{+^{(q+1)}}\right) \\
&\le \frac{q}{q+1}\int_0^T\int_\Omega (p_k - n_k)\left((n_k - M)^{+^{(q+1)}} - (p_k - M)^{+^{(q+1)}}\right) \\
&\quad + M\int_0^T\int_\Omega (p_k - n_k)\left((n_k - M)^{+^q} - (p_k - M)^{+^q}\right) \\
&\quad + \frac{q}{q+1}\int_0^T\int_\Omega C(x)\left((n_k - M)^{+^{(q+1)}} - (p_k - M)^{+^{(q+1)}}\right) \\
&\quad + \int_0^T\int_\Omega \left(\overline{r} + \frac{1}{\tau} + MC(x)\right)\left((n_k - M)^{+^q} - (p_k - M)^{+^q}\right).
\end{aligned} \tag{2.50}$$

Noticing that

$$(p_k - n_k)\Big((n_k - M)^{+^\theta} - (p_k - M)^{+^\theta}\Big) \le 0, \quad \forall \theta > 1 \tag{2.51}$$

and applying Hölder inequality, we further have

$$\begin{aligned} &\|(n_k - M)^+\|^{q+1}_{L^{q+1}(\Omega)} + \left\|(p_k - M)^+\right\|^{q+1}_{L^{q+1}(\Omega)} \\ &\quad \le C(q+1)\int_0^T \left(\|(n_k - M)^+\|^{q+1}_{L^{q+1}(\Omega)} + \left\|(p_k - M)^+\right\|^{q+1}_{L^{q+1}(\Omega)}\right) + C, \end{aligned} \tag{2.52}$$

where C is independent of ε and q. Gronwall's inequality then implies that

$$\|(n_k - M)^+\|_{L^{q+1}(\Omega)} + \left\|(p_k - M)^+\right\|_{L^{q+1}(\Omega)} \le Ce^{Ct} \tag{2.53}$$

for all $q \ge 1$ and $k \ge M$. Since the right-hand side of this equality does not depend on k, we can let $k \to \infty$ and then $q \to \infty$ to obtain the desired upper bound. □

Thus, taking k large enough, we see that $(\psi_\varepsilon, n_\varepsilon, p_\varepsilon)$ solves

$$-\nabla \cdot (\nabla \psi_\varepsilon) = p_\varepsilon - n_\varepsilon + C(x), \tag{2.54}$$

$$(n_\varepsilon)_t - \nabla \cdot \left(\left(\gamma n_\varepsilon^{\gamma-1} + \varepsilon\right)\nabla n_\varepsilon - n_\varepsilon \nabla \psi_\varepsilon\right) = r(n_\varepsilon, p_\varepsilon)(1 - n_\varepsilon p_\varepsilon) + g_\tau(n_\varepsilon, p_\varepsilon, \nabla n_\varepsilon, \nabla p_\varepsilon, \nabla \psi_\varepsilon), \tag{2.55}$$

$$(p_\varepsilon)_t - \nabla \cdot \left(\left(\gamma p_\varepsilon^{\gamma-1} + \varepsilon\right)\nabla p_\varepsilon + p_\varepsilon \nabla \psi\right) = r(n_\varepsilon, p_\varepsilon)(1 - n_\varepsilon p_\varepsilon) + g_\tau(n_\varepsilon, p_\varepsilon, \nabla n_\varepsilon, \nabla p_\varepsilon, \nabla \psi_\varepsilon) \tag{2.56}$$

subject to the initial and boundary conditions (1.4)–(1.6).

Proof of Theorem 2.1. Noticing that the function g_τ is bound, we can obtain the following convergence properties by using the same method as the proof in [8, Theorem 1.1] and Lemma 3.2:

$$\begin{aligned} \left(n_\varepsilon^{(\gamma-1)/2}, p_\varepsilon^{(\gamma-1)/2}\right) &\longrightarrow \left(n^{(\gamma-1)/2}, p^{(\gamma-1)/2}\right) \quad \text{strongly in } L^2(Q_T), \\ (n_\varepsilon, p_\varepsilon) &\longrightarrow (n, p), \quad \text{a.e. in } Q_T, \\ \left(\nabla n_\varepsilon^\gamma, \nabla p_\varepsilon^\gamma\right) &\longrightarrow (\nabla n^\gamma, \nabla p^\gamma) \quad \text{weakly in } L^2\left(0, T; H^1(\Omega)\right), \\ ((n_\varepsilon)_t, (p_\varepsilon)_t,) &\longrightarrow (n_t, p_t) \quad \text{weakly in } L^2(0, T; Y_0^*), \\ \psi_\varepsilon &\longrightarrow \psi \quad \text{weakly in } L^\infty\left(0, T; H^1(\Omega)\right), \\ (n_\varepsilon \nabla \psi_\varepsilon, p_\varepsilon \nabla \psi_\varepsilon) &\longrightarrow (n \nabla \psi, p \nabla \psi) \quad \text{weakly in } L^2(Q_T). \end{aligned} \tag{2.57}$$

In addition, a standard elliptic estimate gives

$$\|\nabla(\psi_\varepsilon - \psi)\|_{L^2(Q_T)} \le \|(p_\varepsilon - p) - (n_\varepsilon - n)\|_{L^2(Q_T)}, \tag{2.58}$$

from which we conclude

$$\nabla\psi_\varepsilon \longrightarrow \nabla\psi \quad \text{strongly in } L^2(Q_T), \tag{2.59}$$

and furthermore

$$\nabla\psi_\varepsilon \longrightarrow \nabla\psi \quad \text{a.e. in } Q_T. \tag{2.60}$$

Next, using $(n_\varepsilon^\gamma - n^\gamma)$ as test function in (2.55), we get

$$\begin{aligned}\int_0^T\int_\Omega \left|\nabla(n_\varepsilon^\gamma - n^\gamma)\right|^2 \le& -\int_0^T \langle n_t, n_\varepsilon^\gamma - n^\gamma\rangle_{Y^*,Y} + \varepsilon\int_0^T\int_\Omega \nabla n_\varepsilon \cdot \nabla\left(n_\varepsilon^\gamma - n^\gamma\right) \\ &+ \int_0^T\int_\Omega \nabla n^\gamma \cdot \nabla\left(n_\varepsilon^\gamma - n^\gamma\right) + \int_0^T\int_\Omega n_\varepsilon \nabla\psi_\varepsilon \cdot \nabla\left(n_\varepsilon^\gamma - n^\gamma\right) \\ &+ \max\{\alpha_{10}, \alpha_{20}\}\int_0^T\int_\Omega \left(\left|\nabla n_\varepsilon^\gamma\right| + \left|\nabla p_\varepsilon^\gamma\right| + (n_\varepsilon + p_\varepsilon)\left|\nabla\psi_\varepsilon\right|\right)\left|n_\varepsilon^\gamma - n^\gamma\right| \longrightarrow 0\end{aligned} \tag{2.61}$$

as $\varepsilon \to 0$, where we have used

$$\begin{aligned}&\varepsilon\int_0^T\int_\Omega \nabla n_\varepsilon \cdot \nabla\left(n_\varepsilon^\gamma - n^\gamma\right) \le \varepsilon^{1/2}\left(\int_0^T\int_\Omega \varepsilon|\nabla n_\varepsilon|^2\right)^{1/2}\left(\int_0^T\int_\Omega \left|\nabla\left(n_\varepsilon^\gamma - n^\gamma\right)\right|^2\right)^{1/2} \le \varepsilon^{1/2}C \longrightarrow 0, \\ &\int_0^T\int_\Omega n_\varepsilon\nabla\psi_\varepsilon \cdot \nabla\left(n_\varepsilon^\gamma - n^\gamma\right) = \frac{1}{\gamma+1}\int_0^T\int_\Omega (p_\varepsilon - n_\varepsilon + C(x))\left(n_\varepsilon^{\gamma+1} - n^{\gamma+1}\right) - \int_0^T\int_\Omega n_\varepsilon\nabla\psi_\varepsilon \cdot \nabla n^\gamma \\ &\qquad + \frac{1}{\gamma+1}\int_0^T\int_\Omega \nabla\psi_\varepsilon \cdot \nabla n^{\gamma+1} \longrightarrow 0,\end{aligned} \tag{2.62}$$

and (2.57)–(2.60). The same argument shows that $\|\nabla p_\varepsilon^\gamma - \nabla p^\gamma\|_{L^2(Q_T)} \to 0$ as $\varepsilon \to 0$.

Thus, there exists a subsequence (not relabeled) such that $\nabla n_\varepsilon^\gamma \to \nabla n^\gamma, \nabla p_\varepsilon^\gamma \to \nabla p^\gamma$ almost everywhere in Q_T as $\varepsilon \to 0$. Then it follows from Vitali's theorem that

$$g_\tau(n_\varepsilon, p_\varepsilon, \nabla n_\varepsilon, \nabla p_\varepsilon, \nabla\psi_\varepsilon) \longrightarrow g_\tau(n, p, \nabla n, \nabla p, \nabla\psi) \quad \text{strongly in } L^2(Q_T). \tag{2.63}$$

Now we can conclude that (ψ, n, p) is the solution of the problem (2.2)–(2.4), (1.4)–(1.6) from the above convergence by standard method and then complete the proof of Theorem 2.1. □

3. Proof of the Main Result

In the last section, we prove that there is at least one global weak solution $(\psi_\tau, n_\tau, p_\tau)$ to the problem (2.2)–(2.4), (1.4)–(1.6) for every given τ. In the following what we need to do is to prove that the limit of $(n_\tau, p_\tau, \psi_\tau)$ is a solution of (1.1)–(1.6). To this end, we first give some uniform estimates for the problem (2.2)–(2.4), (1.4)–(1.6). For simplicity, we drop the subscript τ of $(n_\tau, p_\tau, \psi_\tau)$ and set $\alpha_i = 1, i = 1, 2$.

Lemma 3.1. *For all $u \in H^1(\Omega)$, there holds*

$$\|u\|_{L^s(\Omega)} \le C\|u\|^{\alpha}_{L^2(\Omega)}\|u\|^{1-\alpha}_{H^1(\Omega)}, \tag{3.1}$$

where α,s satisfy

$$0 < 1 - \alpha = N\left(\frac{1}{2} - \frac{1}{s}\right) < 1, \quad 1 < s < \frac{2N}{N-2}. \tag{3.2}$$

This is the well-known Gagliardo-Nirenberg Inequality [15].

Lemma 3.2. *If (ψ, n, p) is the solution of the problem* (2.2)–(2.4), (1.4)–(1.6), *the following estimate holds:*

$$\begin{aligned}
&\int_0^T\int_\Omega (n^\gamma + p^\gamma)|\nabla\psi|^2 \\
&\quad\le \delta\int_0^T\int_\Omega\left(|\nabla(n^\gamma)|^2 + |\nabla(p^\gamma)|^2\right) + C(\delta)\left[\int_0^T\left(\int_\Omega\left(n^{\gamma+1} + p^{\gamma+1}\right)dx\right)^2 + 1\right], \quad \forall\gamma \ge 1,
\end{aligned}\tag{3.3}$$

where δ is a sufficiently small constant.

Proof. First of all, the following L^∞ estimate of ψ

$$\|\psi(t)\|_{L^\infty(\Omega)} \le C\left\{1 + \left(\int_\Omega(n^2(t) + p^2(t))dx\right)^{1/2}\right\}, \quad \forall t \ge 0 \tag{3.4}$$

can follow from the standard techniques in [17]. Then by taking $\psi - \overline{\psi}$ as test function in (2.2), we have

$$\int_\Omega|\nabla(\psi - \overline{\psi})|^2 + \int_\Omega\nabla\overline{\psi}\cdot\nabla(\psi - \overline{\psi}) = \int_\Omega(p - n + C(x))(\psi - \overline{\psi}). \tag{3.5}$$

The assumptions we have made and Poincaré inequality yield

$$\|\nabla(\psi - \overline{\psi})\|_{L^2(\Omega)} \le C\left(1 + \|p - n\|_{L^2(\Omega)}\right) \le C\left(1 + \|n\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\right). \tag{3.6}$$

Using $(n^\gamma - \overline{n}^\gamma)\psi$ as test function for (2.2), and noting that $2 < \gamma + 1$ we get

$$
\begin{aligned}
\int_0^T\int_\Omega n^\gamma|\nabla\psi|^2 &= \int_0^T\int_\Omega \overline{n}^\gamma|\nabla\psi|^2 + \int_0^T\int_\Omega (p-n+C(x))\left[(n^\gamma-\overline{n}^\gamma)\psi\right] - \int_0^T\int_\Omega \psi\nabla(n^\gamma-\overline{n}^\gamma)\cdot\nabla\psi \\
&\le C\|\nabla\psi\|^2_{L^2(Q_T)} + \int_0^T \|\psi\|_{L^\infty(\Omega)}\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right) \\
&\quad + \int_0^T \|\psi\|_{L^\infty(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}\|\nabla(n^\gamma-\overline{n}^\gamma)\|_{L^2(\Omega)} \\
&\le C\int_0^T\int_\Omega\left(n^2+p^2\right) + \int_0^T\left[\left(\int_\Omega(n^2+p^2)\right)^{1/2}\int_\Omega\left(n^{\gamma+1}+n^{\gamma+1}\right)\right] \\
&\quad + \delta\int_0^T\int_\Omega|\nabla(n^\gamma)|^2 + C(\delta)\left[\int_0^T\left(\int_\Omega\left(n^2+p^2\right)\right)^2+1\right] \\
&\le \delta\int_0^T\int_\Omega|\nabla(n^\gamma)|^2 + C(\delta)\left[\int_0^T\left(\int_\Omega(n^{\gamma+1}+p^{\gamma+1})\right)^2+1\right].
\end{aligned}
\tag{3.7}
$$

A similar estimate for $\int_0^T\int_\Omega p^\gamma|\nabla\psi|^2$ follows from a same procedure. Then the proof is completed. □

Lemma 3.3. *If (ψ, n, p) is the solution of the problem* (2.2)–(2.4), (1.4)–(1.6), *there holds that*

$$\|n\|_{L^\infty(0,T_0;L^{\gamma+1}(\Omega))} + \|\nabla(n^\gamma)\|_{L^2(Q_{T_0})} \le C, \tag{3.8}$$

$$\|p\|_{L^\infty(0,T_0;L^{\gamma+1}(\Omega))} + \|\nabla(p^\gamma)\|_{L^2(Q_{T_0})} \le C, \tag{3.9}$$

$$\|\psi\|_{L^2(0,T_0;H^1(\Omega))} + \|\psi\|_{L^\infty(Q_{T_0})} \le C, \tag{3.10}$$

for some sufficiently small T_0; here positive constant C is independent of τ.

Proof. Without loss of generality, we assume $N = 3$ and the $N = 1$ or 2 case is easier.

Case 1. $2 \le \gamma \le 5$. In this case, from Hölder inequality and (3.3), we have

$$
\begin{aligned}
\int_0^T\int_\Omega\left(n^2+p^2\right)|\nabla\psi|^2 &\le C\left(\int_0^T\int_\Omega(n^\gamma+p^\gamma)|\nabla\psi|^2 + \int_0^T\int_\Omega|\nabla\psi|^2\right) \\
&\le \delta\int_0^T\int_\Omega\left(|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right) + C(\delta)\left[\int_0^T\left(\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)\mathrm{d}x\right)^2+1\right].
\end{aligned}
\tag{3.11}
$$

Noting that $2 \le \gamma \le 5$, we can choose a constant q such that $6\gamma/5 \le 2q \le \gamma + 1$. Then by taking $n^{2q-\gamma} - \overline{n}^{2q-\gamma}$ and $p^{2q-\gamma} - \overline{p}^{2q-\gamma}$ as test function in (2.3) and (2.4), respectively, and adding them together, we get

$$\begin{aligned}
&\frac{1}{2q-\gamma+1}\int_\Omega \left(n^{2q-\gamma+1} + p^{2q-\gamma+1}\right) + \frac{\gamma(2q-\gamma)}{q^2}\int_0^T\int_\Omega \left[|\nabla(n^q)|^2 + |\nabla(p^q)|^2\right] \\
&\quad = \frac{1}{2q-\gamma+1}\int_\Omega \left(n_0^{2q-\gamma+1} + p_0^{2q-\gamma+1}\right) + \int_\Omega \left[\overline{n}^{2q-\gamma}(n-n_0) + \overline{p}^{2q-\gamma}(p-p_0)\right] \\
&\qquad + \int_0^T\int_\Omega \left[\nabla\psi \cdot \left(n\nabla\left(n^{2q-\gamma} - \overline{n}^{2q-\gamma}\right) - p\nabla\left(p^{2q-\gamma} - \overline{p}^{2q-\gamma}\right)\right)\right] \\
&\qquad + \int_0^T\int_\Omega \left[\nabla(n^\gamma)\cdot\nabla\left(\overline{n}^{2q-\gamma}\right) + \nabla(p^\gamma)\cdot\nabla\left(\overline{p}^{2q-\gamma}\right)\right] \\
&\qquad + \int_0^T\int_\Omega (R(n,p) + g_\varepsilon)\left[\left(n^{2q-\gamma} - \overline{n}^{2q-\gamma}\right) + \left(p^{2q-\gamma} - \overline{p}^{2q-\gamma}\right)\right] \\
&\quad = I_1 + \cdots + I_4.
\end{aligned} \tag{3.12}$$

We estimate the right-hand side term by term. Due to the equation of ψ and (3.6), we obtain

$$\begin{aligned}
I_2 &= \frac{2q-\gamma}{2q-\gamma+1}\int_0^T\int_\Omega \nabla\psi \cdot \left[\nabla\left(n^{2q-\gamma+1} - \overline{n}^{2q-\gamma+1}\right) - \nabla\left(p^{2q-\gamma+1} - \overline{p}^{2q-\gamma+1}\right)\right] \\
&\quad + \int_0^T\int_\Omega \nabla\psi \cdot \left[(\overline{n}-n)\nabla\left(\overline{n}^{2q-\gamma}\right) - (\overline{p}-p)\nabla\left(\overline{p}^{2q-\gamma}\right)\right] \\
&\le \frac{2q-\gamma}{2q-\gamma+1}\int_0^T\int_\Omega (p-n+C(x))\left[\left(n^{2q-\gamma+1} - p^{2q-\gamma+1}\right) - \left(\overline{n}^{2q-\gamma+1} - \overline{p}^{2q-\gamma+1}\right)\right] \\
&\quad + C\int_0^T \|\nabla\psi\|_{L^2(\Omega)}\left(\|\overline{n}-n\|_{L^2(\Omega)} + \|\overline{p}-p\|_{L^2(\Omega)}\right) \\
&\le C\left[\int_0^T\int_\Omega \left(n^{\gamma+1} + p^{\gamma+1}\right) + 1\right],
\end{aligned} \tag{3.13}$$

where we use that $(p-n)(n^{2q-\gamma+1} - p^{2q-\gamma+1}) \le 0$ and $2q-\gamma+1 \le \gamma+1$ when $\gamma \ge 2$. I_3 and I_4 can be bounded as

$$I_3 \le \frac{\lambda_1}{4}\int_0^T\int_\Omega \left(|\nabla(n^\gamma)|^2 + |\nabla(p^\gamma)|^2\right) + C(\lambda_1), \tag{3.14}$$

$$\begin{aligned}
I_4 &\le \frac{\lambda_1}{4}\int_0^T\int_\Omega \left(|\nabla(n^\gamma)|^2 + |\nabla(p^\gamma)|^2\right) + C\int_0^T\int_\Omega \left(n^2 + p^2\right) \\
&\qquad + C(\lambda_1)\left[\int_0^T\int_\Omega \left(n^{4q-2\gamma} + p^{4q-2\gamma}\right) + \int_0^T\int_\Omega \left(n^2 + p^2\right)|\nabla\psi|^2\right],
\end{aligned} \tag{3.15}$$

where we have used the nonnegativity of n and p. Inserting (3.13)–(3.15) into (3.12) we conclude that

$$\begin{aligned}
&\int_\Omega \left(n^{2q-\gamma+1}+p^{2q-\gamma+1}\right)+\int_0^T\int_\Omega \left[|\nabla(n^q)|^2+|\nabla(p^q)|^2\right] \\
&\quad\le \lambda_1\left[\int_0^T\int_\Omega \left(|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right)+\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)\right] \\
&\qquad + C(\lambda_1)\left[\int_0^T\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left(n^2+p^2\right)|\nabla\psi|^2+1\right].
\end{aligned} \tag{3.16}$$

Similarly, taking $n^\gamma-\overline{n}^\gamma$ and $p^\gamma-\overline{p}^\gamma$ as test function in (2.3) and (2.4), we conclude that

$$\begin{aligned}
&\frac{1}{\gamma+1}\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left[|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right] \\
&=\frac{1}{\gamma+1}\int_\Omega\left(n_0^{\gamma+1}+p_0^{\gamma+1}\right)+\int_\Omega\left[\overline{n}^\gamma(n-n_0)+\overline{p}^\gamma(p-p_0)\right] \\
&\quad+\int_0^T\int_\Omega\left[\nabla(n^\gamma)\cdot\nabla(\overline{n}^\gamma)+\nabla(p^\gamma)\cdot\nabla(\overline{p}^\gamma)\right]+\int_0^T\int_\Omega\nabla\psi\cdot\left[(\overline{n}-n)\nabla(\overline{n}^\gamma)-(\overline{p}-p)\nabla(\overline{p}^\gamma)\right] \\
&\quad+\frac{\gamma}{\gamma+1}\int_0^T\int_\Omega\nabla\psi\cdot\left[\nabla\left(n^{\gamma+1}-\overline{n}^{\gamma+1}\right)-\nabla\left(p^{\gamma+1}-\overline{p}^{\gamma+1}\right)\right]+\int_0^T\int_\Omega\left(R(n,p)+g_\varepsilon\right)\left[(n^\gamma-\overline{n}^\gamma)+(p^\gamma-\overline{p}^\gamma)\right] \\
&\le\lambda_1\left[\int_0^T\int_\Omega\left(|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right)+\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)\right] \\
&\quad+C(\lambda_1)\left[\int_0^T\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left(n^{2\gamma}+p^{2\gamma}\right)+\int_0^T\int_\Omega\left(n^2+p^2\right)|\nabla\psi|^2+1\right].
\end{aligned} \tag{3.17}$$

Choosing λ_1 sufficiently small, and then summing (3.16) and (3.17), we have

$$\begin{aligned}
&\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left[|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right] \\
&\quad+\int_\Omega\left(n^{2q-\gamma+1}+p^{2q-\gamma+1}\right)+\int_0^T\int_\Omega\left[|\nabla(n^q)|^2+|\nabla(p^q)|^2\right] \\
&\quad\le C(\lambda_1)\left[\int_0^T\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left(n^{2\gamma}+p^{2\gamma}\right)+\int_0^T\int_\Omega\left(n^2+p^2\right)|\nabla\psi|^2+1\right].
\end{aligned} \tag{3.18}$$

Now we estimate the term $\int_0^T\int_\Omega n^{2\gamma}$ (or $\int_0^T\int_\Omega p^{2\gamma}$). Due to Lemma 2.2 we get

$$\begin{aligned}\int_0^T\int_\Omega n^{2\gamma} &= \int_0^T\int_\Omega (n^q)^s \le \int_0^T\int_\Omega \|n^q\|_{H^1(\Omega)}^{(1-\alpha)s}\|n^q\|_{L^2(\Omega)}^{\alpha s} \\ &\le \lambda_2\int_0^T \|\nabla(n^q)\|_{L^2(\Omega)}^2 + C(\lambda_2)\left[\int_0^T\left(\int_\Omega n^{2q}\mathrm{d}x\right)^\beta + 1\right],\end{aligned} \tag{3.19}$$

where s, α, β satisfy

$$s = \frac{2\gamma}{q}, \quad 0 < 1-\alpha = 3\left(\frac{1}{2} - \frac{q}{2\gamma}\right) < 1, \quad 0 < (1-\alpha)s < 2, \quad \beta = \frac{\alpha s}{2-(1-\alpha)s} \tag{3.20}$$

Consequently we obtain, taking into account (3.11) and the choice of q,

$$\begin{aligned}&\int_\Omega\left(n^{\gamma+1} + p^{\gamma+1}\right) + \int_0^T\int_\Omega\left[|\nabla(n^\gamma)|^2 + |\nabla(p^\gamma)|^2\right] \\ &\quad\le C(\lambda_1,\lambda_2,\delta)\left[1 + \int_0^T\left(\int_\Omega\left(n^{\gamma+1} + p^{\gamma+1}\right)\mathrm{d}x\right)^{\max\{2,\beta\}}\right]\end{aligned} \tag{3.21}$$

for sufficiently small λ_2 and δ such that $(\lambda_2 + \delta)C(\lambda_1) < 1$, where β only depends on γ.

This proves that

$$\max_{t\in[0,T_0]}\int_\Omega\left[n^{\gamma+1}(t) + p^{\gamma+1}(t)\right] \le C \tag{3.22}$$

for some sufficiently small T_0 by Lemma 2.2 of the appendix in [14] , and thus

$$\int_0^{T_0}\int_\Omega\left[|\nabla(n^\gamma)|^2 + |\nabla(p^\gamma)|^2\right] \le C. \tag{3.23}$$

Using the above estimates and (3.4)–(3.6), we can obtain (3.10).

Case 2. $1 < \gamma < 2$. In this case, the estimates on I_1, I_2, I_3 are the same as Case 1, and

$$\begin{aligned}I_4 &\le \frac{\lambda_1}{4}\int_0^T\int_\Omega\left(|\nabla(n^\gamma)|^2 + |\nabla(p^\gamma)|^2\right) + C\int_0^T\int_\Omega\left(n^2 + p^2\right) \\ &\quad + C(\lambda_1)\left[\int_0^T\int_\Omega\left(n^{4q-3\gamma+2} + p^{4q-3\gamma+2}\right) + \int_0^T\int_\Omega\left(n^\gamma + p^\gamma\right)|\nabla\psi|^2\right].\end{aligned} \tag{3.24}$$

Then we have

$$\begin{aligned}
&\int_\Omega\left(n^{2q-\gamma+1}+p^{2q-\gamma+1}\right)+\int_0^T\int_\Omega\left[|\nabla(n^q)|^2+|\nabla(p^q)|^2\right]\\
&\quad\le\lambda_1\left[\int_0^T\int_\Omega\left(|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right)+\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)\right]\\
&\qquad+C(\lambda_1)\left[\int_0^T\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left(n^{4q-3\gamma+2}+p^{4q-3\gamma+2}\right)+\int_0^T\int_\Omega(n^\gamma+p^\gamma)|\nabla\varphi|^2+1\right].
\end{aligned}\tag{3.25}$$

Next, by a small change in (3.17), we obtain

$$\begin{aligned}
&\frac{1}{\gamma+1}\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left[|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right]\\
&\quad\le\lambda_1\left[\int_0^T\int_\Omega\left(|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right)+\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)\right]\\
&\qquad+C(\lambda_1)\left[\int_0^T\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)+\int_0^T\int_\Omega\left(n^{\gamma+2}+p^{\gamma+2}\right)\right.\\
&\qquad\qquad\left.+\int_0^T\int_\Omega\left(n^{2\gamma}+p^{2\gamma}\right)+\int_0^T\int_\Omega(n^\gamma+p^\gamma)|\nabla\varphi|^2+1\right]\\
&\quad\le\lambda_1\left[\int_0^T\int_\Omega\left(|\nabla(n^\gamma)|^2+|\nabla(p^\gamma)|^2\right)+\int_\Omega\left(n^{\gamma+1}+p^{\gamma+1}\right)\right]\\
&\qquad+C(\lambda_1)\left[\int_0^T\int_\Omega\left(n^{\gamma+2}+p^{\gamma+2}\right)+\int_0^T\int_\Omega(n^\gamma+p^\gamma)|\nabla\varphi|^2+1\right].
\end{aligned}\tag{3.26}$$

Choose q such that

$$\frac{3}{10}(\gamma+2)<q<\frac{\gamma+1}{2},\tag{3.27}$$

and reset

$$s=\frac{\gamma+2}{q},\quad 0<1-\alpha=3\left(\frac{1}{2}-\frac{q}{\gamma+2}\right)<1,\quad 0<(1-\alpha)s<2,\quad \beta=\frac{\alpha s}{2-(1-\alpha)s}.\tag{3.28}$$

By a discuss similar to (3.19), we obtain

$$\int_0^T\int_\Omega n^{\gamma+2}\le\lambda_2\int_0^T\|\nabla(n^q)\|^2_{L^2(\Omega)}+C(\lambda_2)\left[\int_0^T\left(\int_\Omega n^{2q}\mathrm{d}x\right)^\beta+1\right].\tag{3.29}$$

Finally, noting $4q-3\gamma+2<\gamma+2$, from (3.25)–(3.29) and estimate (3.3) we deduce that (3.21) also holds for $1<\gamma<2$. Therefore, we can also obtain estimates (3.8)–(3.10) when $1<\gamma<2$. □

The following lemma is indispensable to prove the $L^2(0,T_0;Y^*)$ estimates of n_t and p_t uniformly in τ.

Lemma 3.4. *If* (ψ,n,p) *is the solution of the problem* (2.2)–(2.4), (1.4)–(1.6), *there holds that*

$$\|n\|_{L^m(Q_{T_0})}+\|p\|_{L^m(Q_{T_0})}\le C,\quad \frac{10}{3}<m<10. \tag{3.30}$$

Proof. Denote by $V_2(Q_T)$ a Banach space in which function v satisfies

$$\|v\|_{V_2(Q_T)}=\sup_{0\le t\le T}\|v(t)\|_{L^2(\Omega)}+\left(\int_0^T\int_\Omega|\nabla v|^2\right)^{1/2}<\infty. \tag{3.31}$$

The following proof is base on Moser's iteration technique [15]. We insert $(n-s)^+$ into (2.3), $(p-s)^+$ into (2.4), integrate over the interval $(0,T_0)$, and add the equations to obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega&\left[(n(t)-s)^{+^2}+(p(t)-s)^{+^2}\right]+\gamma\int_0^{T_0}\int_\Omega n^{\gamma-1}|\nabla(n-s)^+|^2+\gamma\int_0^{T_0}\int_\Omega p^{\gamma-1}\left|\nabla(p-s)^+\right|^2\\&=\int_0^{T_0}\int_\Omega n\nabla\psi\cdot\nabla(n-s)^+-\int_0^{T_0}\int_\Omega p\nabla\psi\cdot\nabla(p-s)^++\int_0^{T_0}\int_\Omega(R(n,p)+g_\varepsilon)\left[(n-s)^++(p-s)^+\right]\\&\le\frac{1}{2}\int_0^{T_0}\int_\Omega[p-n+C(x)]\left[\left(n^2-s^2\right)^+-\left(p^2-s^2\right)^+\right]+\int_0^{T_0}\int_\Omega(g_\varepsilon+\overline{r})\left[(n-s)^++(p-s)^+\right].\end{aligned} \tag{3.32}$$

Since $(p-n)[(n^2-s^2)^+-(p^2-s^2)^+]$ is negative, and choosing $s\ge c_0=\max\{\sup_\Omega\{n_0,p_0\},\sup_{\Sigma_D}\{\overline{n},\overline{p}\}\}+1$, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega&\left[(n(t)-s)^{+^2}+(p(t)-s)^{+^2}\right]+\int_0^{T_0}\int_\Omega\left[|\nabla(n-s)^+|^2+\left|\nabla(p-s)^+\right|^2\right]\\&\le\int_0^{T_0}\int_\Omega C(x)\left[n(n-s)^++p(p-s)^+\right]+\int_0^{T_0}\int_\Omega(g_\varepsilon+\overline{r})\left[(n-s)^++(p-s)^+\right]\\&=I_1+I_2\end{aligned} \tag{3.33}$$

for all $s\ge c_0$.

Applying Hölder inequality, we obtain

$$I_1 \le \|C(x)\|_{L^\infty(\Omega)}\|n\|_{L^2(Q_{T_0})}\left\|(n-s)^+ + (p-s)^+\right\|_{L^{2(N+2)/N}(Q_{T_0})}\left|Q_{T_0}\cap[n>s,p>s]\right|^{1/(N+2)}$$
$$\le \lambda_3\left\|(n-s)^+ + (p-s)^+\right\|^2_{L^{2(N+2)/N}(Q_{T_0})} + C(\lambda_3)\left|Q_{T_0}\cap[n>s,p>s]\right|^{2/(N+2)}, \tag{3.34}$$

$$I_2 \le \left(\|\nabla(n^\gamma+p^\gamma)\|_{L^2(Q_{T_0})} + \|(n+p)\nabla\psi\|_{L^2(Q_{T_0})}\right)$$
$$\times\|(n-s)^+ + (p-s)^+\|_{L^{2(N+2)/N}(Q_{T_0})}\left|Q_{T_0}\cap[n>s,p>s]\right|^{1/(N+2)}$$
$$+\overline{r}\|(n-s)^+ + (p-s)^+\|_{L^{2(N+2)/N}(Q_{T_0})}\left|Q_{T_0}\cap[n>s,p>s]\right|^{(N+4)/2(N+2)}$$
$$\le \lambda_3\|(n-s)^+ + (p-s)^+\|^2_{L^{2(N+2)/N}(Q_{T_0})} + C(\lambda_3)\left|Q_{T_0}\cap[n>s,p>s]\right|^{2/(N+2)}, \tag{3.35}$$

where $|\cdot|$ denotes measure, and $[n>s,p>s]=\{(x,t)\mid n(x,t)>s,p(x,t)>s\}$.

Choose λ_3 sufficiently small, together with (3.33)–(3.35) and $V_2(Q_{T_0}) \hookrightarrow L^{2(N+2)/N}(Q_{T_0})$, then

$$\left\|(n_\varepsilon-s)^+ + (p_\varepsilon-s)^+\right\|^2_{L^{2(N+2)/N}(Q_{T_0})} \le C(\lambda_3)\left|Q_{T_0}\cap[n_\varepsilon>s,p_\varepsilon>s]\right|^{2/(N+2)}. \tag{3.36}$$

On the other hand, we have

$$\int_0^{T_0}\int_\Omega\left[(n-s)^+ + (p-s)^+\right]$$
$$\le \left\|(n_\varepsilon-s)^+ + (p_\varepsilon-s)^+\right\|_{L^{2(N+2)/N}(Q_{T_0})}\left|Q_{T_0}\cap[n_\varepsilon>s,p_\varepsilon>s]\right|^{(N+4)/2(N+2)}. \tag{3.37}$$

Let $\chi(s)=|Q_{T_0}\cap[n_\varepsilon>s,p_\varepsilon>s]|$. From (3.36) and (3.37), we obtain

$$\int_0^{T_0}\int_\Omega\left[(n-s)^+ + (p-s)^+\right] \le C(\lambda_3)\chi(s)^{(N+6)/2(N+2)}, \quad \forall s\ge c_0, \tag{3.38}$$

which proves (3.30) by [15, Lemma 5.2 in Chapter 2]. □

Lemma 3.5. *If* (ψ,n,p) *is the solution of the problem* (2.2)–(2.4), (1.4)–(1.6), *there holds that*

$$\|n_t\|_{L^2(0,T_0;Y^*)} \le C, \tag{3.39}$$

$$\|p_t\|_{L^2(0,T_0;Y^*)} \le C. \tag{3.40}$$

Proof. We only prove (3.39). The proof of (3.40) is completely the same as that for (3.39). Take test function $\phi \in L^2(0, T_0; Y)$ for (2.3). Then by Lemma 3.2–3.4 we conclude from

$$\left|\int_0^{T_0} \langle n_t, \phi\rangle_{Y^*,Y}\right| \leq \left(\|\nabla(n^\gamma)\|_{L^2(Q_{T_0})} + \|n\nabla\psi\|_{L^2(Q_{T_0})}\right)\|\nabla\phi\|_{L^2(Q_{T_0})} + \left[\|\nabla(n^\gamma + p^\gamma)\|_{L^2(Q_{T_0})} + \|(n+p)|\nabla\psi|\|_{L^2(Q_{T_0})} + \|n\|_{L^4(Q_{T_0})}\|p\|_{L^4(Q_{T_0})}\right]\|\phi\|_{L^2(Q_{T_0})} \tag{3.41}$$

that $\|n_t\|_{L^2(0,T_1;Y_0^*)} \leq c$. The proof is complete. □

Proof of Theorem 1.2. By passing to a subsequence if necessary, from Lemma 3.2–3.5, together with compact lemma of Lions [18], we infer that

$$\begin{aligned}
((n_\tau)^\gamma, (p_\tau)^\gamma) &\longrightarrow (n^\gamma, p^\gamma) \quad \text{strongly in } L^2(Q_{T_0}), \\
(n_\tau, p_\tau) &\longrightarrow (n, p), \quad \text{a.e. in } Q_{T_0}, \\
((n_\tau)^\gamma, (p_\tau)^\gamma) &\longrightarrow (n^\gamma, p^\gamma) \quad \text{weakly in } L^2\left(0, T_0; H^1(\Omega)\right), \\
((n_\tau)_t, (p_\tau)_t) &\longrightarrow (n_t, p_t) \quad \text{weakly in } L^2(0, T_0; Y^*), \\
(n_\tau\nabla\psi_\tau, p_\tau\nabla\psi_\tau) &\longrightarrow (n\nabla\psi, p\nabla\psi) \quad \text{weakly in } L^2(Q_{T_0}), \\
\psi_\tau &\longrightarrow \psi \quad \text{weakly}^* \text{ in } L^\infty\left(0, T_0; H^1(\Omega)\right).
\end{aligned} \tag{3.42}$$

Then by the same argument as the proof of Theorem 2.1, we can conclude that (ψ, n, p) is the solution of the problem (1.1)–(1.6) from the convergence of (3.42) and then complete the proof of Theorem 1.2. □

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