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Research Article

Optimal Pricing and Production Master Planning in a Multiperiod Horizon Considering Capacity and Inventory Constraints

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We formulate and solve a single-item joint pricing and master planning optimization problem with capacity and inventory constrains. The objective is to maximize profits over a discrete-time multiperiod horizon. The solution process consists of two steps. First, we solve the single-period problem exactly. Second, using the exact solution of the single-period problem, we solve the multiperiod problem using a dynamic programming approach. The solution process and the importance of considering both capacity and inventory constraints are illustrated with numerical examples.

1. Introduction

In the recent years, interest has increased in problems involving the joint optimization of pricing and production decisions. Such problems may not have been very practical some years ago since manufacturing firms have not traditionally been in close contact with the final consumers, making it hard to predict demand as a function of price. However, nowadays new sales channels such as the internet allow direct sales, making easy to observe end customer behavior [1]. Under these conditions, joint price and production planning problems arise in the manufacturing sector. According to Farnham [2], direct sales have grown 1% per year faster than traditional retail sales for the last ten years and that direct sales reached $30 billion in 2003, making it a significant business sector. Manufacturing firms can now, due to direct sales and the internet, maintain close contact with their customers as is currently done by the airlines, and change prices more quickly and at a much lower cost. In addition, firms that sell to a large number of small retailers can apply pricing models considering the numerous retailers to be their final customers.

Joint pricing and production decisions problems are treated in literature; see, for example, Elmaghraby and Keskinocak [3]. However, problems with both capacity and inventory constraints are not common. A literature review is provided below. According to our experience as consultants, the master planning linear programming models currently in use at several large companies reveal that both capacity and inventory constraints are commonly used in practice. The upper bounds on inventory result from either storage and/or budget limitations or company policies. The inventory constraints are of two types, absolute limits, which are considered in this paper, and days-of-cover requirements, which will be addressed in a future paper.

Food and beverage producers may be able to apply the proposed model. Within the Mexican market, these firms supply a very large number of small stores in addition to large supermarkets. The large number of stores is the set of customers served directly by the producers. The producer and many of the retailers are owned by the same parent corporation, making the application of dynamic pricing to the end consumer feasible. The food and beverage producers' market exhibits a fairly stable demand with seasonal variation and identifiable long-term trends. Due to specialized storage requirements and expiration dates, inventory storage capacity cannot be easily expanded, making upper limits on inventory especially useful. The model we study in this paper is based on the planning models currently in use at such companies (cost minimizing linear programming models) with the addition of the pricing decision, profit maximization, and a focus on a single-item model in a multi period horizon. Furthermore, we do not require the demand pattern to be seasonal, only that the behavior of demand as a function of the price to be known in each period. We consider that a single-item model with multiple time periods is a reasonable starting point that can be used as a base for further work involving more realistic multiitem formulations.

In general, capacity constraints are uncommon in literature and inventory constraints even more so. One notable paper that contains inventory limits but no capacity limits is Cheng [4] who examines an EOQ model with pricing considerations and an inventory constraint. Additionally, pricing problems at the aggregate, master planning level with multiple discrete time periods were not considered before. Extending in this direction, we formulate and solve a multi period horizon with a single-item problem in a joint price and production master panning optimization subject to a capacity and inventory constrains.

Although we consider a deterministic model, it is to be noted that revenue optimization (revenue management) with stochastic elements in the service sector has received considerable attention by many researchers. For an overview see Bitran and Caldentey [5] and Boyd and Bilegan [1]. The most notable examples are the applications in the airline industry [6, 7]. Related techniques have also been applied in hotel [8], restaurant [9, 10], and retail [11, 12] areas. Gallego and Van Ryzin [13] develop a model that applies in the airline, hotel and retail settings. The inclusion of pricing decisions in revenue management models is fairly recent. See Gallego and van Ryzin [14], Feng and Gallego [15], and Bitran and Mondschein [16] for examples of the early work in pricing within a revenue management context. More recently Chan et al. [17] study delayed production and delayed pricing strategies for a multiperiod model.

Our work differs from most of the works in literature review in that we consider both capacity and inventory limits. We also address the problem at a master planning level rather than at the faster paced lot scheduling level where rapid and frequent price changes may not be feasible. It is important to point out that Chan et al. [17] study delayed production and delayed pricing strategies rather than optimal simultaneous determination of both pricing and production, placing their work in a separate category.

The remainder of the paper is organized as follows. Section 2 describes the multiperiod price-optimizing model. Section 3 shows how the multiperiod problem can be simplified in the single-period case with known initial and ending inventories. Section 4 provides a closed form solution of the single-period problem of Section 3, assuming a demand function of the exponential form. Section 5 shows how the result of Section 4 can be used to solve the multiperiod model using a dynamic programming approach. Numerical examples are presented in Section 6 and conclusions are provided in Section 7.

2. Model Description

We formulate a single-item price-optimizing master planning problem. The problem is to determine for each period of a discrete time, finite planning horizon, the optimal sales price, production quantity, and sales amount for a single-item. In each period, a production capacity, a variable cost of production, a fixed cost, a safety stock requirement, and a demand function that returns demand as a function of price are considered. The production capacity, variable cost, fixed cost, and safety stock requirement are allowed to vary in each time period. The demand function is allowed to vary over time but is always of the same parametric form. The following notation is defined:

p_t: sales price in period t,

n_t: production quantity in period t,

s_t: sales quantity in period t,

I_t: inventory in period t,

V_t: variable cost per unit produced in period t,

F: fixed cost per time period,

C_t: production capacity in period t,

$I_t^{\min}$: safety stock requirement in period t,

$I_t^{\max}$: maximum inventory limit in period t,

I_0: given value of initial inventory,

$D_t(p_t)$: demand function in period t.

The demand function is assumed to be continuous, nonincreasing and asymptotically equal to zero. The general multiperiod model for T periods is the following:

$$\max\ Z = \sum_{t=1}^{T} \left(p_t \cdot s_t - V_t \cdot n_t \right) - T \cdot F, \tag{2.1}$$

s.t.

$$s_t \leq D_t(p_t) \quad \forall t, \tag{2.2}$$

$$n_t \leq C_t \quad \forall t, \tag{2.3}$$

$$I_t = I_{t-1} + n_t - s_t \quad \forall t, \tag{2.4}$$

$$I_t \leq I_t^{\max} \quad \forall t, \tag{2.5}$$

$$I_t \geq I_t^{\min} \quad \forall t. \tag{2.6}$$

In addition, all variables are assumed to be nonnegative. For simplicity the nonnegativity constraints are not explicitly expressed. The objective function (2.1) is to maximize profit. Notice that the fixed cost does not play a role in the optimization, it has been included only to clarify that profit is to be maximized. Constraint (2.2) limits the sales amount to the demand. Constraint (2.3) ensures that production will not exceed the available capacity. Constraint (2.4) is an inventory balance equation. Constraints (2.5) and (2.6) keep the inventory between specified maximum and minimum limits. In order to show how to solve the multiperiod problem, we first provide a solution to a simplified problem with a single-period, assuming that the initial and ending inventories, I_0 and I_1, are known and feasible with respect to (2.5) and (2.6).

Notice that although an inventory holding cost parameter is not included in the objective function (2.1), it is possible to model the financial opportunity costs of holding inventory by multiplying the terms of (2.1) by the appropriate discount factors. The resulting objective is then to maximize the present value (NPV) of the future cash flows. For examples of this approach please see Hadley [18], Park and Sharp-Bette [19], Sun and Queyranne [20], and Smith and Martínez-Flores [21]. In Smith and Martínez-Flores [21] it is shown that the traditional approach and net present value (NPV) approach can yield different optimal costs and inventory policies. It is important to mention that the papers listed in Table 1 do not consider the NPV approach. The NPV model, assuming that all cash flows occur at the end of a period and eliminating the constant terms in (2.1), is the following:

$$\sum_{t=1}^{T} (1+r)^{-t} \cdot \left(p_t \cdot s_t - V_t \cdot n_t \right), \tag{2.7}$$

where r models the financial opportunity cost. Although an operational inventory holding cost (cash cost) cannot be modeled in this way, in practice, the financial opportunity cost tends to be by far the largest portion of the inventory holding cost [22], making this modeling technique adequate for a wide range of applications.

Table 1: Inventory models with price and production decisions.

Author(s)	Year	Inventory model	Capacity constrain	Inventory constrain	Multiperiod horizon
Whitin [23]	1955	It was linked the price policy and inventory theory and determined the combined policy that yield the maximum profit	No	No	No
Thomas [24]	1970	Determines simultaneously the price and production decision with a known deterministic demand function	No	No	Yes
Kunreuther and Richard [25]	1971	Determines the price and ordering decision considering a stationary demand curve	No	No	No
Cheng [4]	1990	An economic order quantity (EOQ) model that integrates the product pricing and order sizing decisions with storage space and inventory investment limitations	No	Yes	No
Harris and Pinder [26]	1995	Determines optimal price and capacity decision for a single-period	No	No	No
Kim and Lee [27]	1998	Determines the optimal price, lot size and the capacity decision for a firm with constant price-dependent demands	Yes	No	No
Gilbert [28]	1999	Determines the optimal price and production schedule for a product with seasonal demand	No	No	Yes
Bhattacharjee and Ramesh [29]	2000	Determines the optimal price and lot size for a product with fixed life perish-ability for a certain number of periods	No	No	Yes
Gilbert [30]	2000	Determines the optimal price and production schedule for a product with seasonal price dependent demand	Yes	No	Yes
Zhao and Wang [31]	2002	Coordination of price and production schedules in a decentralized supply chain	No	No	Yes
Chen and Simchi-Levi [32]	2003	Determines the price and productions decisions	No	No	Yes
Deng and Yano [33]	2006	Setting prices and choosing production quantities for a single product over a finite horizon for a capacity-constrained manufacturer facing price-sensitive demands	Yes	No	Yes
Chan et al. [17]	2006	Study delayed production and delayed pricing strategies for a multiple period horizon under a general, nonstationary stochastic demand function with a discrete menu of prices	Yes	No	Yes
This paper	This year	Determines the optimal pricing and production master planning in a multi period horizon considering capacity and inventory constraints	Yes	Yes	Yes

3. The Single-Period Problem with Known Initial and Ending Inventories

The multiperiod model given by (2.1) to (2.6) can be simplified in the single-period case. The single-period model without assuming I_1 is given as

$$\max\ Z = p_1 \cdot s_1 - V_1 \cdot n_1 - F \tag{3.1}$$

s.t.

$$s_1 \leq D_1(p_1), \tag{3.2}$$
$$n_1 \leq C_1, \tag{3.3}$$
$$I_1 = I_0 + n_1 - s_1, \tag{3.4}$$
$$I_1 \leq I_1^{\max}, \tag{3.5}$$
$$I_1 \geq I_1^{\min}. \tag{3.6}$$

As before, all variables are assumed to be nonnegative and the initial inventory (I_0) is assumed to be feasible with respect to (2.5) and (2.6). Now, the problem with given initial and ending inventories, I_0 and I_1, respectively, that are feasible with respect to (2.5) and (2.6) can be formulated as

$$\max\ Z = p_1 \cdot s_1 - V_1 \cdot n_1 - F \tag{3.7}$$

s.t.

$$s_1 = D_1(p_1), \tag{3.8}$$
$$n_1 \leq C_1, \tag{3.9}$$
$$I_1 = I_0 + n_1 - s_1, \tag{3.10}$$
$$n_1 \geq 0. \tag{3.11}$$

Constraints (3.5) and (3.6) can be eliminated because I_0 and I_1 are assumed to be feasible. A proof to justify the equality in (3.8) can be found in the appendix. Using (3.8) and (3.10), formulations (3.7)–(3.11) can be simplified to

$$\max\ Z = (p_1 - V_1) \cdot D(p_1) + V_1 \cdot (I_0 - I_1) - F \tag{3.12}$$

s.t.

$$D_1(p_1) \leq C_1 + I_0 - I_1, \tag{3.13}$$
$$D_1(p_1) \geq I_0 - I_1. \tag{3.14}$$

In the next section a closed form solution to (3.7)–(3.11), assuming a specific parametric form of the demand function, is derived. We drop the subscripts for simplicity.

4. Closed Form Solution with an Exponential Demand Function

We now present an analytic solution assuming an exponential demand function given by

$$D(p) = M \exp\left(-\frac{p}{k}\right), \tag{4.1}$$

where M is the y-intercept (demand) with a price equal to zero and $k > 0$ is a price scaling constant. See Ladany [34] and Smith and Achabal [35] for previous examples of the use of this function to model demand as a function of price. Notice that with the exponential demand function given above, problem (3.12)–(3.14) is infeasible when $I_0 - I_1 > M$, since M is the absolute upper bound on demand and when $I_1 - I_0 > C$, since C is the absolute upper bound on production. The following proposition gives the optimal closed form solution.

Proposition 4.1. *The optimal values of sales price, sales quantity, and production quantity for problem* (3.7)–(3.11) *with* $D(p) = M\exp(-p/k)$ *with* $I_0 - I_1 \le M$ *and* $I_1 - I_0 \le C$ *are given by*

$$p^* = \begin{cases} \min\{\max\{V + k, k[\ln(M) - \ln(C + I_0 - I_1)]\}, & \\ \qquad k[\ln(M) - \ln(I_0 - I_1)]\}, & \text{if } I_1 - I_0 < 0, \\ \max\{V + k, k[\ln(M) - \ln(C + I_0 - I_1)]\}, & \text{if } 0 \le I_1 - I_0 < C, \\ \infty & \text{if } I_1 - I_0 = C, \end{cases} \tag{4.2}$$

$$s^* = D(p^*), \tag{4.3}$$

$$n^* = s^* - I_0 + I_1. \tag{4.4}$$

Proof. The unconstrained version of problem (3.12)–(3.14) using the exponential demand function is given by

$$\max\ Z = (p - V)M \exp\left(-\frac{p}{k}\right) + V\ (I_0 - I_1) - F. \tag{4.5}$$

Problem (4.5) is solved by setting the derivative with respect to p equal to zero,

$$\begin{gathered} \frac{d}{dp}\left[(p - V)\cdot M \cdot \exp\left(-\frac{p}{k}\right)\right] = 0, \\ M\left[\exp\left(-\frac{p}{k}\right) + (p - V)\cdot\left(-\frac{1}{k}\right)\cdot\exp\left(-\frac{p}{k}\right)\right] = 0, \\ M\left[(V - p + k)\cdot\exp\left(-\frac{p}{k}\right)\right] = 0. \end{gathered} \tag{4.6}$$

One solution is $p = V + k$ (the other is at infinity). The solution $p = V + k$ can be shown to be a maximum by verifying that the second derivative is negative at that point. The second derivative is given by

$$\frac{d^2}{dp}\left[(p-V)\cdot M\cdot \exp\left(-\frac{p}{k}\right)\right] = \frac{M}{k^2}(p-V)\exp\left(-\frac{p}{k}\right) - \frac{2M}{k}\exp\left(-\frac{p}{k}\right). \tag{4.7}$$

With $p = V + k$, we obtain

$$-\frac{M}{k}\exp\left(-\frac{V+k}{k}\right), \tag{4.8}$$

which can be seen to be negative by inspection. Problem (4.5) thus has a maximum at $p = V + k$, is strictly decreasing for $p > V + k$ and strictly increasing for $p < V + k$. This result will be used later.

For ease of reference, we define

$$L_1 = D^{-1}(C + I_0 - I_1),$$
$$L_2 = \begin{cases} D^{-1}(I_0 - I_1), & \text{if } I_0 - I_1 > 0, \\ \infty, & \text{if } I_0 - I_1 \leq 0. \end{cases} \tag{4.9}$$

Notice that these quantities are related to the right-hand sides of (3.13) and (3.14). The relationship $L_1 \leq L_2$ can be seen to be true by inspection. Constraints (3.13) and (3.14) can be solved for p to obtain

$$p \geq L_1, \tag{4.10}$$

$$p \leq L_2, \tag{4.11}$$

respectively. Since $L_1 \leq L_2$, three cases are possible.

Case 1. $V + k \leq L_1 \leq L_2$.

Case 2. $L_1 \leq V + k \leq L_2$.

Case 3. $L_1 \leq L_2 \leq V + k$.

In Case 1 the prices given by (4.10) and (4.11) at equality are both to the right of the unconstrained maximum at $V + k$. Therefore (4.10) is the binding constraint that determines the solution to the problem. In Case 2 the unconstrained solution at $V + k$ is between the prices given by (4.10) and (4.11). Neither constraint is binding so the solution is at $V + k$. In Case 3 the prices given by (4.10) and (5.1) are both to the left of the unconstrained maximum at $V + k$. Therefore (4.11) is the binding constraint that determines the optimal price. It can be easily verified that (4.2) provides the correct answer in each case. Expression (4.3) follows from the proof of (3.8) given in the appendix and (4.4) follows from (3.10). □

5. Solving the Multiperiod Problem

To solve the multiperiod model a dynamic programming solution approach employing the result of Proposition 4.1 is developed in this section. In order to simplify the procedure for dynamic programming, we allow only integer inventory quantities. This is well justified because the inventory quantities would be integer values in a real application. Letting $t$represent the time period, the recursive relationship for backward induction is

$$f_t^*(I_{t-1}) = \max_{I_t=I_t^{\min},\ldots,I_t^{\max}} f_t(I_{t-1}, I_t), \tag{5.1}$$

with

$$f_t(I_{t-1}, I_t) = p_t \cdot s_t - V_t \cdot n_t + f_{t+1}^*(I_t), \tag{5.2}$$

where $f_t(I_{t-1}, I_t)$ is the contribution of periods from time t until the end of the horizon given period t begins with I_{t-1} inventory, ends with I_t inventory and optimal decisions are made thereafter. The value of $f_{T+1}^*(I_T)$ is by definition equal to zero and p_t, s_t, and n_t are defined as follows:

$$p_t = \begin{cases} \min\{\max\{V_t + k_t, k_t[\ln(M_t) - \ln(C_t + I_{t-1} - I_t)]\}, \\ \qquad k_t[\ln(M_t) - \ln(I_{t-1} - I_t)]\}, & \text{if } I_t - I_{t-1} < 0, \\ \max\{V_t + k_t, k_t[\ln(M_t) - \ln(C_t + I_{t-1} - I_t)]\}, & \text{if } 0 \le I_t - I_{t-1} < C_t, \\ \infty, & \text{if } I_t - I_{t-1} = C_t, \end{cases} \tag{5.3}$$

$$s_t = D_t(p_t), \tag{5.4}$$

$$n_t = s_t - I_{t-1} + I_t. \tag{5.5}$$

When a discounted cash flow approach is used, (5.2) becomes

$$f_t(I_{t-1}, I_t) = (1+r)^{-t} \cdot (p_t \cdot s_t - V_t \cdot n_t) + f_{t+1}^*(I_t). \tag{5.6}$$

In the implementation of the method, when $s_t = 0$, which can occur when the ending inventory is greater than the initial inventory by an amount exactly equal to the available capacity, the price is not relevant and is set equal to any positive constant in order to correctly evaluate the objective function. When $I_{t-1} - I_t > M_t$, which makes the problem infeasible, the objective function is set to a negative number, effectively eliminating such a combination from further consideration. The problem is also infeasible when $I_t - I_{t-1} > C_t$ since C_t is the absolute upper bound on production. These cases are explicitly excluded from consideration.

Table 2: For $t = 3$.

I_2	I_3 0	1	2	3	$f_3^*(I_2)$	I_3^*
0	33.288	29.726	23.526	13.263	33.288	0
1	35.311	33.288	29.726	23.526	35.311	0
2	37.252	35.311	33.288	29.726	37.252	0
3	39.193	37.252	35.311	33.288	39.193	0

Table 3: For $t = 2$.

I_1	I_2 0	1	2	3	$f_2^*(I_1)$	I_2^*
0	36.472	36.534	36.515	36.223	36.534	1
1	38.433	38.495	38.475	38.456	38.495	1
2	40.314	40.456	40.436	40.416	40.456	1
3	41.442	42.337	42.397	42.377	42.397	2

Table 4: For $t = 1$.

I_0	I_1 0	1	2	3	$f_1^*(I_1)$	I_1^*
1	44.124	44.105	44.086	44.047	44.124	0

6. Numerical Examples

In this section, some numerical examples will be presented to illustrate the dynamic programming solution approach on a small three-period problem. Let, $r = 0.01$, $I_0 = 1$, $M_1 = 10$, $M_2 = 12$, $M_3 = 15$, $k_1 = 3$, $k_2 = 2$, $k_3 = 8$, and $C_t = 4$, $V_t = 2$, $I_t^{\min} = 0$, $I_t^{\max} = 3$ for all t. Table 2 for $t = 3$ is populated using (5.1)–(5.5). The value of $f_4^*(\cdot)$ is by definition equal to zero.

Table 3 for $t = 2$ is populated similarly.

Table 4 for $t = 1$ is populated similarly.

The optimal solution is shown in Table 5. The optimal sales prices, sales quantities, and production quantities can be found using (5.3), (5.4), and (5.5), respectively. Notice that the sales and production quantities are not integer values. In practical master planning applications that are solved using linear programming, noninteger values are acceptable approximations due to the aggregate nature of the products, the medium to long-term time horizons involved, and the typically large quantities planned to be produced.

Three additional illustrative examples will be presented to highlight some of the behavior of the model. The following example illustrates how it is possible to have an optimal solution in which, although it is feasible to produce and sell the optimal quantity in one period when that period is considered in isolation, the sale in that period will be limited to allow greater sales in a later more profitable period. Let $r = 0.01$, $I_0 = 0$, $M_1 = 100$, $M_2 = 100$, $M_3 = 610$, $k_1 = 5.1$, $k_2 = 5.1$, $k_3 = 8$, and $C_t = 21$, $V_t = 10$, $I_t^{\min} = 0$, $I_t^{\max} = 40$ for all t. The optimal solution is shown in Table 6. Notice how although it is feasible to sell 5.18 units in periods 1 and 2 (which would be optimal for those periods taken in isolation), the optimal solution is to limit sales in the first two periods to allow greater sales in the last period.

Table 5: Optimal solution for example 1.

T	I_t	s_t	p_t	n_t
1	0	1.89	5.0	0.89
2	1	1.62	4.0	2.62
3	0	4.30	10.0	3.30

Table 6: Optimal solution for example 2.

T	I_t	s_t	p_t	n_t
1	17	4.0	16.42	21.0
2	34	4.0	16.42	21.0
3	0	55.0	19.25	21.0

Table 7: Optimal solution for example 3.

T	I_t	s_t	p_t	n_t
1	0	5.18	15.10	5.18
2	28	5.18	15.10	33.18
3	0	63.00	18.16	35.00

Table 8: Optimal solution for example 4.

T	I_t	s_t	p_t	n_t
1	16	5.00	15.28	21.00
2	28	5.18	15.10	17.18
3	0	63.00	18.16	35.00

Our next example illustrates what we call *horizon decoupling* which can help solve problems with many time periods when production costs are constant or decrease over time. It is worth noticing that the *horizon decoupling* is an example of a regeneration point, which is a fundamental construct of planning horizon theory. See, for instance, Chand et al. [36] for a review of literature on planning horizon theory. This example is identical to the previous one with the exception that $C_2 = C_3 = 35$. The optimal solution is shown in Table 7. Notice that the capacity in the last period is not enough to produce its optimal (when considered in isolation with infinite capacity) sales amount of 64.30 units. It, therefore, remains coupled to previous periods. Further notice that the two last periods do have between them enough capacity to produce their optimal (when each period is considered in isolation with infinite capacity) sales amounts of $(64.30 + 5.18 = 69.48)$. They, therefore, decouple from previous periods and can be solved independently of any previous periods. This property of the problem may allow problems with long planning horizons to be split into several smaller problems with shorter planning horizons that can be solved more easily. Notice, however, that if we let $V_1 = 9$, the optimal solution calls for inventory to be accumulated at the end of period 1 to take advantage of the lower production cost. The optimal solution with $V_1 = 9$ is shown in Table 8. Thus, it cannot be assumed that the planning horizon will decouple when production costs are increasing over time. It is important to note that if the setup costs are included, the planning horizon results would change.

Table 9: Optimal solution for example 5.

T	I_t	s_t	p_t	n_t
1	1	4.0	16.09	5.0
2	3	3.0	17.53	5.0
3	0	8.0	24.37	5.0

Table 10: Projected plan assuming aggregate capacity.

T	I_t	s_t	p_t	n_t
1	0	2.52	18.4	2.52
2	0	2.52	18.4	2.52
3	0	9.96	22.4	9.96

The last example we present illustrates the value of solving a joint pricing and production problem taking into account capacity and inventory constraints. The parameters of the problem are the following: $r = 0.01$, $I_0 = 0$, $M_1 = 100$, $M_2 = 100$, $M_3 = 110$, $k_1 = 5$, $k_2 = 5$, $k_3 = 7$, and $C_t = 5$, $V_t = 10$, $I_t^{\min} = 0$, $I_t^{\max} = 2$ for all t. The optimal solution is shown in Table 10. The optimal objective value is \$161.96. Now assume the marketing department sets prices using the same data but assuming that the aggregate capacity over the next three periods is equal to 15. That is, an aggregate capacity limit is imposed rather than a period by period limit. The projected plan under these assumptions is shown in Table 9. The projected objective value would be \$165.87. Now assuming that the marketing department executed to the planned prices, but production is now constrained by the real inventory and capacity limits, the greatest possible profit is only \$97.96, well below both the projected plan and the optimal plan. Now suppose that marketing creates a pricing plan taking into account the period by period capacity limits but fails to consider the inventory limits. The projected plan is shown in Table 11. The projected objective value would be \$165.36. Now assuming that marketing executes to the planned prices, but production is now constrained by the real inventory and capacity limits, the greatest possible profit is only \$97.44, also well below both the projected plan and the optimal plan. These examples show that neglecting to consider capacity and/or inventory constraints can have very significant detrimental effects on the profitability of the firm.

7. Conclusions and Recommendations for Further Research

In conclusion, we derive an exact solution to the single-period price-optimizing master planning problem with deterministic demand and inventory and capacity constraints for the case with known initial and ending inventories. In addition, we show how to solve the multiperiod version of the problem using a dynamic programming approach. Our direct observation of the planning models in use at a variety of industries shows that the types of constraints we consider are commonly used in practice but largely missing in literature. We also show that inventory holding costs can be included in the model by discounting the terms of the objective function. The numerical examples presented serve to highlight the importance of taking into account capacity and inventory constraints when generating a pricing and production plan. The implication for practitioners is that potentially significantly higher profits can be obtained through price optimization, making sure to consider the firms capacity and inventory constraints.

Table 11: Projected plan assuming no constraints on inventory.

T	I_t	s_t	p_t	n_t
1	2	3.0	17.53	5.0
2	5	2.0	19.56	5.0
3	0	10.0	22.36	5.0

It is worth noting that our proposed model has three main advantages. First, our model considers both capacity and inventory limits. Our consulting experience shows that firms take both types of constraints into account, making their inclusion desirable. Second, we address the problem at a master planning level, where setups are usually not considered. The previously published works address similar problems at a lot scheduling level despite the fact the price changes (with the exception of discounts) are often not feasible in the short term. Third, our model considers the net present value approach instead of the traditional approach.

The research presented in this paper may be extended in several ways. One extension is to solve the model with an upper bound on the price or, alternatively, on the allowable change in price between periods, which is a realistic market scenario. In addition, solution approaches could be developed for multiitem and stochastic versions of the problem. Models with days-of-cover constraints would also be relevant research topics as would be the inclusion of setup costs in a mixed integer formulation. An additional recommendation is to investigate a dynamic control version of the problem, which would recast the problem from a planning level to an operational level. Additional possible extensions are to reformulate the model to include demand learning effects [37], and to model the supply chain with a supplier-buyer relationship as two-player nonzero sum differential game [38].

Appendix

Justification of the Equality in Constraint (3.8)

Claim 1. For problem (3.7)–(3.11), with (3.8) rewritten as $s_1 \leq D(p_1)$ and assuming that $D(p)$ is a demand function that is continuous, nonincreasing and asymptotically equal to zero, if p^*, s^*, and n^* are optimal, then $s^* = D(p^*)$.

Proof. Assume that $s^* \neq D(p^*)$. This yields two cases. The first is that $s^* > D(p^*)$. This case can be ignored since it is impossible for sales to exceed demand. The remaining case is that $s^* < D(p^*)$. For optimality we require that $s^*p^* - Vn^* \geq sp - Vn$ for any feasible choice of s, p, and n (notice that p is not bounded from above by (3.8)–(3.11)). However, if $s^* < D(p^*)$, given that $D(p)$ is continuous, nonincreasing and asymptotically equal to zero, there exists a feasible $p^s > p^*$ such that $s^* = D(p^s)$. This implies that $s^*p^* - Vn^* < s^*p^s - Vn^*$, which contradicts the optimality of p^*. □

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