Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 341982, 14 pages
doi:10.1155/2010/341982

Research Article

Convergence Analysis of Preconditioned AOR Iterative Method for Linear Systems

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Received 26 June 2009; Revised 21 February 2010; Accepted 13 May 2010

Academic Editor: Paulo Batista Gonçalves

M-(H-)matrices appear in many areas of science and engineering, for example, in the solution of the linear complementarity problem (LCP) in optimization theory and in the solution of large systems for real-time changes of data in fluid analysis in car industry. Classical (stationary) iterative methods used for the solution of linear systems have been shown to convergence for this class of matrices. In this paper, we present some comparison theorems on the preconditioned AOR iterative method for solving the linear system. Comparison results show that the rate of convergence of the preconditioned iterative method is faster than the rate of convergence of the classical iterative method. Meanwhile, we apply the preconditioner to H-matrices and obtain the convergence result. Numerical examples are given to illustrate our results.

1. Introduction

In numerical linear algebra, the theory of M- and H-matrices is very important for the solution of linear systems of algebra equations by iterative methods (see, e.g., [1–14]). For example, (a) in the linear complementarity problem (LCP) (see [5, Section 10.1] for specific applications), where we are interested in finding a $z \in R^n$ such that $z \geq 0$, $Mz + q \geq 0$, $z^T(Mz + q) = 0$, with $M \in R^{n\times n}$ and $q \in R^n$ given, a sufficient condition for a solution to exist, and to be found by a modification of an iterative method, especially of SOR, is that M is an H-matrix, with $m_{i,i} > 0$, $i = 1, \ldots, n$ [15]; (b) in fluid analysis, in the car modeling design [16, 17], it was observed that large linear systems with an H-matrix coefficient A are solved iteratively much faster if A is postmultiplied by a suitable diagonal matrix D, with $d_{i,i} > 0$, $i = 1, \ldots, n$, so that AD is strictly diagonally dominant. We consider the following linear system:

$$Ax = b, \tag{1.1}$$

where A is an $n \times n$ square matrix, x and b are two n-dimensional vectors. For any splitting, $A = M - N$ with the nonsingular matrix M, the basic iterative method for solving the linear system (1.1) is as follows:

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b, \quad i = 0, 1, 2, \ldots. \tag{1.2}$$

Without loss of generality, let $A = I - L - U$ and $a_{i,1} \neq 0$, $i = 2, \ldots, n$, where L and U are strictly lower triangular and strictly upper triangular matrices of A, respectively. Then the iterative matrix of the AOR iterative method [18] for solving the linear system (1.1) is

$$T_{\gamma,\omega} = (I - \gamma L)^{-1}\left[(1-\omega)I + (\omega - \gamma)L + \omega U\right], \tag{1.3}$$

where ω and γ are nonnegative real parameters with $\omega \neq 0$.

To improve the convergence rate of the basic iterative methods, several preconditioned iterative methods have been proposed in [8, 12, 13, 19–24]. We now transform the original system (1.1) into the preconditioned form

$$PAx = Pb, \tag{1.4}$$

where P is a nonsingular matrix. The corresponding basic iterative method is given in general by

$$x^{i+1} = M_P^{-1}N_P x^i + M_P^{-1}Pb, \quad i = 0, 1, 2, \ldots, \tag{1.5}$$

where $PA = M_P - N_P$ is a splitting of PA.

Milaszewicz [19] presented a modified Jacobi and Gauss-Seidel iterative methods by using the preconditioned matrix $P = I + S$, where

$$P = (I + S) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{bmatrix}. \tag{1.6}$$

The author [19] suggests that if the original iteration matrix is nonnegative and irreducible, then performing Gaussian elimination on a selected column of iteration matrix to make it zero will improve the convergence of the iteration matrix.

In 2003, Hadjidimos et al. [4] considered the generalized preconditioner used in this case is of the form

$$P(\alpha) = (I + S_\alpha) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_2 a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_n a_{n1} & 0 & \cdots & 1 \end{bmatrix}, \tag{1.7}$$

where $\alpha = (\alpha_2, \ldots, \alpha_n)^T \in R^{n-1}$ with $\alpha_i \in [0,1]$, $i = 2, \ldots, n$, constants. The selection of α's will be made from the $(n-1)$-dimensional nonnegative cone K_{n-1} in such a way that none of the diagonal elements of the preconditioned matrix $\widetilde{A} = P(\alpha)A$ vanishes. They discussed the convergence of preconditioned Jacobi and Gauss-Seidel when a coefficient matrix A is an M-matrix.

In this paper, we consider the preconditioned linear system of the form

$$\widetilde{A}x = \widetilde{b}, \tag{1.8}$$

where $\widetilde{A} = (I + S_\alpha)A$ and $\widetilde{b} = (I + S_\alpha)b$. It is clear that $S_\alpha L = 0$. Thus, we obtain the equality

$$\widetilde{A} = (I + S_\alpha)A = (I + S_\alpha)(I - L - U) = I - S_D - L - S_L + S_\alpha - U - S_U, \tag{1.9}$$

where S_D, S_L, and S_U are the diagonal, strictly lower, and strictly upper triangular parts of the matrix $S_\alpha U$, respectively. If we apply the AOR iterative method to the preconditioned linear system (1.8), then we get the preconditioned AOR iterative method whose iteration matrix is

$$\widetilde{T}_{\gamma,\omega} = \left(\widetilde{D} - \gamma\widetilde{L}\right)^{-1}\left[(1-\omega)\widetilde{D} + (\omega - \gamma)\widetilde{L} + \omega\widetilde{U}\right]. \tag{1.10}$$

This paper is organized as follows. Section 2 is preliminaries. Section 3 will discuss the convergence of the preconditioned AOR method and obtain comparison theorems with the classical iterative method when a coefficient matrix is a Z-matrix. In Section 4 we apply the preconditioner to H-matrices and obtain the convergence result. In Section 5 we use numerical examples to illustrate our results.

2. Preliminaries

We say that a vector x is nonnegative (positive), denoted $x \geq 0$ $(x > 0)$, if all its entries are nonnegative (positive). Similarly, a matrix B is said to be nonnegative, denoted $B \geq 0$, if all its entries are nonnegative or, equivalently, if it leaves invariant the set of all nonnegative vectors. We compare two matrices $A \geq B$, when $A - B \geq 0$, and two vectors $x \geq y$ $(x > y)$ when $x - y \geq 0$ $(x - y > 0)$. Given a matrix $A = (a_{i,j})$, we define the matrix $|A| = (|a_{i,j}|)$. It follows that $|A| \geq 0$ and that $|AB| \leq |A||B|$ for any two matrices A and B of compatible size.

Definition 2.1. A matrix $A = (a_{i,j}) \in R^{n\times n}$ is called a Z-matrix if $a_{i,j} \leq 0$ for $i \neq j$. A matrix A is called a nonsingular M-matrix if A is a Z-matrix and $A^{-1} \geq 0$.

Definition 2.2. A matrix A is an H-matrix if its comparison matrix $\langle A \rangle = (\overline{a}_{i,j})$ is an M-matrix, where $\overline{a}_{i,j}$ is

$$\overline{a}_{i,i} = |a_{i,i}|, \qquad \overline{a}_{i,j} = -\left|a_{i,j}\right|, \quad i \neq j. \tag{2.1}$$

Definition 2.3 (see [1]). The splitting $A = M - N$ is called an H-splitting if $\langle M \rangle - |N|$ is an M-matrix and an H-compatible splitting if $\langle A \rangle = \langle M \rangle - |N|$.

Definition 2.4. Let $A = (a_{i,j}) \in R^{n\times n}$. $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called

(a) convergent if $\rho(M^{-1}N) < 1$

(b) regular if $M^{-1} \geq 0$ and $N \geq 0$

(c) nonnegative if $M^{-1}N \geq 0$

(d) M-splitting if M is a nonsingular M-matrix and $N \geq 0$.

Lemma 2.5 (see [1]). *Let $A = M - N$ be a splitting. If the splitting is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leq \rho(\langle M\rangle^{-1}|N|) < 1$. If the splitting is an H-compatible splitting and A is an H-matrix, then it is an H-splitting and thus convergent.*

Lemma 2.6 (Perron-Frobenius theorem). *Let $A \geq 0$ be an irreducible matrix. Then the following hold:*

(a) *A has a positive eigenvalue equal to $\rho(A)$.*

(b) *A has an eigenvector $x > 0$ corresponding to $\rho(A)$.*

(c) *$\rho(A)$ is a simple eigenvalue of A.*

Lemma 2.7 (see [3, 25]). *Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ $(= 1)$ if and only if A is a nonsingular (singular) M-matrix. If A is irreducible, then here is a positive vector x such that $M^{-1}Nx = \rho(M^{-1}N)x$.*

Lemma 2.8 (see [5]). *Let $A \geq 0$ be a nonnegative matrix. Then the following hold.*

(a) *If $Ax \geq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\rho(A) \geq \beta$.*

(b) *If $Ax \leq \gamma x$ for a vector $x > 0$, then $\rho(A) \leq \gamma$; moreover, if A is irreducible and if $\beta x \leq Ax \leq \gamma x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\beta < \rho(A) < \gamma$ and $x > 0$.*

3. Convergence Theorems for Z-Matrix

We first consider the convergence of the iteration matrix $\widetilde{T}_{\gamma,\omega}$ of the preconditioned linear system (1.8) when the coefficient matrix is a Z-matrix.

Particularly, we consider $\alpha_i = 1$, $i = 2, \dots, n$. Define

$$\overline{A} = (I + S_1)A = (I + S_1)(I - L - U) = I - D' - L - L' + S_1 - U - U', \tag{3.1}$$

where D', L', and U' are diagonal, strictly lower triangular, and strictly upper triangular parts of the matrix S_1U, respectively. Then the preconditioned AOR method is expressed as follows:

$$\overline{T}_{\gamma,\omega} = \left(\overline{D} - \gamma\overline{L}\right)^{-1}\left[(1-\omega)\overline{D} + (\omega-\gamma)\overline{L} + \omega\overline{U}\right], \tag{3.2}$$

where $\overline{D} = I - D'$, $\overline{L} = L + L' - S_1$, and $\overline{U} = U + U'$ are the diagonal, strictly lower, and strictly upper triangular matrices obtained from $\overline{A}$, respectively.

Lemma 3.1. *Let $A = (a_{i,j}) \in R^{n\times n}$ be a Z-matrix. Then $(I + S_\alpha)A$ is also a Z-matrix.*

Proof. Since $\widetilde{A} = (\widetilde{a}_{i,j}) = (I + S_\alpha)A$, we have

$$\widetilde{a}_{i,j} = \begin{cases} a_{i,j}, & i = 1, \\ a_{i,j} - \alpha_i a_{i,1} a_{1,j}, & i = 2, \dots, n. \end{cases} \tag{3.3}$$

It is clear that $\widetilde{A}$ is a Z-matrix for any $\alpha_i \in [0,1]$, $i = 2, \dots, n$. □

Lemma 3.2. *Let $T_{\gamma,\omega}$ and $\widetilde{T}_{\gamma,\omega}$ be defined by* (1.3) *and* (1.10)*. Assume that $0 \le \gamma \le \omega \le 1$ $(\omega \ne 0, \gamma \ne 1)$. If A is an irreducible Z-matrix with $a_{i1}a_{1i} < 1$, $i = 2, \dots, n$, for $\alpha_i \in (0,1)$, $i = 2, \dots, n$, then $T_{\gamma,\omega}$ and $\widetilde{T}_{\gamma,\omega}$ are nonnegative and irreducible.*

Proof. Since $A = I - L - U$ is irreducible. Then for $\alpha_i \in (0,1)$, $i = 2, \dots, n$, we have that $\widetilde{A} = (I + S_\alpha)A = \widetilde{D} - \widetilde{L} - \widetilde{U}$ is also irreducible. Observe that

$$T_{\gamma,\omega} = (1 - \omega)I + \omega(1 - \gamma)L + \omega U + T, \tag{3.4}$$

where T is a nonnegative matrix. As A is an irreducible Z-matrix and $\omega \ne 0$, $\gamma \ne 1$, it is easy to show that $T_{\gamma,\omega}$ is nonnegative and irreducible. By assumption, $\widetilde{D}, \widetilde{L}$, and $\widetilde{U}$ are all nonnegative and thus $\widetilde{T}_{\gamma,\omega}$ is nonnegative. Observe that $\widetilde{T}_{\gamma,\omega}$ can be expressed as

$$\widetilde{T}_{\gamma,\omega} = (1 - \omega)I + \omega(1 - \gamma)\widetilde{D}^{-1}\widetilde{L} + \omega\widetilde{D}^{-1}\widetilde{U} + \widetilde{T}, \tag{3.5}$$

where $\widetilde{T}$ is a nonnegative matrix. Since $\omega \ne 0$, $\gamma \ne 1$, and $\widetilde{A}$ is irreducible, $\omega(1-\gamma)\widetilde{D}^{-1}\widetilde{L}+\omega\widetilde{D}^{-1}\widetilde{U}$ is irreducible. Hence, $\widetilde{T}_{\gamma,\omega}$ is irreducible from (3.5). □

Our main result in this section is as follows.

Theorem 3.3. *Let $T_{\gamma,\omega}$ and $\widetilde{T}_{\gamma,\omega}$ be defined by* (1.3) *and* (1.10)*. Assume that $0 \le \gamma \le \omega \le 1$ $(\omega \ne 0, \gamma \ne 1)$. If A is an irreducible Z-matrix with $a_{i1}a_{1i} < 1$, $i = 2, \dots, n$, for $\alpha_i \in (0,1)$, $i = 2, \dots, n$, then*

(a) *for $\alpha_i \in (0,1)$, $\rho(\widetilde{T}_{\gamma,\omega}) < \rho(T_{\gamma,\omega}) < 1$ if $\rho(T_{\gamma,\omega}) < 1$;*

(b) *for $\alpha_i \in [0,1]$, $\rho(\widetilde{T}_{\gamma,\omega}) = \rho(T_{\gamma,\omega}) = 1$ if $\rho(T_{\gamma,\omega}) = 1$;*

(c) *for $\alpha_i \in (0,1)$, $\rho(\widetilde{T}_{\gamma,\omega}) > \rho(T_{\gamma,\omega}) > 1$ if $\rho(T_{\gamma,\omega}) > 1$.*

Proof. Let $A = I - L - U$ be irreducible. It is clear that $I - \gamma L$ is an M-matrix and $(1 - \omega)I + (\omega - \gamma)L + \omega U \ge 0$. So $A = (I - \gamma L) - [(1 - \omega)I + (\omega - \gamma)L + \omega U]$ is an M-splitting of A. From Lemma 2.7, there exists a positive vector x such that

$$T_{\gamma,\omega}x = \lambda x, \tag{3.6}$$

where λ denotes the spectral radius of $T_{\gamma,\omega}$. Observe that $T_{\gamma,\omega} = (I-\gamma L)^{-1}[(1-\omega)I+(\omega-\gamma)L+\omega U]$; we have

$$[(1-\omega)I + (\omega-\gamma)L + \omega U]x = \lambda(I-\gamma L)x, \tag{3.7}$$

which is equivalent to

$$(\lambda-1)(I-\gamma L)x = \omega(L+U-I)x. \tag{3.8}$$

Let $S_\alpha U = S_D + S_L + S_U$, where S_D, S_L, and S_U are the diagonal, strictly lower, and strictly upper triangular parts of $S_\alpha U$, respectively. It is clear that $S_\alpha L = 0$, so

$$\widetilde{A} = \widetilde{D} - \widetilde{L} - \widetilde{U} = (I - S_D) - (L + S_L - S_\alpha) - (U + S_U), \tag{3.9}$$

where

$$\widetilde{D} = I - S_D, \qquad \widetilde{L} = L + S_L - S_\alpha, \qquad \widetilde{U} = U + S_U. \tag{3.10}$$

From (3.8) and (3.10), we have

$$\begin{aligned}
\widetilde{T}_{\gamma,\omega}x - \lambda x &= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(1-\omega)\widetilde{D} + (\omega-\gamma)\widetilde{L} + \omega\widetilde{U} - \lambda\left(\widetilde{D}-\gamma\widetilde{L}\right)\right]x \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(1-\omega-\lambda)\widetilde{D} + (\omega-\gamma+\lambda\gamma)\widetilde{L} + \omega\widetilde{U}\right]x \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(1-\omega-\lambda)(I-S_D) + (\omega-\gamma+\lambda\gamma)(L+S_L-S_\alpha) + \omega(U+S_U)\right. \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(1-\omega-\lambda)I + (\omega-\gamma+\lambda\gamma)L + \omega U\right. \\
&\quad \left. -(1-\omega-\lambda)S_D + (\omega-\gamma+\lambda\gamma)S_L - (\omega-\gamma+\lambda\gamma)S_\alpha + \omega S_U\right]x \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(\lambda-1)S_D + \omega S_D + (\lambda-1)\gamma S_L + \omega S_L + \omega S_U - (\omega-\gamma+\lambda\gamma)S_\alpha\right]x \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(\lambda-1)S_D + (\lambda-1)\gamma S_L + \omega S_\alpha U - \omega S_\alpha + \omega S_\alpha L - (\lambda-1)\gamma S_\alpha\right]x \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(\lambda-1)S_D + (\lambda-1)\gamma S_L - (\lambda-1)\gamma S_\alpha + \omega S_\alpha(U+L-I)\right]x \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(\lambda-1)S_D + (\lambda-1)\gamma S_L - (\lambda-1)\gamma S_\alpha + (\lambda-1)S_\alpha(I-\gamma L)\right]x \\
&= \left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[(\lambda-1)S_D + (\lambda-1)\gamma S_L - (\lambda-1)\gamma S_\alpha + (\lambda-1)S_\alpha\right]x \\
&= (\lambda-1)\left(\widetilde{D}-\gamma\widetilde{L}\right)^{-1}\left[S_D + (1-\gamma)S_\alpha + \gamma S_L\right]x.
\end{aligned} \tag{3.11}$$

Since $a_{i,1}a_{1,i} < 1$, $i = 2, \dots, n$, then $\widetilde{D} - \gamma\widetilde{L}$ is an M-matrix. Notice that $S_D \geq 0$, $S_\alpha \geq 0$, and $S_L \geq 0$. If $\lambda < 1$, then from (3.11), we have $\widetilde{T}_{\gamma,\omega}x \leq \lambda x$. As $x > 0$, Lemma 2.8 implied that

$\rho(\widetilde{T}_{\gamma,\omega}) \leq \lambda = \rho(T_{\gamma,\omega})$. For the case of $\lambda = 1$ and $\lambda > 1$, $\widetilde{T}_{\gamma,\omega}x = \lambda x$ and $\widetilde{T}_{\gamma,\omega}x \geq \lambda x$ are obtained from (3.11), respectively. Hence, Theorem 3.3 follows from Lemmas 2.8 and 3.2. □

We next consider the case of $\alpha_i = 1$, $i = 2, \ldots, n$; the convergence theorem is given as follows see [26, 27].

Theorem 3.4. *Let $T_{\gamma,\omega}$ and $\overline{T}_{\gamma,\omega}$ be defined by* (1.3) *and* (1.10)*. Assume that A is an irreducible Z-matrix and $A(2:n, 2:n)$ is an irreducible submatrix of A deleting the first row and the first column. Then for $0 \leq \gamma \leq \omega \leq 1$ $(\omega \neq 0, \gamma \neq 1)$ and $a_{i1}a_{1i} < 1$, $i = 2, \ldots, n$, we have*

(a) $\rho(\overline{T}_{\gamma,\omega}) < \rho(T_{\gamma,\omega}) < 1$ *if* $\rho(T_{\gamma,\omega}) < 1$;

(b) $\rho(\overline{T}_{\gamma,\omega}) = \rho(T_{\gamma,\omega}) = 1$ *if* $\rho(T_{\gamma,\omega}) = 1$;

(c) $\rho(\overline{T}_{\gamma,\omega}) > \rho(T_{\gamma,\omega}) > 1$ *if* $\rho(T_{\gamma,\omega}) > 1$.

Proof. Let $A = I - L - U$ be irreducible. It is clear that $I - \gamma L$ is an M-matrix and $(1-\omega)I + (\omega - \gamma)L + \omega U \geq 0$. So $A = (I - \gamma L) - [(1-\omega)I + (\omega-\gamma)L + \omega U]$ is an M-splitting of A. From Lemma 2.7, there exists a positive vector x such that

$$T_{\gamma,\omega}x = \lambda x, \tag{3.12}$$

where λ denotes the spectral radius of $T_{\gamma,\omega}$. Observe that $T_{\gamma,\omega} = (I-\gamma L)^{-1}[(1-\omega)I + (\omega-\gamma)L + \omega U]$; we have

$$\left[(1-\omega)I + (\omega-\gamma)L + \omega U\right]x = \lambda(I - \gamma L)x, \tag{3.13}$$

which is equivalent to

$$(\lambda - 1)(I - \gamma L)x = \omega(L + U - I)x. \tag{3.14}$$

Similar to the proof of the equality (3.11), we have

$$\begin{aligned} \overline{T}_{\gamma,\omega}x - \lambda x &= \left(\overline{D} - \gamma\overline{L}\right)^{-1}\left[(1-\omega)\overline{D} + (\omega-\gamma)\overline{L} + \omega\overline{U} - \lambda\left(\overline{D} - \gamma\overline{L}\right)\right]x \\ &= \left(\overline{D} - \gamma\overline{L}\right)^{-1}\left[(1-\omega-\lambda)\overline{D} + (\omega - \gamma + \lambda\gamma)\overline{L} + \omega\overline{U}\right]x. \end{aligned} \tag{3.15}$$

Since $\overline{D} = I - D'$, $\overline{L} = L + L' - S_1$, and $\overline{U} = U + U'$, then we have

$$\overline{T}_{\gamma,\omega}x - \lambda x = (\lambda - 1)\left(\overline{D} - \gamma\overline{L}\right)^{-1}(D' + \gamma L' + (1-\gamma)S_1)x. \tag{3.16}$$

By computation, we have

$$\overline{T}_{\gamma,\omega} = (1-\omega)I + \omega(1-\gamma)\overline{D}^{-1}\overline{L} + \omega\overline{D}^{-1}\overline{U} + \overline{H} = \begin{bmatrix} 1-\omega & \overline{T}_{1,2} \\ 0 & \overline{T}_{2,2} \end{bmatrix}, \tag{3.17}$$

where $\overline{H}$ is a nonnegative matrix, $\overline{T}_{1,2} \geq 0$ is a $1\times(n-1)$ matrix, and $\overline{T}_{2,2} \geq 0$ is an $(n-1)\times(n-1)$ matrix. As A is irreducible, then at least one $a_{1,i} \neq 0$ and $\overline{T}_{1,2}$ is nonzero. Since $A(2:n,2:n)$ is irreducible, it is clear that $\overline{A}(2:n,2:n)$ is irreducible. Since $\omega \neq 0$ and $\gamma \neq 1$, from (3.17), we have that $\overline{T}_{2,2}$ is irreducible. Let

$$u = \left(\overline{D} - \gamma\overline{L}\right)^{-1}\left(D' + \gamma L' + (1-\gamma)S_1\right)x, \qquad v = \left(\overline{D} - \gamma\overline{L}\right)^{-1}u. \tag{3.18}$$

From (3.18), and $x > 0$, we know that $u \geq 0$, and the first component of u is zero. Hence $v \geq 0$ and its first component is zero. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \tag{3.19}$$

where $x_1 \in R^1 > 0$, $x_2 \in R^{n-1} > 0$, and $v_2 \in R^{n-1} \geq 0$ being a nonzero vector. From (3.16) and (3.17), we have

$$\overline{T}_{\gamma,\omega}x - \lambda x = (\lambda - 1)v. \tag{3.20}$$

That is,

$$(1-\omega)x_1 + \overline{T}_{1,2}x_2 = \lambda x_1, \tag{3.21}$$

$$\overline{T}_{2,2}x_2 - \lambda x_2 = (\lambda - 1)v_2. \tag{3.22}$$

If $\lambda < 1$, from (3.22) and v_2 is a nonzero vector, we have

$$\overline{T}_{2,2}x_2 < \lambda x_2, \qquad (\lambda - 1)v_2 \neq 0. \tag{3.23}$$

Since $\overline{T}_{2,2}$ is irreducible, from Lemma 2.8, we have

$$\rho\left(\overline{T}_{2,2}\right) < \lambda. \tag{3.24}$$

Since $x_2 > 0$ and $\overline{T}_{1,2}$ is a nonzero nonnegative vector, from (3.21), we have $(1-\omega)x_1 < \lambda x_1$. Namely,

$$1 - \omega < \lambda. \tag{3.25}$$

It is clear that $\rho(\overline{T}_{\gamma,\omega}) = \max\{1-\omega, \rho(\overline{T}_{2,2})\}$. Hence, we have

$$\rho\left(\overline{T}_{\gamma,\omega}\right) < \lambda. \tag{3.26}$$

For the case of $\lambda > 1$, $\overline{T}_{2,2}x_2 \geq \lambda x_2$ is obtained from (3.22) and equality is excluded. Hence $\rho(\overline{T}_{\gamma,\omega}) > \lambda$ follows from Lemma 2.8 and $\overline{T}_{2,2}$ is irreducible. Since $A = (I - \gamma L) - [(1-\omega)I + (\omega - \gamma)L + \omega U]$ is an M-splitting of A, from Lemma 2.7, we know that $\lambda = 1$ if and only if A is a singular M-matrix. So $\overline{A} = (I + S_1)A$ is a singular M-matrix. Since $\overline{A} = (\overline{D} - \gamma\overline{L}) - [(1-\omega)\overline{D} + (\omega-\gamma)\overline{L} + \omega\overline{U}]$ is an M-splitting of $\overline{A}$; from Lemma 2.7 again, we have $\overline{T}_{\gamma,\omega} = 1$, which completes the proof. □

In Theorem 3.4, if we let $\omega = \gamma$, then can obtain some results about SOR method. For the similarity of proof of the Theorem 3.4, we only give the convergence result of the SOR method.

Theorem 3.5. *Let T_ω and $\overline{T}_\omega$ be defined by* (1.3) *and* (1.10)*. Assume that A is an irreducible Z-matrix and $A(2:n, 2:n)$ is an irreducible submatrix of A deleting the first row and the first column. Then for $0 \leq \omega \leq 1$ $(\omega \neq 0)$ and $a_{i1}a_{1i} < 1$, $i = 2, \ldots, n$, we have*

(a) $\rho(\overline{T}_\omega) < \rho(T_\omega) < 1$ *if* $\rho(T_\omega) < 1$;

(b) $\rho(\overline{T}_\omega) = \rho(T_\omega) = 1$ *if* $\rho(T_\omega) = 1$;

(c) $\rho(\overline{T}_\omega) > \rho(T_\omega) > 1$ *if* $\rho(T_\omega) > 1$.

4. AOR Method for H-Matrix

In this Section, we will consider AOR method for H-matrices. For convenience, we still use some notions and definitions in Section 2.

Lemma 4.1 (see [7]). *Let A be an H-matrix with unit diagonal elements, defining the matrices $S_D \doteq \operatorname{diag}(0, \alpha_2 a_{2,1}a_{1,2}, \ldots, \alpha_n a_{n,1}a_{1,n})$ and $S_\alpha U \doteq S_D + S_L + S_U$, where S_L and S_U are the strictly lower and strictly upper triangular components of $S_\alpha U$, respectively; then $\widetilde{A} = (I + S_\alpha)A = M_\alpha - N_\alpha$, $M_\alpha = I - S_D - L - S_L + S_\alpha$, and $N_\alpha = U + S_U$. Let $u = (u_1, \ldots, u_n)^T$ be a positive vector such that $\langle A\rangle u > 0$; assume that $a_{i1} \neq 0$ for $i = 2, \ldots, n$, and*

$$\alpha_i' = \frac{u_i - \sum_{j=2}^{i-1} |a_{i,j}| u_j - \sum_{j=i+1}^{n} |a_{i,j}| u_j + |a_{i,1}| u_1}{|a_{i,1}| \sum_{j=1}^{n} |a_{1,j}| u_j}; \tag{4.1}$$

then $\alpha_i' > 1$ for $i = 2, \ldots, n$ and for $0 \leq \alpha_i < \alpha_i'$, the splitting $\widetilde{A} = M_\alpha - N_\alpha$ is an H-splitting and $\rho(M_\alpha^{-1}N_\alpha) < 1$ so that the iteration (1.3) *converges to the solution of* (1.1).

Lemma 4.2. *Let $A = (a_{i,j})$ be an H-matrix, and let $\alpha' = \min\{\alpha_i'\}$, $i = 2, \ldots, n$, where α_i' is defined as Lemma 4.1. Then for any $\alpha \in [0, \alpha']$, $\widetilde{A} = (I + S_\alpha)A$ is also an H-matrix.*

Proof. The conclusion is easily obtained by Lemma 4.1 [7]. □

Lemma 4.3. *Let $0 \le \gamma \le \omega \le 1$ $(\omega \ne 0, \gamma \ne 1)$. Then $\widetilde{A} = \widetilde{M} - \widetilde{N}$ is an H-compatible splitting.*

Proof. Let $\langle \widetilde{A} \rangle = (\overline{a}_{i,j})$ and $\langle \widetilde{M} \rangle - |\widetilde{N}| = (b_{i,j})$, where $\widetilde{M} = (1/\omega)(\widetilde{D} - \gamma \widetilde{L})$ and $\widetilde{N} = (1/\omega)[(1-\omega)\widetilde{D} + (\omega - \gamma)\widetilde{L} + \omega \widetilde{U}]$. Since

$$\widetilde{a}_{i,j} = \begin{cases} a_{i,j}, & i = 1, \\ a_{i,j} - \alpha_i a_{i,1} a_{1,j}, & i = 2, \ldots, n, \end{cases} \tag{4.2}$$

we have that

(a) if $i = j$, then

$$\begin{gathered} \overline{a}_{i,j} = |1 - \alpha_i a_{i,1} a_{1,i}|, \\ b_{i,j} = \frac{1}{\omega}[|1 - \alpha_i a_{i,1} a_{1,i}| - (1-\omega)|1 - \alpha_i a_{i,1} a_{1,i}|] = |1 - \alpha_i a_{i,1} a_{1,i}|; \end{gathered} \tag{4.3}$$

(b) if $i \ne j$, then

$$\overline{a}_{i,j} = -\left|a_{i,j} - \alpha_i a_{i,1} a_{1,j}\right|, \tag{4.4}$$

since $\langle \widetilde{M} \rangle - |N| = (1/\omega)\langle \widetilde{D} - \gamma \widetilde{L} \rangle - (1/\omega)|(1-\omega)\widetilde{D} + (\omega-\gamma)\widetilde{L} + \omega \widetilde{U}|$; observe that if $i < j$, we have

$$b_{i,j} = \frac{1}{\omega}\left(0 - \omega\left|-a_{i,j} + \alpha_i a_{i,1} a_{1,j}\right|\right) = -\left|a_{ij} - \alpha_i a_{i,1} a_{1,j}\right|. \tag{4.5}$$

if $i > j$, we have

$$b_{i,j} = \frac{1}{\omega}\left[-\left|\gamma\left(a_{i,j} - \alpha_i a_{i,1} a_{1,j}\right)\right| - (\omega - \gamma)\left|-a_{i,j} + \alpha_i a_{i,1} a_{1,j}\right|\right] = -\left|a_{i,j} - \alpha_i a_{i,1} a_{1,j}\right|; \tag{4.6}$$

Hence, we have $\langle \widetilde{A} \rangle = \langle \widetilde{M} \rangle - |\widetilde{N}|$, that is, $\widetilde{A} = \widetilde{M} - \widetilde{N}$ is an H-compatible splitting. □

Theorem 4.4. *Let the assumption of Lemma 4.2 holds. Then for any $\alpha \in [0, \alpha']$ and $0 \le \gamma \le \omega \le 1$ $(\omega \ne 0, \gamma \ne 1)$, we have $\rho(\widetilde{T}_{\gamma,\omega}) < 1$.*

Proof. By Lemmas 2.5, 4.2, and 4.3, the conclusion is easily obtained. □

5. Numerical Examples

In this Section, we give three numerical examples to illustrate the results obtained in Sections 3 and 4.

Table 1: Spectral radius of the iteration matrices $\rho(T_{\gamma,\omega})$ and $\rho(\overline{T}_{\gamma,\omega})$ with different values of ω and γ for Example 5.1.

ω	γ	$\rho(T_{\gamma,\omega})$	$\rho(\overline{T}_{\gamma,\omega})$	ω	γ	$\rho(T_{\gamma,\omega})$	$\rho(\overline{T}_{\gamma,\omega})$
0.4	0.1	0.9983	0.9840	0.8	0.7	0.9952	0.9559
0.4	0.4	0.9980	0.9815	0.8	0.8	0.9949	0.9529
0.5	0.2	0.9977	0.9790	0.9	0.7	0.9946	0.9504
0.5	0.4	0.9975	0.9768	0.9	0.9	0.9938	0.9431
0.6	0.4	0.9970	0.9722	1	0.8	0.9936	0.9411
0.6	0.6	0.9966	0.9689	1	0.9	0.9931	0.9367

Table 2: CPU time and the iteration number of the basic and the preconditioned Gauss-Seidel method for Example 5.1.

n	IT (GS)	CPU (GS)	IT (PGS)	CPU (PGS)
60	232	0.0780	229	0.0780
90	340	0.2030	337	0.2030
120	446	0.5000	443	0.4380
150	551	4.5780	548	4.5470
180	655	9.5930	652	9.5000
210	758	36.7190	755	30.0470

Example 5.1. Consider a $n \times n$ matrix of A of the form

$$A = \begin{bmatrix} 1 & c_1 & c_2 & c_3 & c_1 & \cdots \\ c_3 & 1 & c_1 & c_2 & \ddots & c_1 \\ c_2 & c_3 & \ddots & \ddots & \ddots & c_3 \\ c_1 & \ddots & \ddots & 1 & c_1 & c_2 \\ c_3 & \ddots & c_2 & c_3 & 1 & c_1 \\ \vdots & c_3 & c_1 & c_2 & c_3 & 1 \end{bmatrix}, \tag{5.1}$$

where $c_1 = -2/n$, $c_2 = -1/n + 1$, and $c_3 = -1/n + 2$. It is clear that the matrix A satisfies the assumptions of Theorem 3.3. Numerical results for this matrix A are given in Table 1.

We consider Example 5.1; if we let $c_1 = -2/n$, $c_2 = 0$, and $c_3 = -1/n + 2$, it is clear to show that A is an M-matrix. The initial approximation of x^0 is taken as a zero vector, and b is chosen so that $x = (1, 2, \ldots, n)^T$ is the solution of the linear system (1.1). Here $\|x^{k+1} - x^k\|/\|x^{k+1}\| \leq 10^{-6}$ is used as the stopping criterion.

All experiments were executed on a PC using MATLAB programming package.

In order to show that the preconditioned AOR method is superior to the basic AOR method. We consider $\omega = \gamma = 1$, that is, the AOR method is reduced to the Gauss-Seidel method. In Table 2, we report the CPU time (T) and the number of iterations (IT) for the basic and the preconditioned Gauss-Seidel method. Here GS represents the restarted Gauss-Seidel method; the preconditioned restarted Gauss-Seidel method is noted by PGS.

Table 3: CPU time and the iteration number with various values of α_i for Example 5.2.

n	$\alpha_i = 0.5$	$\alpha_i = 0.8$	$\alpha_i = 1$	$\alpha_i = 1.2$	$\alpha_i = 2$	$\alpha_i = 0$
64	1	1	1	1	1	1
	0.0601	0.0488	0.0587	0.0524	0.0501	0.0629
81	1	1	1	1	1	1
	0.0522	0.0504	0.0532	0.0524	0.0569	0.0635
100	1	1	1	1	1	1
	0.0577	0.0547	0.0486	0.0555	0.0563	0.0663

Example 5.2. Consider the two-dimensional convection-diffusion equation

$$-\Delta u + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = f \tag{5.2}$$

in the unit squire Ω with Dirichlet boundary conditions see [28].

When the central difference scheme on a uniform grid with $N \times N$ interior nodes (N^2) is applied to the discretization of the convection-diffusion equation (3.5), we can obtain a system of linear equations (1.1) of the coefficient matrix

$$A = I \otimes P + Q \otimes I, \tag{5.3}$$

where $\otimes$ denotes the Kronecker product,

$$P = \operatorname{tridiag}\left(-\frac{2+h}{8}, 1, -\frac{2-h}{8}\right), \qquad Q = \operatorname{tridiag}\left(-\frac{1+h}{4}, 1, -\frac{1-h}{4}\right) \tag{5.4}$$

are $N \times N$ tridiagonal matrices, and the step size is $h = 1/N$.

It is clear that the matrix A is an M-matrix, so it is an H-matrix. Numerical results for this matrix A are given in Table 3.

From Table 3, for $\alpha_i \in [0, \alpha_i')$, it can be seen that the convergence rate of the preconditioned Gauss-Seidel iterative method ($\omega = \gamma = 1$) is faster than the other preconditioned iterative method for H-matrices. And iteration numbers are not changed by the change of α_i; the iteration time slightly changed by the change of α_i. However, it is difficult to select the optical parameters α_i and this needs a further study.

Example 5.3. We consider a symmetric Toeplitz matrix

$$T_n = \begin{bmatrix} a & b & c & \cdots & b \\ b & a & b & \cdots & c \\ c & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & c & b & \cdots & a \end{bmatrix}, \tag{5.5}$$

where $a = 1$, $b = 1/n$, and $c = 1/n - 2$. It is clear that T_n is an H-matrix. The initial approximation of x^0 is taken as a zero vector, and b is chosen so that $x = (1, 2, \ldots, n)^T$ is

Table 4: CPU time and the iteration number of the basic and the preconditioned AOR method for Example 5.3.

n	ω	γ	IT (AOR)	T (AOR)	IT (PAOR)	T (PAOR)
90	0.9	0.5	15	0.3196	11	0.0390
120	0.9	0.5	15	0.1526	10	0.0306
180	0.9	0.5	15	0.1407	11	0.1096
210	0.9	0.5	15	0.2575	11	0.1920
300	0.9	0.5	15	1.2615	10	0.7709
400	0.9	0.5	15	3.2573	11	2.3241

the solution of the linear system (1.1). Here $\|x^{k+1} - x^k\|/\|x^{k+1}\| \leq 10^{-6}$ is used as the stopping criterion see [29].

All experiments were executed on a PC using MATLAB programming package.

We get Table 4 by using the preconditioner $P(\alpha)$. We report the CPU time (T) and the number of iterations (IT) for the basic and the preconditioned AOR method. Here AOR represents the restarted AOR method; the preconditioned restarted AOR method is noted by PAOR.

Remark 5.4. In Example 5.3, we let $\alpha_i > 1$, $i = 2, \dots, n-1$. From Table 4, if α_i is appropriate, the convergence of the preconditioned AOR iterative method can be improved. However, it is difficult to select the optical parameters α_i and this needs a further study.

Acknowledgments

The authors express their thanks to the editor Professor Paulo Batista Gonçalves and the anonymous referees who made much useful and detailed suggestions that helped them to correct some minor errors and improve the quality of the paper. This project is granted financial support from Natural Science Foundation of Shanghai (092R1408700), Shanghai Priority Academic Discipline Foundation, the Ph.D. Program Scholarship Fund of ECNU 2009, and Foundation of Zhejiang Educational Committee (Y200906482) and Ningbo Nature Science Foundation (2010A610097).

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