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Research Article

A Constitutive Formulation for the Linear Thermoelastic Behavior of Arbitrary Fiber-Reinforced Composites

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The linear thermoelastic behavior of a composite material reinforced by two independent and inextensible fiber families has been analyzed theoretically. The composite material is assumed to be anisotropic, compressible, dependent on temperature gradient, and showing linear elastic behavior. Basic principles and axioms of modern continuum mechanics and equations belonging to kinematics and deformation geometries of fibers have provided guidance and have been determining in the process of this study. The matrix material is supposed to be made of elastic material involving an artificial anisotropy due to fibers reinforcing by arbitrary distributions. As a result of thermodynamic constraints, it has been determined that the free energy function is dependent on a symmetric tensor and two vectors whereas the heat flux vector function is dependent on a symmetric tensor and three vectors. The free energy and heat flux vector functions have been represented by a power series expansion, and the type and the number of terms taken into consideration in this series expansion have determined the linearity of the medium. The linear constitutive equations of the stress and heat flux vector are substituted in the Cauchy equation of motion and in the equation of conservation of energy to obtain the field equations.

1. Introduction

Generally, composite materials are separated into natural composites and artificial composites. While natural and artificial composites have functional similarities, they differ greatly in terms of methods of production and purposes of use. Natural composites are not the result of a manufacturing and production method implemented by humans for a certain purpose. Having extremely fine and complex subsystems, structures of this kind comprise known natural structure elements by combination in a certain distribution at a time and on grounds determined according to a universal program of sometimes microscopic and other times macroscopic level. Artificial composites appear as a product of a certain manufacturing process produced by intellectual capabilities of the human mind to create a material with superior characteristics for a certain application purpose. Dealing with microlevel study for natural and artificial composite structure elements is a subject of micromechanics [1].

Generally, studies of composite materials are divided into two main branches, namely, micromechanical and macromechanical analyses. The micromechanical analysis aims to uncover certain mechanical characteristics relating to the general behavior of composite materials using the physical and mechanical properties of matrix and reinforcement materials as a starting point. Micromechanical methods can be separated into three, that is, the energy method, elasticity method, and material mechanics method [2]. Composites are broadly used in civilian and military aircraft, aerospace technologies, automotive industry, sea vehicles primarily in ships, pneumatic vessels, power transmission axles, and orthopedic devices [3].

In our previous study [4], viscoelastic composites of a single fiber family have been studied assuming that the medium has a discontinuity surface. In our study [5], it has been assumed that a viscoelastic medium with two different inextensible fiber families does not have a discontinuity surface. Again, in the studies in [6, 7], the exposure of a viscoelastic medium to the effect of electrical and magnetic fields in addition to its reinforcement by a single fiber family has been researched in the form of separate studies. Furthermore, in his study [8], Usal has examined the electromechanical behavior of a piezoelectric viscoelastic medium with two fiber families. Since temperature was assumed to be constant in all of the previous studies, temperature change has not been taken into consideration. In this study, constitutive equations have been obtained that indicate the stress and heat distribution determining the thermoelastic behavior of a composite material reinforced by two arbitrary independent and inextensible fiber families. Since the temperature is not constant, a temperature gradient has been included in the calculations as an independent constitutive variable.

Researchers like those in [9–11] have made progress in the studies they have conducted on the formulation of thermoelastic effect on a variety of materials. A study by Nowacki fills a large gap in thermoelasticity and its applications [12].

The thermal properties can play a significant role in affecting the design and manufacture of composite structures in their industrial applications [13]. The subject of thermoelastic behavior of composites has been studied by a number of different researchers [14–21]. Fiber-reinforced composite materials belong to a very important class of materials which are often employed in a wide variety of industrial applications. Typically, these composite materials consist of a fabric structure where the fibers are continuously arranged in a matrix material, and, at the macroscopic level, these composite materials exhibit strong directional dependencies. The vehicle tyres furnishe a typical example of technological application of such man-made composites [21].

Due to some technological requirements, it is aspired that specific construction elements have rather elastic properties, provided that they have high durability in certain directions. Fiber-reinforced composite materials are produced by sticking fibers in a polymeric matrix which is elastic but with low strength. These fibers are manufactured from high-strength graphite or bor. They can be easily bent due to the very small size of their cross-section and it can be assumed that these fibers show a continuous distribution in a medium. Assuming inextensibility of the fibers is a reasonable approach since the rigidity of the fibers is very high compared to the rigidity of the matrix. Inextensibility of the fibers is broadly accepted in practice for formulation purposes. Thus, fiber families are assumed to be inextensible ($\lambda_a^2 = C_{KL}A_K A_L = 1$, $\lambda_z^2 = C_{KL}Z_K Z_L = 1$) [22]. On the other hand, the composite material taken into consideration in this work is assumed to be compressible and shows linear elastic behavior. In a class of engineered fiber composites for structural load-bearing components in civil or aerospace applications, an assumption of linear elastic behavior is suitable and this class of composites belongs to a compressible material response.

2. Kinematics of Fibers Deformation

It is assumed that an element from two different continuous fiber families is placed on each point of the composite material. Before deformation and after deformation, these fiber families are represented by continuous unit vectors $\mathbf{A}(\mathbf{X})$, $\mathbf{Z}(\mathbf{X})$, $\mathbf{a}(\mathbf{x})$, and $\mathbf{z}(\mathbf{x})$, respectively. The fibers deform along with the material; that is, fibers do not have a relative motion with respect to the material in which they are embedded. Relationships given below are true for an A-fiber family [23, 24]

$$a_k dl = x_{k,\,K} A_K dL, \qquad a_k \frac{dl}{dL} = x_{k,K} A_K. \tag{2.1}$$

Rates of extension of fiber family A can be defined as follows:

$$\lambda_a \equiv \left(\frac{dl}{dL}\right)_A. \tag{2.2}$$

If expression (2.2) is substituted into (2.1), the following expression is obtained:

$$a_k = \lambda_a^{-1} x_{k,K} A_K. \tag{2.3}$$

Deformation geometry of the fiber family A is expressed by relationship (2.3). Because vectors $\mathbf{A}$ and $\mathbf{a}$ here represent unitized vectors of the fiber family A, operations are true

$$|\mathbf{A}| = |\mathbf{a}| = 1, \qquad a_k a_k = 1 = \lambda_a^{-2} x_{k,K} x_{k,L} A_K A_L = \lambda_a^{-2} C_{KL} A_K A_L. \tag{2.4}$$

Accordingly, the form is found to be

$$\lambda_a^2 = C_{KL} A_K A_L. \tag{2.5}$$

Relationships that are true for the Z-fiber family can be expressed as follows [23, 24]:

$$z_k = \lambda_z^{-1} x_{k,K} Z_K, \qquad \lambda_z \equiv \left(\frac{dl}{dL}\right)_Z, \qquad \lambda_z^2 = C_{KL} Z_K Z_L, \tag{2.6}$$

where dL and dl are, respectively, arc length of fiber before and after deformation, A_K and Z_K are fiber unit vector components before deformation, a_k and z_k are fiber unit vector components after deformation, $x_{k,K} = \partial x_k / \partial X_K$ is deformation gradient, λ_a and λ_z are rates of extension of fiber families, and $C_{KL} = x_{k,K} x_{k,L}$ is Green deformation tensor.

3. Thermomechanic Balance Equations

Balance equations, mass, linear momentum, angular momentum, energy balances, and entropy inequality have been summarized in [22, 25]. We have the following:

Conservation of mass:

$$\dot{\rho} + \rho v_{k,k} = 0, \qquad \rho(\mathbf{x}, t) = \frac{\rho_0(\mathbf{X})}{J(\mathbf{x}, t)} \text{ (conservation of mass in material representation)}, \tag{3.1}$$

balance of linear mometum:

$$\rho \dot{v}_p = \rho \mathrm{f}_p + t_{rp,r}, \tag{3.2}$$

balance of moment of momentum:

$$\varepsilon_{krp} t_{rp} = 0, \qquad t_{rp} = t_{pr}, \tag{3.3}$$

conservation of energy:

$$\rho \dot{\varepsilon} = t_{kl} d_{kl} + q_{k,k} + \rho h, \tag{3.4}$$

Clausius-Duhem inequality:

$$\rho \theta \dot{\eta} - \nabla \cdot \mathbf{q} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta - \rho \mathrm{h} \geqslant 0. \tag{3.5}$$

Here, v stands for the velocity field in a continuous medium, ρ_0 for mass density before deformation, ρ for mass density after deformation, $J \equiv \det[x_{k,K}] = \rho_0 / \rho(\mathbf{x}, t)$ for jacobian, $\dot{v}$ for acceleration, t_{lk} for stress tensor, f_k for the mechanical volumetric force per unit of mass, ε for internal energy density per unit of mass, q_k for heat flux vector, h for heat source per unit of mass, η for entropy density per unit of mass, $\theta(\mathbf{X}, t)$ for the absolute temperature of a material point $\mathbf{X}$ at a moment t, and ε_{ijk} for permutation tensor.

4. Thermodynamic Constraints and Modeling Constitutive Equations

Taking (ρh) from the local energy (3.4) and substituting it in the entropy inequality (3.5) will give us the following:

$$-\rho \left(\dot{\varepsilon} - \theta \dot{\eta}\right) + t_{kl} d_{lk} + \frac{1}{\theta} q_k \theta_{,k} \geq 0. \tag{4.1}$$

Since the material derivative of the entropy density in this expression cannot be controlled inside a thermodynamic process, a defined Legendre transformation like the one provided below can be used to transform the derivative of these values into the controllable value θ

$$\psi \equiv \varepsilon - \theta\eta. \tag{4.2}$$

As a result, the entropy inequality is transformed as follows, expressed in new terms:

$$-\rho\left(\dot{\psi} + \dot{\theta}\eta\right) + t_{kl}d_{lk} + \frac{1}{\theta}q_k\theta_{,k} \geq 0. \tag{4.3}$$

Entropy inequality is obtained as follows in the material form [26]:

$$-\left(\dot{\Sigma} + \rho_0\dot{\theta}\eta\right) + \frac{1}{2}T_{KL}\dot{C}_{KL} + \frac{1}{\theta}\theta_{,K}Q_K \geq 0. \tag{4.4}$$

Terms relating to the new values have been provided below:

$$\Sigma \equiv \rho_0\psi, \tag{4.5}$$

$$\dot{C}_{KL} = 2\mathrm{d}_{kl}x_{k,K}x_{l,L} \Longrightarrow d_{kl} = \frac{1}{2}\dot{C}_{KL}X_{K,k}X_{L,l}, \tag{4.6}$$

$$T_{KL} \equiv JX_{K,k}X_{L,l}t_{kl} \Longrightarrow t_{kl} = J^{-1}x_{k,K}x_{l,L}T_{KL}, \tag{4.7}$$

$$Q_K \equiv JX_{K,k}q_k \Longrightarrow q_k = J^{-1}x_{k,K}Q_K, \tag{4.8}$$

$$G_K \equiv \theta_{,K} = x_{k,K}\theta_{,k} \Longrightarrow g_k \equiv \theta_{,k} = X_{K,k}\theta_{,K}. \tag{4.9}$$

Here, Σ stands for thermodynamic stress potential, ψ for generalized free energy density, d_{kl} for deformation (strain) rate tensor, $X_{K,k} = \partial X_K/\partial x_k$ for the deformation gradient of the reverse motion, T_{KL} for the stress tensor on material coordinates, Q_K for the heat flux vector on material coordinates, and G_K for the temperature gradient on material coordinates.

To be able to use the inequality (4.4), we need to know the independent variables on which the thermodynamic potential Σ depends. Arguments of Σ and the variables they depend on have been found using constitutive axioms based on the selected material. According to the axioms of causality and determinism [22, 25], our stress potential, as a response functional at a material point X at a time t, can be written as follows:

$$\Sigma(\mathbf{X}, t) = \Sigma\left[\mathbf{x}\left(\mathbf{X}', t'\right),\ \theta\left(\mathbf{X}', t'\right), \mathbf{X}\right] \quad X \in V \ \ -\infty < t' \leq t. \tag{4.10}$$

Here, t' is any point in time between now and the past. $\mathbf{X}'$ stands for all material points other than $\mathbf{X}$.

Using the results of causality, determinism, objectivity, smooth neighborhood, and admissibility axioms [22, 25], the arguments on which Σ depends in a composite with two fiber families exposed to mechanical loading and temperature change can be expressed as follows:

$$\Sigma(X_K, t) = \Sigma\ [C_{KL}(X_K, t), G_K(X_K, t), A_K(X_K), Z_K(X_K), \theta(X_K, t), X_K]. \tag{4.11}$$

Assuming that the materials are homogenous, $\mathbf{X}$ will be eliminated from among the arguments given in the expression (4.11) on which Σ depends. Because the fiber vectors A_K and Z_K do not depend on time, the following expression is obtained by taking the material derivative of expression (4.11).

$$\dot{\Sigma} = \frac{\partial \Sigma}{\partial C_{KL}} \dot{C}_{KL} + \frac{\partial \Sigma}{\partial \theta_{,K}} \dot{G} + \frac{\partial \Sigma}{\partial \theta} \dot{\theta}. \tag{4.12}$$

Substituting this expression in (4.4) gives us the following inequality:

$$\frac{1}{2}\left(T_{KL} - 2\frac{\partial \Sigma}{\partial C_{KL}}\right)\dot{C}_{KL} - \rho_0\left(\eta + \frac{1}{\rho_0}\frac{\partial \Sigma}{\partial \theta}\right)\dot{\theta} - \frac{\partial \Sigma}{\partial G_K}\dot{G}_K + \frac{1}{\theta} G_K Q_K \geq 0. \tag{4.13}$$

Since we are able to arbitrarily replace the arguments in inequality (4.13) from θ to $\dot{\theta}$, from C_{KL} to $\dot{C}_{KL}$, and from G_K to $\dot{G}_K$, for the inequality (4.13) to be satisfied, the coefficients of $\dot{\theta}$, $\dot{C}_{KL}$ and $\dot{G}_K$ will be zero. The coefficient of G_K cannot be zero as due to G_K's presence in the arguments of Σ, it cannot be arbitrarily replaced. Equalizing the coefficients of $\dot{C}_{KL}$, $\dot{\theta}$, and $\dot{G}_K$ to zero will give us the following expressions:

$$T_{KL} = 2\frac{\partial \Sigma}{\partial C_{KL}}, \tag{4.14}$$

$$\eta = -\frac{1}{\rho_0}\frac{\partial \Sigma}{\partial \theta}, \tag{4.15}$$

$$\frac{\partial \Sigma}{\partial G_K} = 0. \tag{4.16}$$

It is understood from expression (4.16) that the stress potential does not depend on G_K. Therefore, arguments on which the stress potential depends are expressed as follows:

$$\Sigma = \Sigma(C_{KL}, A_K, Z_K, \theta). \tag{4.17}$$

Thus, inequality (4.13) is reduced to the following form:

$$\frac{1}{\theta}\ G_K Q_K \geq 0. \tag{4.18}$$

For the heat flux vector, expression (4.18) gives the Clausius-Duhem inequality and the following expression indicates the arguments on which the heat flux vector depends

$$Q_K = Q_K(C_{KL}, A_K, Z_K, G_K, \theta). \tag{4.19}$$

In consideration of expression (4.19), inequality (4.18) is written down as follows:

$$G_K Q_K(C_{KL}, A_K, Z_K, G_K, \theta) \geq 0 \quad \text{or } \mathbf{Q}\ (\mathbf{C}, \mathbf{A}, \mathbf{Z}, \mathbf{G}, \theta, \mathbf{X}) \cdot \mathbf{G} \geq 0. \tag{4.20}$$

In inequality (4.20), when $G_K = 0$, Q_K must also be equal to zero. Accordingly, maintaining the order of independent constitutive variables in expression (4.20), the following expression should be written:

$$Q_K(C_{KL}, A_K, Z_K, 0, \theta) = 0. \tag{4.21}$$

On the other hand, internal energy density ε can be written as follows from the expressions (4.2), (4.5), and (4.15):

$$\varepsilon = \frac{1}{\rho_0}\left(\Sigma - \frac{\partial \Sigma}{\partial \theta}\theta\right). \tag{4.22}$$

From the constitutive equations offered by expressions (4.14) and (4.19), it is understood that the stress is derived from the stress potential Σ, while the heat flux vector appears as a vectorial form with known arguments independent of the stress potential. Thus, the explicit forms of Σ and Q_K, which appear as constitutive functions with definite arguments, should be determined. However, constraints imposed on the constitutive functions of the material in question by the material symmetry axiom should firstly be revised.

Let the symmetry group of the material be the full orthogonal group (isotropic material) or any of its subgroups (anisotropic material). Let $\underline{\underline{S}} = [S_{KL}]$ be any arbitrary matrix representing the orthogonal transformation of material coordinates (or rigid configurations of the material medium according to the reference coordinate frame) and pertaining to the symmetry group of the medium. According to the material symmetry axiom, constitutive functionals under each transformation

$$X'_K = S_{KL} X_L, \qquad X_L = S^T_{LK} X'_K, \qquad \underline{\underline{S}}^{-1} = \underline{\underline{S}}^T \tag{4.23}$$

established using the orthogonal matrix $\underline{\underline{S}}$ should remain form invariant. Mathematically, this indicates the validity of the transformations

$$\Sigma\left(\underline{\underline{S}}\ \underline{\underline{C}}\ \underline{\underline{S}}^T, \underline{\underline{S}}\ \underline{A}, \underline{\underline{S}}\ \underline{Z}, \theta\right) = \Sigma\ \left(\underline{\underline{C}}, \underline{A}, \underline{Z}, \theta\right), \tag{4.24}$$

$$\underline{Q}\left(\underline{\underline{S}}\ \underline{\underline{C}}\ \underline{\underline{S}}^T, \underline{\underline{S}}\ \underline{A},\ \underline{\underline{S}}\ \underline{Z},\ \underline{\underline{S}}\ \underline{G}, \theta\right) = \underline{Q}\left(\underline{\underline{C}}, \underline{A}, \underline{Z}, \underline{G}, \theta\right). \tag{4.25}$$

The following conditions should be satisfied since the fiber families are assumed to be inextensible [22, 27]:

$$\lambda_a^2 = C_{KL} A_K A_L = 1, \qquad \lambda_z^2 = C_{KL} Z_K Z_L = 1. \tag{4.26}$$

Thus, the constitutive equation for the stress is obtained as follows in spatial and material coordinates:

$$t_{kl} = \Gamma_a a_k a_l + \Gamma_z z_k z_l + 2x_{k,K} x_{l.L} \frac{\partial \Sigma}{\partial C_{KL}},$$
$$T_{KL} = \Gamma_a A_K A_L + \Gamma_z Z_K Z_L + 2\frac{\partial \Sigma}{\partial C_{KL}}. \tag{4.27}$$

In these expressions, Γ_a and Γ_z are Lagrange coefficients and are defined by field equations and boundary conditions.

In this study, the matrix material has been considered as an anisotropic medium. In the scope of this approach, the stress potential and heat flux vector functions are expanded in power series in terms of the components of arguments on which they depend, giving us the thermoelastic behavior of the composite medium. The reference position of the medium has been selected at a uniform temperature T_0 in a stress-free natural condition, and it has been assumed that the medium moves away from that position by small displacements and deformations and small changes in temperature. By referring to small changes in temperature, we mean $\theta = T_0 + T$, $T_0 > 0$, $|T| \ll T_0$ [22]. The type and the number of terms taken in the series expansion have been determined based on the linearity condition of the medium. Moreover, because the matrix material remains insensitive to change of direction along the fibers, expressions of vector fields representing the fiber distribution through outer products in even numbers as arguments should be considered. The linear constitutive equation of stress has been obtained by taking the derivative of the stress potential according to its deformation tensor. Field equations have been obtained by substituting the linear constitutive equations of the stress and heat flux vector in the Cauchy motion equation and in the equation of conservation of energy.

5. Determination of Stress Constitutive Equation in Linear Thermoelasticity

Since the relation $C_{KL} = \delta_{KL} + 2E_{KL}$ exists between the Green deformation tensor and strain tensor, E_{KL} can be expressed as $E_{KL} \cong \widetilde{E}_{KL} \equiv (1/2)(U_{K,L} + U_{L,K})$ in a linear theory, and the arguments of the stress potential given by expression (4.17) can be written down as follows:

$$\Sigma = \Sigma\left(\widetilde{E}_{KL}, A_K, Z_K, \theta\right). \tag{5.1}$$

Assuming that this function is analytic in terms of the $\widetilde{\mathbf{E}}, \mathbf{A}, \mathbf{Z}$ values, if this function is expanded in Taylor series around $\widetilde{\mathbf{E}} = \mathbf{0}$, $\mathbf{A} = \mathbf{0}$, $\mathbf{Z} = \mathbf{0}$, the expression will be obtained for the stress potential:

$$\begin{aligned}\Sigma\left(\widetilde{E}_{KL}, A_S, Z_Y, \theta\right) = {} & \Sigma_0(\theta, \mathbf{X}) + \Sigma_{KL}(\theta, \mathbf{X})\widetilde{E}_{KL} + \frac{1}{2}\Sigma_{KLMN}(\theta, \mathbf{X})\widetilde{E}_{KL}\widetilde{E}_{MN} + \lambda_{SN}(\theta, \mathbf{X})A_S A_N \\ & + \Omega_{YN}(\theta, \mathbf{X})Z_Y Z_N + \zeta_{\mathrm{KLSN}}(\theta, \mathbf{X})E_{KL}A_S A_N \\ & + \kappa_{KLYN}(\theta, \mathbf{X})E_{KL}Z_Y Z_N + \cdots .\end{aligned} \tag{5.2}$$

Coefficients in this equation have been defined as follows:

$$\Sigma_0 = \Sigma\left(\underline{\underline{0}}, \underline{0}\right), \qquad \Sigma_{KL} \equiv \left.\frac{\partial \Sigma}{\partial \widetilde{E}_{KL}}\right|_0, \qquad \Sigma_{KLMN} \equiv \left.\frac{\partial^2 \Sigma}{\partial \widetilde{E}_{KL} \partial \widetilde{E}_{MN}}\right|_0, \qquad \lambda_{SN} \equiv \frac{1}{2} \left.\frac{\partial^2 \Sigma}{\partial A_S \partial A_N}\right|_0,$$
$$\Omega_{YN} \equiv \frac{1}{2} \left.\frac{\partial^2 \Sigma}{\partial Z_Y \partial Z_N}\right|_0, \qquad \zeta_{KLSN} \equiv \frac{1}{6} \left.\frac{\partial^3 \Sigma}{\partial E_{KL} \partial A_S \partial A_N}\right|_0, \qquad \kappa_{KLYN} \equiv \frac{1}{6} \left.\frac{\partial^3 \Sigma}{\partial E_{KL} \partial Z_Y \partial Z_N}\right|_0. \tag{5.3}$$

Due to the symmetry of the $\widetilde{E}_{KL}$ tensor and nondependence on order of the derivatives in the definitions in the expressions (5.3), these coefficients bear the symmetry characteristics indicated below:

$$\Sigma_{KL} = \Sigma_{LK}, \qquad \Sigma_{KLMN} = \Sigma_{LKMN} = \Sigma_{KLNM} = \Sigma_{MNKL}, \qquad \lambda_{SN} = \lambda_{NS}, \qquad \Omega_{YN} = \Omega_{NY},$$
$$\zeta_{KLSN} = \zeta_{LKSN} = \zeta_{KLNS}, \qquad \kappa_{KLYN} = \kappa_{LKYN} = \kappa_{KLNY}. \tag{5.4}$$

The following relations can be written down for the linear theory in continuum mechanics [22]:

$$E_{KL} \cong \widetilde{E}_{KL} \equiv \frac{1}{2}(U_{K,L} + U_{L,K}), \quad e_{kl} \cong \widetilde{e}_{kl} = \epsilon_{kl} \equiv \frac{1}{2}(u_{k,l} + u_{l,k}), \quad \epsilon_{kl} \cong \lambda_{kK} \lambda_{lL} \widetilde{E}_{KL},$$
$$\widetilde{E}_{KL} \cong \lambda_{kK} \lambda_{lL} \widetilde{e}_{kl}, \quad x_{k,K} = \lambda_{kK} + u_{k,K}, \quad X_{K,k} = \Lambda_{Kk} - U_{K,k}, \quad x_{k,K} x_{l,L} = \lambda_{kK} \lambda_{lL},$$
$$X_{K,k} X_{L,l} = \lambda_{kK} \lambda_{lL}, \quad E_{KL} \cong \widetilde{E}_{KL} \equiv \lambda_{kK} \lambda_{lL} \widetilde{e}_{kl} = \frac{1}{2} \lambda_{kK} \lambda_{lL} (u_{k,l} + u_{l,k}),$$
$$x_{p,P} x_{r,R} A_K A_L = x_{p,P} x_{r,R} X_{K,k} X_{L,l} a_k a_l \lambda_a^2 = \lambda_{pP} \lambda_{rR} \lambda_{kK} \lambda_{lL} a_k a_l \quad (\text{for } \lambda_a = 1), \tag{5.5}$$
$$x_{p,P} x_{r,R} Z_K Z_L = x_{p,P} x_{r,R} X_{K,k} X_{L,l} z_k z_l \lambda_z^2 = \lambda_{pP} \lambda_{rR} \lambda_{kK} \lambda_{lL} z_k z_l \quad (\text{for } \lambda_z = 1),$$
$$d_{pr} = \frac{\partial E_{PR}}{\partial t} X_{P,r} X_{R,r} = \frac{\partial \epsilon_{pr}}{\partial t}, \qquad d_{pr} = \frac{\partial (u_{p,r})}{\partial t}, \qquad \dot{\varepsilon} \approx \frac{\partial \varepsilon}{\partial t},$$
$$J^{-1} \cong 1 - u_{k,k}, \qquad \rho \cong \rho_0 (1 - u_{k,k}).$$

The expression of the spatial form of stress for compressible media with inextensible fiber families can be written down as indicated below:

$$t_{pr} = \Gamma_a a_k a_l + \Gamma_z z_k z_l + (1 - u_{k,k}) \frac{\partial \Sigma}{\partial \epsilon_{pr}}. \tag{5.6}$$

In the linear theory, arguments on which Σ depends can be expressed in spatial form as follows:

$$\Sigma = \Sigma(\epsilon_{kl}, a_k, z_k, \theta, \mathbf{X}). \tag{5.7}$$

Assuming this function is analytic in terms of $\in_{kl}, a_k, z_k$ and expanding it in the Taylor series around $\in_{kl} = 0,\ a_s = 0,\ z_y = 0$ will give us the following expression:

$$\begin{aligned}\Sigma(\in_{kl}, a_s, z_y, \theta, \mathbf{X}) = \Sigma_0(\theta, \mathbf{X}) + \Sigma_{kl}(\theta, \mathbf{X})\in_{kl} + \frac{1}{2}\Sigma_{klmn}(\theta, \mathbf{X})\in_{kl}\in_{mn} + \lambda_{sn}(\theta, \mathbf{X}) a_s a_n \\ + \Omega_{yn}(\theta, \mathbf{X}) z_y z_n + \zeta_{klsn}(\theta, \mathbf{X})\in_{kl} a_s a_n + \kappa_{klyn}(\theta, \mathbf{X})\in_{kl} z_y z_n + \cdots .\end{aligned} \tag{5.8}$$

The spatial material tensors Σ_{kl}, Σ_{klmn}, λ_{sn}, Ω_{yn}, ζ_{klsn}, and κ_{klyn} in (5.8) bear the same properties as the material tensors of the material Σ_{KL}, Σ_{KLMN}, λ_{SN}, Ω_{YN}, ζ_{KLSN}, and κ_{KLYN} and are defined as follows:

$$\begin{gathered}\Sigma_{kl} = \lambda_{kK}\lambda_{lL}\Sigma_{KL}, \qquad \Sigma_{klmn} \equiv \lambda_{kK}\lambda_{lL}\lambda_{mM}\lambda_{nN}\Sigma_{KLMN}, \qquad \lambda_{sn} = \lambda_{sS}\lambda_{nN}\lambda_{SN}, \\ \Omega_{yn} = \lambda_{yY}\lambda_{nN}\Omega_{YN}, \qquad \zeta_{klsn} \equiv \lambda_{kK}\lambda_{lL}\lambda_{sS}\lambda_{nN}\zeta_{KLSN}, \qquad \kappa_{klyn} \equiv \lambda_{kK}\lambda_{lL}\lambda_{yY}\lambda_{nN}\kappa_{KLYN}.\end{gathered} \tag{5.9}$$

In order to obtain a correct formulation of the linear theory, expression (5.8) should be quadratic at most in terms of the endlessly small expansion tensor $\in_{kl}$ and temperature change T. For this purpose, the coefficients dependent on θ in the expression (5.8) have been defined as follows, respectively:

$$\begin{aligned}\Sigma_0(\theta, \mathbf{X}) &= \Sigma_0(T_0 + T, \mathbf{X}) = \rho_0(\mathbf{X})\psi_0(T_0, \mathbf{X}) - \rho_0(\mathbf{X})\eta_0(T_0, \mathbf{X})T - \frac{1}{2}\rho_0(\mathbf{X})\frac{1}{T_0}C(T_0, \mathbf{X})T^2 + \cdots , \\ \Sigma_{kl}(\theta, \mathbf{X}) &= \gamma_{kl}\ (T_0, \mathbf{X}) - \beta_{kl}(T_0, \mathbf{X})T + \cdots , \\ \lambda_{sn}(\theta, \mathbf{X}) &= \Lambda_{sn}(T_0, \mathbf{X}) - \mu_{sn}(T_0, \mathbf{X})T + \cdots , \\ \Omega_{yn}(\theta, \mathbf{X}) &= \Omega_{yn}(T_0, \mathbf{X}) - \pi_{yn}(T_0, \mathbf{X})T + \cdots , \\ \Sigma_{klmn}(\theta, \mathbf{X}) &= \Sigma_{klmn}(T_0 + T, \mathbf{X}) = \Sigma_{klmn}(T_0, \mathbf{X}), \\ \zeta_{klsn}(\theta, \mathbf{X}) &= \zeta_{klsn}(T_0 + T, \mathbf{X}) = \zeta_{klsn}(T_0, \mathbf{X}), \\ \kappa_{klyn}(\theta, \mathbf{X}) &= \kappa_{klyn}(T_0 + T, \mathbf{X}) = \kappa_{klyn}(T_0, \mathbf{X}).\end{aligned} \tag{5.10}$$

Coefficients in this equations have been defined as follows:

$$\begin{gathered}\left.\frac{\partial\psi_0(T, \mathbf{X})}{\partial T}\right|_{T=T_0} \equiv -\eta_0(T_0, \mathbf{X}), \qquad \left.\frac{\partial^2\psi_0(T, \mathbf{X})}{\partial T^2}\right|_{T=T_0} \equiv -\frac{1}{T_0}C(T_0, \mathbf{X}), \\ \gamma_{kl}(T_0, \mathbf{X}) \equiv \Sigma_{kl}(T_0, \mathbf{X}) = \gamma_{lk}(T_0, \mathbf{X}), \qquad \beta_{kl}(T_0, \mathbf{X}) \equiv -\left.\frac{\partial\Sigma_{kl}(T, \mathbf{X})}{\partial T}\right|_{T=T_0} = \beta_{lk}(T_0, \mathbf{X}), \\ \Lambda_{sn}(T_0, \mathbf{X}) \equiv \Lambda_{ns}(T_0, \mathbf{X}), \qquad \mu_{sn}(T_0, \mathbf{X}) \equiv -\left.\frac{\partial\lambda_{sn}(T, \mathbf{X})}{\partial T}\right|_{T=T_0} = \mu_{ns}(T_0, \mathbf{X}), \\ \Omega_{yn}(T_0, \mathbf{X}) \equiv \Omega_{ny}(T_0, \mathbf{X}), \qquad \pi_{yn}(T_0, \mathbf{X}) \equiv -\left.\frac{\partial\Omega_{yn}(T, \mathbf{X})}{\partial T}\right|_{T=T_0} = \pi_{ny}(T_0, \mathbf{X}).\end{gathered} \tag{5.11}$$

In these expressions, $\psi_0(T_0,\mathbf{X})$, $\eta_0(T_0,\mathbf{X})$, and $C(T_0,\mathbf{X})$ are scalar; $\gamma_{kl}(T_0,\mathbf{X})$, $\beta_{kl}(T_0,\mathbf{X})$, $\Lambda_{sn}(T_0,\mathbf{X})$, $\mu_{sn}(T_0,\mathbf{X})$, $\Omega_{yn}(T_0,\mathbf{X})$, $\pi_{yn}(T_0,\mathbf{X})$, $\Sigma_{klmn}(T_0,\mathbf{X})$, $\zeta_{klsn}(T_0,\mathbf{X})$, and $\kappa_{klyn}(T_0,\mathbf{X})$ are tensorial material constants, and these coefficients depend on the initial temperature T_0 of the medium and medium particles in heterogeneous materials. In homogenous materials, the dependence on $\mathbf{X}$ is eliminated. In order to simplify notation, we will not indicate the arguments $(T_0,\mathbf{X})$ of such coefficients. Substituting the expressions (5.11) and (5.10) in (5.8) gives us the following expression:

$$\begin{aligned}\Sigma(\in_{kl}, a_s, z_y, T_0 + T, \mathbf{X}) = {} & \rho_0\psi_0 - \rho_0\eta_0 T - \frac{\rho_0 c}{2T_0}T^2 + \gamma_{kl}\in_{kl} - \beta_{kl}T\in_{kl} + \Lambda_{sn}a_s a_n \\ & - \mu_{sn}T a_s a_n + \Omega_{yn}z_y z_n - \pi_{yn}T z_y z_n + \frac{1}{2}\Sigma_{klmn}\in_{kl}\in_{mn} \\ & + \zeta_{klsn}\in_{kl}a_s a_n + \kappa_{klyn}\in_{kl}z_y z_n + \cdots .\end{aligned} \tag{5.12}$$

If derivative in (5.6) is taken from (5.12) and used in substitution, the following expression is obtained:

$$t_{pr} = \Gamma_a a_p a_r + \Gamma_z z_p z_r + (1 - u_{k,k})\left(-\beta_{pr}T + \Sigma_{prmn}\in_{mn} + \zeta_{prsn}a_s a_n + \kappa_{pryn}z_y z_n\right). \tag{5.13}$$

Due to the $\Sigma_{prmn} = \Sigma_{prnm}$ symmetry property of the coefficient Σ_{prmn} in this expression, the constitutive equation given by expression (5.13) can be converted into the following form in terms of linear constituents of the displacement gradient:

$$\begin{aligned}t_{pr} = {} & \Gamma_a a_p a_r + \Gamma_z z_p z_r - \beta_{pr}T + \Sigma_{prmn}u_{m,n} + \zeta_{prsn}a_s a_n + \kappa_{pryn}z_y z_n \\ & - \zeta_{prsn}a_s a_n u_{k,k} - \kappa_{\text{pryn}}z_y z_n u_{k,k}.\end{aligned} \tag{5.14}$$

In a composite material reinforced by two arbitrary independent and inextensible fiber families, the medium is assumed to be anisotropic, compressible, homogeneous, dependent on temperature gradient and showing linear elastic behavior. Equation (5.14) is the linear constitutive equation of stress. First and second terms on the right part of (5.14) are caused by the inextensibility of the fibers. Γ_a and Γ_z-fiber stretch, both are determined through field equations and boundary conditions. These two terms are reaction stresses and cannot be expressed by any constitutive equation. The third term expresses the temperature effect, and the fourth term expresses the contribution of the elastic deformation to the stress. Regarding the fifth and the sixth term, two interpretations are possible. The first one states that if the medium is not loaded in any way (T = constant, $E_{KL} = 0$), it will not switch to stress and therefore physically it should be $\zeta_{prsn} = 0$ and $\kappa_{pryn} = 0$, because being loaded with fibers is not sufficient for a medium to automatically get stressed. Another interpretation can be as follows. No parameters have been used related with cross-section thickness of fibers neither in this study nor in other studies examining macroscopic behavior of fiber-reinforced media. In other words, distribution of fibers is present in the medium only as a topologic object that just causes anisotropy, which means it is completely geometric. In this regard, there is no constraint that would prevent us from reinforcing fiber on the molecular scale. Therefore, if a distribution can be practically placed into the medium in the form of a molecular chain, it is possible to suggest that this will alter the present ionic distribution and stress the medium with no other effect. In this case, coefficients ζ_{prsn} and κ_{pryn} in fifth and sixth terms will be different from zero and will thus gain a physical meaning. These terms can be interpreted as internal stress contribution stimulated by dislocation. The seventh and eighth terms show the stress formed by interaction of the deformation field with contributions of fiber fields.

If it is assumed that the medium is without fibers, expression (5.14) will be reduced to retain the third and fourth terms indicating the contribution to the stress of the temperature and strain tensor. Accordingly, in this study, terms of (5.14) have been obtained under the mentioned assumptions and are reduced to generally known classical expressions in special cases. This supports the opinion proving the reliability of the model we have created. These new terms are the expressions of constitutive equation on spatial coordinates for stress on a mathematical model created for fabricated composites, specifically for materials involving a distribution of two absolutely arbitrary fibers.

6. Determination of Heat Flux Vector Constitutive Equation in Linear Thermoelasticity

Here, the approach assumed for the stress potential has been assumed for the heat flux vector. Accordingly, the heat flux vector can be found through a power series expansion in terms of components of the arguments on which it depends, around a reference location selected as the natural condition. Considering that $\mathbf{E}$ can be substituted by $\tilde{\mathbf{E}}$ in the linear theory, arguments on which the heat flux vector depends, entropy inequality, and constraint caused by this inequality have been found as follows:

$$Q_R = Q_R\left(\underline{\underline{\tilde{E}}}, \underline{A}, \underline{Z}, \underline{G}, \theta, \underline{X}\right), \tag{6.1}$$

$$\mathbf{Q}\left(\tilde{\mathbf{E}}, \mathbf{A}, \mathbf{Z}, \mathbf{G}, \theta, \mathbf{X}\right) \cdot \mathbf{G} \geq 0, \tag{6.2}$$

$$Q_R = Q_R\left(\tilde{\mathbf{E}}, \mathbf{A}, \mathbf{Z}, , 0, \theta, \mathbf{X}\right) = 0. \tag{6.3}$$

Expanding the function (6.1) in a Taylor series around $\tilde{\mathbf{E}} = \mathbf{0}$, $\mathbf{A} = \mathbf{0}$, $\mathbf{Z} = \mathbf{0}$, $\mathbf{G} = \mathbf{0}$ will give us the following expression:

$$\begin{aligned} Q_R\left(\tilde{\mathbf{E}}, \mathbf{A}, \mathbf{Z}, \mathbf{G}, \theta, \mathbf{X}\right) &= Q_R(\mathbf{G}, \theta, \mathbf{X}) + \left.\frac{\partial Q_R\left(\tilde{\mathbf{E}}, \mathbf{G}, \theta, \mathbf{X}\right)}{\partial E_{LM}}\right|_{\mathbf{E}=0} \tilde{E}_{LM} \\ &+ \left.\frac{\partial Q_R(\mathbf{A}, \mathbf{G}, \theta, \mathbf{X})}{\partial A_L A_M}\right|_{\mathbf{A}=\mathbf{0}} A_L A_M + \left.\frac{\partial Q_R(\mathbf{Z}, \mathbf{G}, \theta, \mathbf{X})}{\partial Z_L Z_M}\right|_{\mathbf{Z}=\mathbf{0}} Z_L Z_M. \end{aligned} \tag{6.4}$$

Here, the definitions as in the following are used

$$\begin{aligned}
Q_R(\mathbf{G},\theta,\mathbf{X}) &\equiv Q_R(\theta,\mathbf{X}) + \left.\frac{\partial Q_R(\mathbf{G},\theta,\mathbf{X})}{\partial G_L}\right|_{\mathbf{G}=\mathbf{0}} G_L = B_R(\theta,\mathbf{X}) + B_{RL}(\theta,\mathbf{X})G_L + \cdots, \\
H_{RLM}(\mathbf{G},\theta,\mathbf{X}) &\equiv \left.\frac{\partial Q_R\left(\widetilde{\mathbf{E}},\mathbf{G},\theta,\mathbf{X}\right)}{\partial E_{LM}}\right|_{\mathbf{E}=\mathbf{0}} = B_{RLM}(\theta,\mathbf{X}) + \cdots, \\
Y_{RLM}(\mathbf{G},\theta,\mathbf{X}) &\equiv \left.\frac{\partial Q_R(\mathbf{A},\mathbf{G},\theta,\mathbf{X})}{\partial A_L A_M}\right|_{\mathbf{A}=\mathbf{0}} = D_{RLM}(\theta,\mathbf{X}) + \cdots, \\
N_{RLM}(\mathbf{G},\theta,\mathbf{X}) &\equiv \left.\frac{\partial Q_R(\mathbf{Z},\mathbf{G},\theta,\mathbf{X})}{\partial Z_L Z_M}\right|_{\mathbf{Z}=\mathbf{0}} = F_{RLM}(\theta,\mathbf{X}) + \cdots.
\end{aligned} \tag{6.5}$$

Using the above-mentioned definitions in the series expansion given by expression (6.4) can give us the following expression:

$$\begin{aligned}
Q_R\left(\widetilde{\mathbf{E}},\mathbf{A},\mathbf{Z},\mathbf{G},\theta,\mathbf{X}\right) = {} & B_R(\theta,\mathbf{X}) + B_{RL}(\theta,\mathbf{X})G_L + B_{RLM}(\theta,\mathbf{X})\widetilde{E}_{LM} \\
& + D_{RLM}(\theta,\mathbf{X})A_L A_M + F_{RLM} Z_L Z_M + \cdots.
\end{aligned} \tag{6.6}$$

Due to the symmetry of the tensor $\widetilde{\mathbf{E}}$ and independence of derivatives in the definitions in expressions (6.5) from the order, these coefficients bear the symmetry characteristics given below:

$$B_{RLM} = B_{RML}, \qquad D_{RLM} = D_{RML}, \qquad F_{RLM} = F_{RML}. \tag{6.7}$$

Since $\mathbf{G} = \mathbf{0} \Rightarrow \mathbf{Q} = \mathbf{0}$ due to the constraint in (6.3), the following expression can be written down from the relation (6.6):

$$\underline{0} = B_R(\theta,\mathbf{X}) + B_{RLM}(\theta,\mathbf{X})\widetilde{E}_{LM} + D_{RLM}(\theta,\mathbf{X})A_L A_M + F_{RLM}(\theta,\mathbf{X})Z_L Z_M + \cdots. \tag{6.8}$$

Since expression (6.8) is zero for any arbitrary deformation measure, coefficients in this equation should be zero. Therefore,

$$B_R(\theta,\mathbf{X}) = B_{RLM}(\theta,\ \mathbf{X}) = D_{RLM}(\theta,\mathbf{X}) = F_{RLM}(\theta,\mathbf{X}) = 0. \tag{6.9}$$

Accordingly, (6.6) is reduced to the following form:

$$Q_R = Q_R\left(\widetilde{\mathbf{E}},\mathbf{A},\mathbf{Z},\mathbf{G},\theta,\mathbf{X}\right) = B_{RL}(\theta,\mathbf{X})G_L = B_{RL}(\theta,\mathbf{X})\theta_{,L}. \tag{6.10}$$

Substituting expression (6.10) in the inequality (6.2) gives us the following expression:

$$B_{RL}(\theta,\mathbf{X})\theta_{,L}\theta_{,R} \geq 0 \quad \text{or } B_{RL}(\theta,\mathbf{X})G_L G_R \geq 0. \tag{6.11}$$

Therefore, the tensor $B_{RL}(\theta, \mathbf{X})$ should satisfy the following condition for any temperature gradient:

$$B_{RL}\theta_{,R}\theta_{,L} \geq 0 \quad \text{or } B_{(RL)}\theta_{,R}\theta_{,L} \geq 0. \tag{6.12}$$

The B_{RL} tensor is named conductivity coefficient tensor. Inequality (6.12) tells us that the symmetric part of this tensor is positive definite. For the linear theory, the coefficient B_{RL} is expressed as follows in similarity to the coefficient Σ_{PR}:

$$B_{RL}(\theta, \mathbf{X}) = B_{RL}(T_0 + T, \mathbf{X}) = B_{RL}(T_0, \mathbf{X}) + \left. \frac{\partial B_{RL}(T, \mathbf{X})}{\partial T} \right|_0 T + \cdots . \tag{6.13}$$

Moreover, the coefficient $\theta_{,L}$ can be written down as follows:

$$\theta_{,L} = (T_0 + T)_{,L} = T_{,L}. \tag{6.14}$$

Substituting expression (6.14) in (6.10) and omitting the nonlinear term $(T)(T_{,L})$, the heat flux vector is written down as follows:

$$Q_R = B_{RL}(T_0, \mathbf{X})T_{,L}. \tag{6.15}$$

The expression of the spatial form of heat flux vector for compressible media can be written down as indicated below:

$$q_r = (1 - u_{k,k})Q_R x_{r,R}. \tag{6.16}$$

If (6.15) is substituted into expression (6.16), using expressions (5.5) and omitting the nonlinear term $(u_{k,k}T_{,l})$, the spatial form of the heat flux vector follows as

$$q_r = B_{rl}(T_0, \mathbf{X})T_{,l}. \tag{6.17}$$

The spatial tensor B_{rl} of the material in (6.17) has the same symmetry characteristics as the tensor B_{RL} and is defined as follows:

$$B_{rl} \equiv \lambda_{rR}\lambda_{lL}B_{RL}. \tag{6.18}$$

Equation (6.17) is the Fourier heat transfer law, which defines linear heat transfer, and it can be written down as follows in the vectorial form:

$$\mathbf{q} = \mathbf{B}\nabla T. \tag{6.19}$$

7. Determination of Field Equations

Before proceeding to obtain the field equations, let us discuss the meaning of the tensor β_{pr} in (5.14). Firstly, let us define the tensor Σ^{-1}_{prmn}, which is the reversed tensor Σ_{prmn} and has the same symmetry properties as this tensor, as follows:

$$\Sigma_{prmn}\Sigma^{-1}_{mnkl} \equiv \frac{1}{2}\left(\delta_{pk}\delta_{rl} + \delta_{pl}\delta_{rk}\right), \qquad \Sigma^{-1}_{prmn} = \Sigma^{-1}_{rpmn} = \Sigma^{-1}_{mnpr} = \Sigma^{-1}_{prnm}. \tag{7.1}$$

The tensor α_{pr} comprised of thermal expansion coefficients that can be easily measured physically can be defined as follows:

$$\alpha_{pr} \equiv \Sigma^{-1}_{prmn}\beta_{mn} = \alpha_{rp}. \tag{7.2}$$

To find the reverse of expression (7.2), let us multiply both sides of the equation by the tensor Σ_{klpr}. Then, using a suitable index replacement, the following can be written down:

$$\beta_{pr} = \Sigma_{prmn}\alpha_{mn}. \tag{7.3}$$

Substituting expression (7.3) in (5.14) will give us the following:

$$\begin{aligned} t_{pr} &= \Gamma_a a_p a_r + \Gamma_z z_p z_r + \zeta_{prsn}(a_s a_n - u_{k,k} a_s a_n) + \kappa_{pryn}(z_y z_n - u_{k,k} z_y z_n) \\ &\quad + \Sigma_{prmn}(u_{m,n} - \alpha_{mn} T). \end{aligned} \tag{7.4}$$

The following expression can be written down in regard to a linear theory:

$$\rho \dot{v}_k \cong \rho_0(1 - u_{k,k})\frac{\partial v_k}{\partial t} \cong \rho_0 \frac{\partial}{\partial t}\frac{\partial u_k}{\partial t} \cong \rho_0 \frac{\partial^2 u_k}{\partial t^2}. \tag{7.5}$$

In a linear theory, the Cauchy equations of motion can be written as follows substituting the expressions (7.5) and (5.5) in (3.2):

$$t_{kl,l} + \rho_0(1 - u_{l,l})\mathrm{f}_\mathrm{k} = \rho_0 \frac{\partial^2 u_k}{\partial t^2}. \tag{7.6}$$

Considering that the medium is homogenous and omitting the term $(\rho_0 u_{l,l}\mathrm{f}_\mathrm{k})$, let us calculate the divergence of the stress given by (7.4) and substitute it in (7.6) to obtain the following field equation under the above-mentioned assumptions:

$$\begin{aligned} \rho_0 \frac{\partial^2 u_p}{\partial t^2} &= \Sigma_{prmn}(u_{m,nr} - \alpha_{mn} T_{,r}) + \rho_0 \mathrm{f}_\mathrm{p} + (\Gamma_a)_{,r} a_p a_r + \Gamma_a\left(a_{p,r} a_r + a_p a_{r,r}\right) \\ &\quad + (\Gamma_z)_{,r} z_p z_r + \Gamma_z\left(z_{p,r} z_r + z_p z_{r,r}\right) + \zeta_{prsn}(a_{s,r} a_n + a_s a_{n,r}) \\ &\quad - \zeta_{prsn}(a_{s,r} a_n + a_s a_{n,r}) u_{k,k} - \zeta_{prsn} u_{k,kr} a_s a_n + \kappa_{pryn}\left(z_{y,r} z_n + z_y z_{n,r}\right) \\ &\quad - \kappa_{pryn}\left(z_{y,r} z_n + z_y z_{n,r}\right) u_{k,k} - \kappa_{pryn} u_{k,kr} z_y z_n. \end{aligned} \tag{7.7}$$

Expression (7.7) gives us a field equation with the unknowns u_k, Γ_a, Γ_z. The solution of this field equation under initial and boundary conditions forms the mathematical structure of a boundary value problem to consider.

Because $\theta = T_0 + T$ and $\partial T/\partial\theta = 1$, the entropy and the internal energy density given in expressions (4.15) and (4.22) can be written down as follows:

$$\eta = -\frac{1}{\rho_0}\frac{\partial\Sigma}{\partial T}\frac{\partial T}{\partial\theta} = -\frac{1}{\rho_0}\frac{\partial\Sigma}{\partial T}, \tag{7.8}$$

$$\varepsilon = \frac{1}{\rho_0}\left[\Sigma - (T_0 + T)\frac{\partial\Sigma}{\partial T}\right]. \tag{7.9}$$

Substituting (7.8) in expression (7.9) will give us the following expression:

$$\varepsilon = \frac{\Sigma}{\rho_0} + (T_0 + T)\eta. \tag{7.10}$$

Taking the derivative of Σ given by expression (5.12) according to T and substituting it in (7.8) after related operations will allow us to express entropy in terms of the displacement gradient component as follows:

$$\eta = \eta_0 + \frac{cT}{T_0} + \frac{\beta_{kl}}{\rho_0}u_{k,l} + \frac{\mu_{sn}}{\rho_0}a_s a_n + \frac{\pi_{yn}}{\rho_0}z_y z_n. \tag{7.11}$$

Let us now substitute expressions (5.12) and (7.11) in (7.10) and make necessary arrangements to obtain the internal energy density as follows:

$$\begin{aligned}\varepsilon = \varepsilon_0 &+ c\left(T + \frac{T^2}{2T_0}\right) + \frac{T_0\beta_{kl}}{\rho_0}u_{k,l} + \frac{1}{2\,\rho_0}\Sigma_{klmn}u_{k,l}u_{m,n}\\ &+ \frac{1}{\rho_0}\,\left[(\Lambda_{sn} + \zeta_{klsn}u_{k,l})a_s a_n + (\Omega_{yn} + \kappa_{klyn}u_{k,l})z_y z_n\right].\end{aligned} \tag{7.12}$$

ε_0 coefficient in this equation has been defined as $\varepsilon_0 = \psi_0 + T_0\eta_0$, where ε_0, ψ_0, and η_0 are, respectively, internal energy density, free energy density, and entropy density in natural condition. Taking a material derivative of expression (7.12) and considering that $\rho = \rho_0\ (1 - u_{k,k})$ give us the following expression:

$$\begin{aligned}\rho\dot{\varepsilon} &= \rho_0(1 - u_{m,m})\dot{\varepsilon}\\ &= \rho_0 c\left(1 + \frac{T}{T_0}\right)\frac{\partial T}{\partial t} + T_0\beta_{kl}\frac{\partial u_{k,l}}{\partial t} + \Sigma_{klmn}\frac{\partial u_{k,l}}{\partial t}u_{m,n} + \zeta_{klsn}\frac{\partial u_{k,l}}{\partial t}a_s a_n\\ &\quad + \kappa_{klyn}\frac{\partial u_{k,l}}{\partial t}z_y z_n - \zeta_{klsn}\frac{\partial u_{k,l}}{\partial t}u_{m,m}a_s a_n - \kappa_{klyn}\frac{\partial u_{k,l}}{\partial t}u_{m,m}z_y z_n\end{aligned} \tag{7.13}$$

The term $q_{r,r}$ is obtained as follows from (6.17):

$$q_{r,r} = B_{rl,r}T_l + B_{(rl)}T_{,lr} = B_{(rl)}T_{,lr}. \tag{7.14}$$

Let us now substitute the expressions (7.13), (7.14), (7.4), and (5.5) in the equation conservation of energy given by expression (3.4) and make necessary arrangements to make the following field equations linear in terms of $u_{k,l}$ and T:

$$\rho_0 c\frac{\partial T}{\partial t} + \left(T_0\beta_{kl} - \Gamma_a a_k a_l - \Gamma_z z_k z_l\right)\frac{\partial u_{k,l}}{\partial t} = \beta_{(kl)}T_{,lk} + \rho_0(1 - u_{l,l})h. \tag{7.15}$$

In a composite material reinforced by two arbitrary independent and inextensible fiber families, where the medium is assumed to be anisotropic, compressible, homogeneous, dependent on temperature gradient, and showing linear elastic behavior, (7.15) is a heat transfer equation.

8. Conclusions

As an approach in this study, the stress potential and heat flux vector functions have been assumed to be analytic and expanded in Taylor series in terms of their arguments on which they depend. The type and the number of terms taken in the series expansion have been determined based on the assumption that mechanical interactions and temperature changes are linear. On the other hand, since the matrix material has to remain insensitive to directional changes along fibers, even-numbered exterior products of vector fields representing fiber distributions have been considered. The reference position of the medium has been selected at the uniform temperature T_0 and stress-free natural condition, from which position the medium has been assumed to move away by small displacements, and small temperature changes. Accordingly, the forms in spatial coordinates of the constitutive equations of the stress and the heat flux vector have been presented by (5.14) and (6.17). The constitutive equation of the stress expressed by (5.14) in terms of the tensor α_{pr} comprised of thermal expansion coefficients has been expressed by (7.4). To obtain field equations, constitutive equation of the stress given by (7.4) has been substituted into the Cauchy equation of motion, yielding field equation (7.7). Values in the equation of conservation of energy given by expression (3.4) have been substituted into (3.4), yielding field equation (7.15). Solution of the field equations along with initial and boundary conditions in conformity with the structure of the problem to be used in practice will constitute the structure of a boundary value problem to consider. Unknowns in the field equations (7.7) and (7.15), $\mathbf{u}(\mathbf{x}, t)$, Γ_a, and Γ_z. Γ_a and Γ_z, which are Lagrange coefficients, can be calculated using the field equations and boundary conditions. After $\mathbf{u}$ has been designated, the stress distribution is obtained from (7.4). After the stress distribution is found as a tensor field, the stress vector at a desired cross-section can be easily calculated from the expression $\mathbf{t}_{(\mathbf{n})_r} = n_p\, t_{pr}$. Here, it needs to be considered that the fiber distributions $a_k(\mathbf{x}, t)$ and $z_k(\mathbf{x}, t)$ after deformation for inextensible fibers in terms of fiber distributions before deformation are $a_k = x_{k,K}A_K(\mathbf{X})$ and $z_k = x_{k,K}Z_K(\mathbf{X})$.

Besides, considering the equations of motion (7.7), we can see the internal thermo-mechanical forces affecting the medium. Type of the terms on the right is in the dimension of force per unit of volume. The first term on the right represents force created by the elastic deformation, the second term is force created by the temperature gradient, the third term is mechanical body force, and the fourth and the sixth terms are similar to molecular considerations in (5.14). The fifth, seventh, eighth, and eleventh terms are forces caused by curvature of the fibers. The ninth and twelfth terms are forces caused by interaction of fibers and their curvature with the deformation field. The tenth and thirteenth terms represent forces caused by interaction of the deformation field with distributions of fiber fields. In other words, by drawing a free body diagram of a material element in the medium, it is possible to see all such force contributions acting on the element. As stated before, our field equations where we can apply initial and boundary conditions for the aforementioned media are (7.7) and (7.15). A more detailed discussion of these equations will be contained in future works.

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