Research Article

Dynamical Aspects of Macroscopic and Quantum Transitions due to Coherence Function and Time Series Events

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This study presents the application of dynamical equations able to generate alternating deformations with increasing amplitude and delayed pulses in a certain material medium. It is considered that an external force acts at certain time interval (similar to a time series) upon the material medium in the same area. Using a specific differential equation (considering nonzero initial values and using a function similar to the coherence function between the external force and the deformations inside the material), it results that modulated amplitude oscillations appear inside the material. If the order of the differential dynamical equation is higher, supplementary aspects as different delayed pulses and multiscale behaviour can be noticed. These features are similar to non-Markov aspects of quantum transitions, and for this reason the mathematical model is suitable for describing both quantum phenomena and macroscopic aspects generated by sequence of pulses. An example of a quantum system, namely, the Hydrogen atom, is discussed.

1. Introduction

For simulating the generation of specific deformations inside a material medium under the action of external forces it can be considered that some short wavelength vibrations appear in the area where the force acts. The corresponding deformation is generated inside the material medium, using linear differential equations or equations with partial derivatives (similar to the wave equation or to the equation of diffusion). Yet such linear equations cannot explain the distance between the space area where the external force acts and the space area where fracture phenomena appear. Using differential equations of higher order, some slow variations of deformation along a certain direction could be obtained. Due to the fact that the mathematical model should explain the sharp deformations at a certain distance of the point of space where the force acts (leading to fracture phenomena), some different types

of differential equations had been investigated. An analysis of oscillations generated by step functions acting upon second-order systems working at limt of stability was presented in [1].

Better qualitative results were obtained using dynamical equations able to generate practical test functions (similar to wavelets) and delayed pulses (when a free term which corresponds to an external force is added) [2] for justifying fracture phenomena appearing in a certain material medium [3]. It has been considered that an external force (described by a short wavelength sine function multiplied by a gaussian function) acts upon the material medium in a certain area. Using a specific differential equation (able to generate symmetrical functions for a null free term) for describing the generation of the corresponding deformation along an axis inside the material medium, it has been shown that a significant deformation could appear at a certain distance. This significant deformation justifies the fracture phenomena, while the inner structure of the material cannot allow significant sharp deformations without breaking.

However, this mathematical model cannot explain the breaking effect of a sequence of external pulses (e.g., applied as transverse force upon a beam fixed at both ends), when the time interval between these pulses is large enough so as the final effect not to be considered a superposition of individual effects of each pulse. It is well known from practice that workers using traditional tools have to apply some medium-power shocks at certain intervals upon a beam fixed at both ends, before a final great-power shock to be applied for breaking the material. Each medium-power shock generates specific damping vibrations inside the material medium, and the subsequent shock has to be applied right before the annihilation of these damping vibrations by the fluctuations of the external medium (the noise). Thus a certain degree of coherence for the effects of external pulses can be achieved, for the maximum possible value of the amplitude of fundamental harmonic corresponding to envelope of generated vibrations (if the time interval between external pulses is shortened, the final value of envelope function at the end of this interval is no more equal to zero and the difference between extreme values of envelope function decreases—thus the amplitude of fundamental harmonic component decreases also). This implies the use of some non-Markov aspects, while the memory of the previous similar events should be involved (similar to [4]).

For this reason, some specific differential equations based on the coherence function between the generated deformation and the alternating input should be taken into consideration. Since this coherence function vanishes if the output equals zero, the initial condition should be set at a small nonzero value.

2. Mathematical Model for Generating Amplitude-Modulated Deformations

As was shown in the previous paragraph, a differential equation modelling aspects similar to quantum phenomena should be based on the use of coherence function as free term. For the beginning, the external command (corresponding to the sequence of external pulses) should be considered as a superposition of cosine functions and we will analyze just the output generated by a certain cosine function (with the period set to the value $T = 1$). The free term of the differential equation is represented by the coherence function

$$Ch(x) = \int_{x_{\text{in}}}^{x} y(t) \cos \omega t \, dt = \int_{x_{\text{in}}}^{x} y(t) \cos 2\pi t \, dt, \qquad (2.1)$$
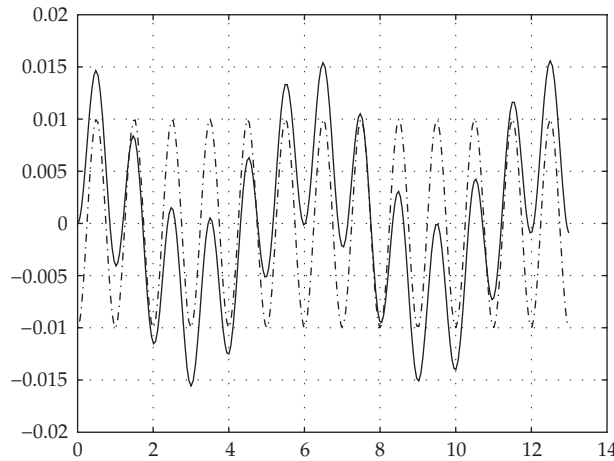
Figure 1: Output generated for second-order system by cosine function.

where variable x corresponds to time. The initial value y_0 of the output function $y(x)$ is set to a small nonzero value $y_0 = 0.3$ at the initial time moment $x_0 = 0$. By choosing an undamped second-order system with time constant $T_0 = 1$ (the period being $2\pi \approx 6T$ so as very weak resonance aspects to appear) it results the equation

$$y'' = -\frac{1}{T_0^2}y + \int_0^x y(t) \cos \omega t \, dt = -y + \int_0^x y(t) \cos 2\pi t \, dt. \tag{2.2}$$

Using simulation in MATLAB (based on Runge-Kutta functions) it results for the derivative of the output $y'(x)$ (denoted as $z(x)$) the function represented in Figure 1. The continuous line corresponds to function $z(x) = y'(x)$; the discontinuous line corresponds to the function $f(x) = -0.1 \cos 2\pi x$ necessary for studying the correlation between $z(x)$ and $f(x)$.

It can be easily noticed that $z(x)$ and $f(x)$ exhibit in-phase oscillations. However, the oscillations corresponding to $z(x)$ present a supplementary aspect: the local maximum values and the local minimum values of each oscillation (the envelopes) are both amplitude-modulated by a periodical signal with a time period six times greater than the period T of the external command. This means that for three successive oscillations the local maximum values increase, and for next three successive oscillations the local maximum values decrease (the peak-to-peak value for each oscillations being the same). This aspect can be put in correspondence with an increasing velocity of particles vibrating in a bar under the influence of an external alternating force, generating fracture phenomena after a few oscillations (when the velocity becomes higher than a certain threshold value). It can be also put in correspondence with quantum aspects, where a transition can appear just after a certain number of oscillations for the wave-trains of particles involved in interaction. Supplementary simulations performed for greater time periods of the proper oscillations of the second order system show that the local maximum values and the local minimum values of each oscillation (the envelopes) are in fact amplitude-modulated by a periodical signal corresponding to the proper oscillations of the second order system, so the number of successive oscillations for which the local maximum increases can vary from 2–3 to some higher values.
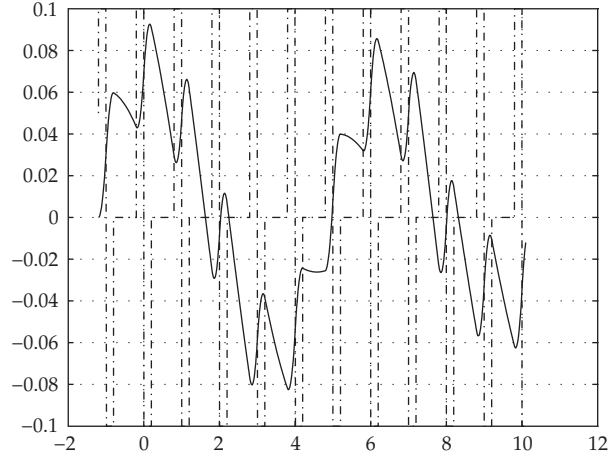
Figure 2: Output generated for second-order system by sequence of alternating pulses.

At next step, the external command should be considered as a sequence of rectangular pulses similar to $\delta'(x - k)$. Let us consider that the same second order system has the same initial conditions at the initial moment of time $x_{\text{in}} = -1.2$ and that the command function is represented by a sequence of step pulses $s(x)$ with amplitude $A = 5$ for short time before $x_k = -1, 0, 1, 2, 3 \ldots$ and with amplitude $A = -5$ short time after these time moments $x_k = -1, 0, 1, 2, 3 \ldots$. This means

$$s = 5 \quad \text{for } x \in [-1.2, -1] \cup [-0.2, 0] \cup [0.8, 1] \cup [1.8, 2] \cup [2.8, 3] \ldots,$$

$$s = -5 \quad \text{for } x \in [-1, -0.8] \cup [0, 0.2] \cup [1, 1.2] \cup [2, 2.2] \cup [3, 3.2] \ldots. \tag{2.3}$$

Using simulation in MATLAB (based on Runge-Kutta functions) it results for the derivative of the output $y'(x)$ (denoted as $z(x)$) the function represented in Figure 2. The continuous line corresponds to function $z(x) = y'(x)$; the discontinuous line corresponds to the function $f(x) = 0.02s(x)$ necessary for studying the correlation between $z(x)$ and $f(x)$. It can be noticed that $z(t)$ is a saw-tooth function. The external command $s(x)$ and $z(x)$ are no more in-phase functions, but the main feature of Figure 1 is still present: an alternance of three increasing local maximum values and of three decreasing local maximum values of $z(t)$ can be noticed, with the period of these alternances being six time greater than the period of the external command $s(x)$ (represented by a sequence of pulses similar to $\delta'(x-k)$, $k \in -1, 0, 1 \ldots$. It should be emphasized the fact that these aspects cannot be noticed if the maximum and minimum values of external command $s(x)$ are close to unity.

If the time period of the proper oscillations of the second order system decreases so as $T_0 \approx T$ the well-known phenomena of beat (interference between two oscillations of slightly different frequencies generating amplitude-modulated oscillations with a frequency corresponding to the difference between the two frequencies) or resonance (oscillations with amplitude $A \to \infty$) appear. For $T_0 < T$, some proper oscillations of the second order system modulated by the external cosine function can be noticed. While none of these mathematical models can generate oscillations with increasing local maximum values for a few oscillations of the external command function, they are not useful for modeling transitions.
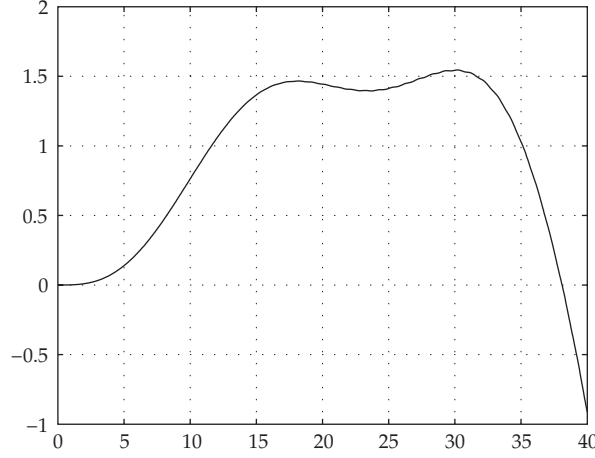
Figure 3: First positive alternance of $z(x)$ generated for six-order system by cosine function.

3. Mathematical Model for Generating Multiscale Delayed Deformations

The analysis can be extended by investigating differential equations able to generate delayed phenomena, similar to non-Markov aspects of transitions in quantum physics (where the intutive model requires a certain interaction time between the wave-trains corresponding to interacting particles before a suddenly emerging transition phenomenon to occur). For this purpose, the order of the differential equation should be increased, while a higher order implies usually a greater transient time. So as new aspects to be revealed accurately, the order of the differential equation is increased three times and the time constant T_0 is increased $\sqrt{10}$ times from unity value for avoiding any resonance effect. It results the differential equation

$$y^{(6)} = -\frac{1}{T_0^2}y^{(4)} + \int_0^x y(t)\cos\omega t\, dt = -0.1y^{(4)} + \int_0^x y(t)\cos 2\pi t\, dt \tag{3.1}$$

for an external command represented by the same oscillating function $\cos 2\pi t$ (the period T equals 1). The initial moment of time is set to $x_{\text{in}} = 0$ and the initial value of $y(x)$ is set to $y_0 = 0.3$ (as in te case of second order differential equation presented at the beginning of previous paragraph).

The simulation performed in Matlab for $z(x) = y'(x)$ reveals new aspects. On the time interval $x \in (0, 40)$ it can be noticed a positive value for $x \in (0, 37)$ with a maximum value $Mv_1 \approx 1.6$ units for $x_1 \approx 30$ (the delay time being $td_1 \approx 5$) as shown in Figure 3.

On the time interval $x \in (0, 100)$ it can be noticed at first sight (due to the scale of vertical axis) a negative value for $x \in (0, 85)$ (the positive values for $x \in (0, 37)$ being hard to be noticed) with a minimum value $Mv_2 \approx -40$ units for $x_2 \approx 75$ (the delay time can be approximated as $td_2 \approx 50$) as shown in Figure 4.

On the time interval $x \in (0, 150)$ it can be noticed at first sight (due to the scale of vertical axis) a positive value for $x \in (0, 130)$ (the negative values for $x \in (37, 85)$ being hard to be noticed) with a maximum value $Mv_3 \approx 1000$ units for $x_3 \approx 120$ (the delay time can be approximated as $td_3 \approx 100$) as shown in Figure 5.
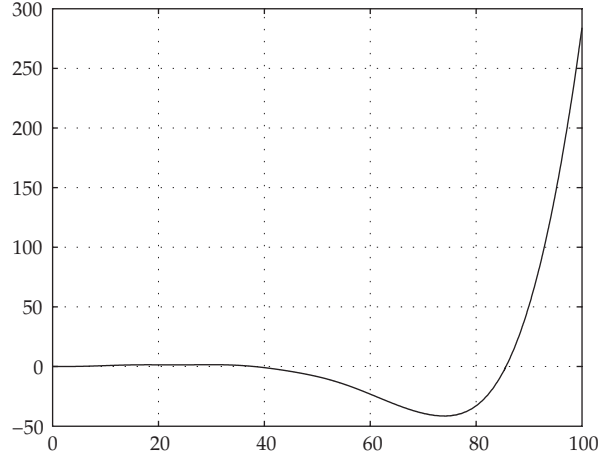
Figure 4: First negative alternance of $z(x)$ generated for six-order system by cosine function.
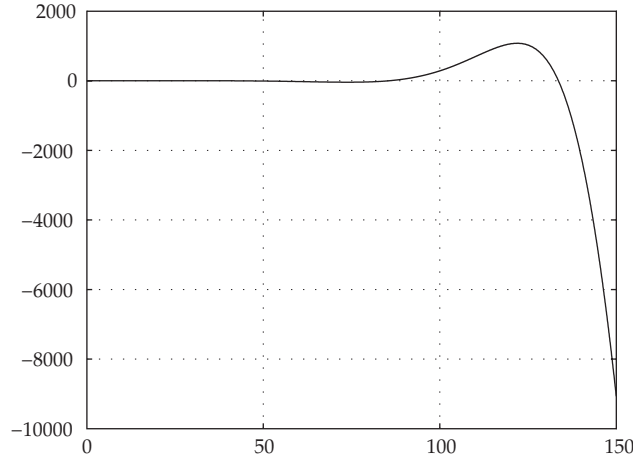

Figure 5: Second positive alternance of $z(x)$ generated for six-order system by cosine function.

On the time interval $x \in (0, 200)$ it can be noticed at first sight (due to the scale of vertical axis) a negative value for $x \in (0, 185)$ (the positive values for $x \in (0, 37)$ and $x \in (85, 130)$ being hard to be noticed) with a minimum value $Mv_4 \approx 30000$ units for $x_4 \approx 170$ (the delay time can be approximated as $td_4 \approx 140$) as shown in Figure 6.

On the time interval $x \in (0, 250)$ it can be noticed at first sight (due to the scale of vertical axis) a positive value for $x \in (0, 230)$ (the negative values for $x \in (37, 85)$ and $x \in (130, 185)$ being hard to be noticed) with a maximum value $Mv_5 \approx 800000$ units for $x_5 \approx 220$ (the delay time can be approximated as $td_5 \approx 180$) as shown in Figure 7.

And so on, it can be noticed that different alternances of $z(t)$ can be noticed, depending on the time interval selected for analysis. The ratio between peak values corresponding to consecutive alternances is about $|Mv_{k+1}/Mv_k| \approx 25$–$30$ and the time interval between moments corresponding to peak values for consecutive alternances is about $t_{k+1} - t_k \approx 45$–50 (approximately two times greater than the period $2\pi T_0 = 2\sqrt{10}\pi$ of the proper oscillations). A similar value can be noticed for the difference between delay times for two consecutive
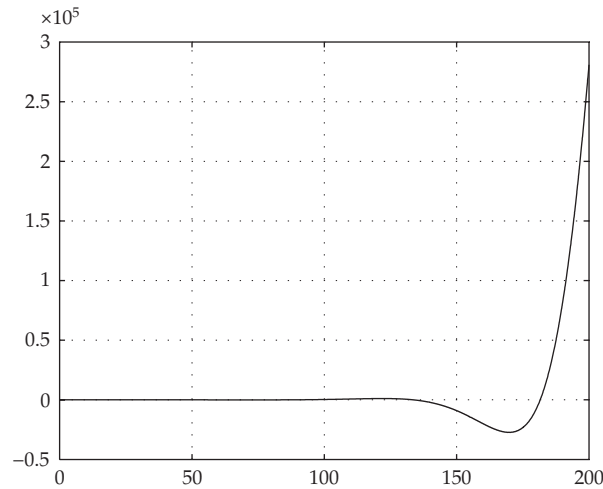
Figure 6: Second negative alternance of $z(x)$ generated for six-order system by cosine function.
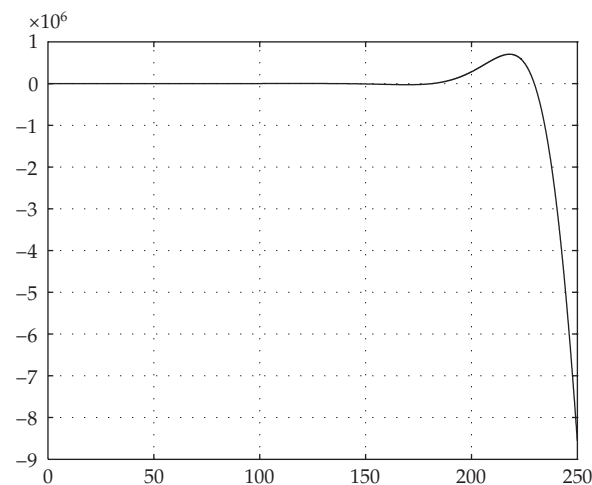

Figure 7: Third positive alternance of $z(x)$ generated for six-order system by cosine function.

alternances ($td_{k+1} - td_k \approx$ 40–50). As a consequence, a higher-order differential equation with a free term corresponding to the coherence function between an external cosine command and the output $y(x)$ generates for $z(x) = y'(x)$ alternances with the same temporal pattern and with an increasing amplitude according to a geometrical progression. For this reason, the higher-order differential equation previously presented is suitable for modeling multiscale phenomena and for explaining multiscale threshold transitions—see also [5, 6].

4. Quantum-Mechanical Case Study: The Hydrogen Atom

Aspects mentioned in the previous paragraphs have shown that both macroscopic and quantum transitions require a certain set of events (similar to a time series events) or a certain repetitive phenomenon for generating the coherence function which drives the dynamics of

internal oscillations. When the peak values of these internal oscillations are above a certain threshold, a possible transition to another state could appear.

However, these aspects should be valid also for standing waves corresponding to stable quantum or macroscopic states. This means that a certain wavefunction corresponding to a noninteracting particle should be described in an intuitive manner by a zero energy state and should be noticed by an external observer as a term of a quantum series (each term of such a series representing a quantum noninteracting state). For this purpose, the well-known case of the hydrogen atom should be analyzed at first.

The principle of mass-energy equivalence normally never comes to mind when the quantum-mechanical analysis of the hydrogen atom is undertaken. It is well known that by applying Schrödinger's equation to the problem of the electron in the hydrogen atom, the Balmer energy levels are obtained by means of a purely classical (i.e., nonrelativistic) analysis [7, 8]. There is, however, a very interesting connection between the problem of the hydrogen atom and the principle of mass-energy equivalence that was previously unexplored. If we write the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{4.1}$$

and its solution

$$\psi = \psi_0 \exp\left(-\frac{i}{\hbar} \int H dt\right), \tag{4.2}$$

where H is the total energy of the moving particle, namely, the electron, we must ask what conclusion can we make if we assume that the electron is in a stable orbit around the nucleus? Obviously, we must assume that the wave function $\psi = \psi_0$ (i.e., a constant, or stable wave function that does not evolve over time; even though it is dependent on the radius and the spherical harmonics). This, of course, is well known, since the electron's wave function in the hydrogen atom represents a standing wave and has no time dependence. Hence, the conclusion that inevitably emerges in this case is that the total energy of the electron H must be equal to zero everywhere along the path of the electron. In view of some fundamental research on the principle of mass-energy equivalence that was previously published by the author [9, 10], this conclusion, as a matter of fact, is not surprising.

In the earlier publications by the author, it was demonstrated that a number of fundamental problems in quantum mechanics cannot be understood on the basis of the relativistic law of mass-energy equivalence, $H = mc^2$. The problem of the hydrogen atom is one such problem. It was further demonstrated that $H = mc^2$ can be regarded as a *special case* of a more general law of mass-energy equivalence that does in fact explain that category of problems that the relativistic law fails to explain. That general law is $H = mv^2$, where the relativistic constant c^2 has been replaced by v^2, with v being the velocity of the moving particle (see [9–11] for a complete historical accounting of the origin and the applications of that law). We will now proceed to solve the problem of the total energy of the electron in the hydrogen atom and demonstrate that the general mass-energy equivalence law $H = mv^2$ correlates with and explains the result predicted by Schrödinger's equation. We will further demonstrate that a new "zero-energy wavefunction" that will be obtained under that law is actually a quantized version of the classical wavefunction that has been known for decades.

4.1. The Law of Mass-Energy Equivalence and the "Zero-Energy" Wave Equation

It is not difficult to see how the general mass-energy equivalence law $H = mv^2$ (which, admittedly, may seem strange to the readers who are not familiar with it) correlates with the result predicted by Schrödinger's equation. In the hydrogen atom, the electron is in equilibrium due to the equality of the two forces

$$\frac{e^2}{r^2} = \frac{mv^2}{r},\tag{4.3}$$

where e^2/r^2 is the Coulomb electrostatic force (here, $e^2 = q^2/4\pi\epsilon_0$, where q is the electron's charge), and where mv^2/r is the centrifugal force. But the electrostatic potential V acting on the electron is equal to $-e^2/r$, by definition. From the above equation, it is therefore clear that $V = -mv^2$. If we now assume that the total energy of the free electron is given by the quantity $+mv^2$, then it must be further clear that the total energy of the bound electron must be equal to zero (due to the addition of the electrostatic potential V). (It is to be pointed out that this conclusion concerns the TOTAL ENERGY of the electron. In practice, the atom is observed to emit and absorb energy during bound-state transitions because such transitions involve only kinetic energy and potential energy changes. Mass-energy equivalence obviously does not play a role in electronic bound-state transitions. That is why the present conclusions are not in disagreement with the classical theory or with experimental results). This is the classical view according to Bohr's theory. Let us now examine the view according to the Schrödinger Hamiltonian theory.

The classical Schrödinger Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V.\tag{4.4}$$

This Hamiltonian represents the sum Kinetic Energy + Potential Energy, and it is the Hamiltonian used to derive the Balmer energy levels and the classical wave function of the electron. If we want to write the Hamiltonian in a manner that takes mass-energy equivalence into account, the Hamiltonian will be written as follows:

$$H = -\frac{\hbar^2}{m}\nabla^2 + V,\tag{4.5}$$

where we have replaced the kinetic energy $1/2mv^2$ by the total energy mv^2. But since the total energy must be equal to zero, then we have the following wave equation:

$$-\frac{\hbar^2}{m}\nabla^2\psi_0 + V\psi_0 = 0.\tag{4.6}$$

We will now demonstrate that the wave function ψ_0 that satisfies this zero-energy wave equation is the same as the wave function derived through the classical analysis, with the surprising restriction that the wave function itself must be radially quantized!

4.2. The Connection between the Zero-Energy Wave Equation and the Classical Wave Equation

For the purpose of comparison, we write the classical equation that is based on the Schrödinger Hamiltonian together with the new wave equation that incorporates mass-energy equivalence:

$$-\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi_0 + V\psi_0 = W\psi_0 \quad \text{(classical)},$$

$$-\left(\frac{\hbar^2}{m}\right)\nabla^2\psi_0 + V\psi_0 = 0 \quad \text{(total energy)},$$

(4.7)

where W represents the Balmer energy levels and where $V = -e^2/r$ is the potential of the nucleus. While the two equations obviously seem to be two very different equations, we will now demonstrate that the second equation does *indeed* revert to the first equation if ψ_0 is restricted to be a radially quantized function, rather than a continuous function! We first write the zero-energy equation as follows:

$$\left(\frac{\hbar^2}{m}\right)\nabla^2\psi_0 = V\psi_0 = -\frac{e^2}{r}\psi_0.$$

(4.8)

Since the radial distance r takes only quantized values as multiples of the Bohr radius, $a = \hbar^2/me^2$, we substitute for r in the equation by using this quantity, getting

$$\left(\frac{\hbar^2}{m}\right)\nabla^2\psi_0 = -e^2\frac{me^2}{\hbar^2}\psi_0 = -\frac{me^4}{\hbar^2}\psi_0.$$

(4.9)

Dividing both sides of the equation by 2 gives

$$\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi_0 = -\frac{me^4}{2\hbar^2}\psi_0.$$

(4.10)

It is not difficult to verify that the coefficient of ψ_0 on the r.h.s. of the equation is the Balmer energy W. That is, we have the result that

$$\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi_0 = W\psi_0.$$

(4.11)

Now, by virtue of (4.11), the zero-energy wave equation in (4.7) can be finally written as

$$-\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi_0 + V\psi_0 = +\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi_0 = W\psi_0.$$

(4.12)

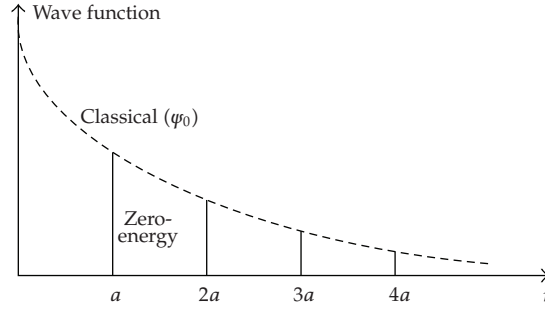This last equation is of course the classical wave equation.

Figure 8: The classical wave function ψ_0 and the quantized solution of the zero-energy wave equation. The latter exists only at integer multiples of the Bohr radius and inhabits the space defined by the former.

If we decompose the zero-energy wave equation into its radial and spherical-harmonic components, it becomes a simple matter to verify that the classical unnormalized wave function

$$\psi_0(r) = \exp\left(-\frac{me^2}{\hbar^2}r\right) \tag{4.13}$$

will indeed satisfy the radial wave equation at $r = na$, or integer multiples of the Bohr radius (see the proof in the appendix). The fact that the classical wave function satisfies the zero-energy wave equation at multiples of the Bohr radius can be understood physically as follows: the classical wave function is a continuous, differentiable function that defines the boundary of a space that theoretically extends from $r = 0$ to $r = \infty$ (see the plot in Figure 8). The solution of the zero-energy wave equation, on the other hand, is a discrete, sparse set in r that is defined *only* at integer multiples of the Bohr radius (see Figure 8). This discrete function therefore inhabits the space defined by the classical wave function (a simple analogy might be a wave in a plastic sheet on top of which tiny droplets of mercury always flow to the minimum of that wave, as if the wave was a "potential well"). This is not a surprise, since, as was concluded earlier, the total energy of the electron is equal to zero at multiples of the Bohr radius. The discrete solution, therefore, is indeed a solution in which the minimum energy principle is manifested; as opposed to the classical solution in which only the kinetic and potential energies are accounted for.

5. Conclusions

This study has presented the application of dynamical equations able to generate alternating deformations with increasing amplitude and delayed pulses in a certain material medium. It has been considered that an external force acts at certain time interval (similar to a time series) upon the material medium in the same area. Using a specific differential equation (considering nonzero initial values and using a function similar to the coherence function between the external force and the deformations inside the material), certain modulated amplitude oscillations were generated. For a higher order of the differential dynamical, some delayed pulses and a specific multiscale behaviour could be noticed. These features are similar to non-Markov aspects of quantum transitions, and for this reason the mathematical model is suitable for describing both quantum phenomena and

macroscopic aspects generated by sequence of pulses. The amplitude modulation of local maximum/minimum values can be put in correspondence with PDE equations (see [12]) and with propagating wavelets through dispersive media [13]. The multiscale analysis of delayed pulses is similar to multiscale analysis of wave propagation [14] and breaking phenomena can be put in correspondence with studies upon localized fractals (as in [15]) and could be extended for solving natural integral equations (as in [16]). A time-series application considering the Hydrogen atom as a case study was presented.

Appendix

Solution of the Zero-Energy Wave Equation and the Quantization Condition

To solve (4.6) for ψ_0, we must replace the operator ∇^2 by its equivalent expression in spherical coordinates and substitute the potential V by the traditional quantity $-e^2/r$. The process of replacing ∇^2 in (4.6) by its equivalent expression in spherical coordinates is well known in the literature [7, 8], and we simply write the result

$$\left(-\frac{\hbar^2}{m}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{mr^2} - \frac{e^2}{r}\right)(r\psi_0(r)) = 0. \tag{A.1}$$

Here, $\psi_0(r)$ is the radial component of ψ_0 and l is the orbital quantum number. Typically, a second equation is needed to solve for the spherical-harmonic component of ψ_0, but since this solution is well known in the literature it will not be discussed here. The usual approach for solving (A.1) is to let the product $r\psi_0(r)$ be equal to another function, say $\Gamma(r)$. Equation (A.1) is then rewritten as

$$-\Gamma''(r) + \left(\frac{l(l+1)}{r^2} - \frac{me^2}{\hbar^2}\cdot\frac{1}{r}\right)\Gamma(r) = 0. \tag{A.2}$$

In the classical solution, the Balmer series for hydrogen is obtained by simply setting $l = 0$. While the above equation cannot be solved for the Balmer energy, setting $l = 0$ results in

$$\Gamma''(r) + \frac{me^2}{\hbar^2}\cdot\frac{1}{r}\Gamma(r) = 0. \tag{A.3}$$

We now note that the quantity $\hbar^2/me^2$ represents the Bohr radius, a. We will follow however the standard procedure of replacing a by na, where n is the principal quantum number. We therefore rewrite the above equation as follows:

$$\Gamma''(r) + \frac{1}{na}\cdot\frac{1}{r}\Gamma(r) = 0. \tag{A.4}$$

Solving this simple differential equation is a simple but rather lengthy and uninformative mathematical exercise. It can be quickly verified, however, that the classical wave function

$$\Gamma(r) = r\psi_0(r) = r \exp\left(-\frac{r}{na}\right) \tag{A.5}$$

does in fact satisfy (A.4), provided that the radial distance r in the final expression is replaced by an integer multiple of the Bohr radius, or na.

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