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*Letter to the Editor*

# A Note on the Paper "Multipoint BVPs for Second-Order Differential Equations with Impulses" by Xuxin Yang, Zhimin He, and Jianhua Shen

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We give a counter example to the comparison principle for the multipoint BVPs (by Xuxin Yang, Zhimin He, and Jianhua Shen, in Mathematical Problems in Engineering, Volume 2009, Article ID 258090, doi:10.1155/2009/258090). Then we suggest and prove a corrected version of the comparison principle.

## 1. Introduction and Preliminaries

Consider the following multipoint BVPs [1]:

$$\begin{gathered} -u''(t) = f(t, u(t), u(\theta(t))), \quad t \neq t_k,\ t \in J = [0,1], \\ \Delta u'(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) - au'(0) = cu(\eta), \qquad u(1) + bu'(1) = du(\xi), \end{gathered} \tag{1.1}$$

where $0 \leq \theta(t) \leq t, \theta \in C(J), 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $f$ is continuous everywhere except at $\{t_k\} \times R^2$; $f(t_k^+, \cdot, \cdot)$ and $f(t_k^-, \cdot, \cdot)$ exist with $f(t_k^-, \cdot, \cdot) = f(t_k, \cdot, \cdot)$; $I_k \in C(R, R)$, and $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $a \geq 0$, $b \geq 0$, $0 \leq c \leq 1$, $0 \leq d \leq 1$, $a + c > 0, b + d > 0, 0 < \eta, \xi < 1$.

Let $PC(J) = \{x : J \to R; x(t)$ be continuous everywhere expect for some $t_k$ at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k) = x(t_k^-), k = 1, 2, \ldots, m\}$; $PC^1(J) = \{x \in PC(J) : x'(t)$ is continuous everywhere expect for some $t_k$ at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k) =$

$x'(t_k^-), k = 1,2,\ldots,m\}$. Let $J^- = J \setminus \{t_k, k = 1,2,\ldots,m\}$, and $E = PC^1(J,R) \cap C^2(J^-,R)$. a function $x \in E$ is called a solution of BVPS (1.1) if it satisfies (1.1).

The purpose of this note is to point out that the results basing on the comparison principle [1, Theorem 2.1] are not true. Then we give a new comparison principle.

## 2. Problem and Statement

The authors [1] proved some existence results for multipoint BVPs (1.1) by use of the following comparison principle [1, Theorem 2.1].

Assume that $u \in E$ satisfies

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) \leq 0, \quad t \neq t_k, \quad t \in J = [0,1], \\ \Delta u'(t_k) \geq L_k u(t_k), \quad k = 1,2,\ldots,m, \\ u(0) - au'(0) \leq cu(\eta), \qquad u(1) + bu'(1) \leq du(\xi), \end{gathered} \tag{2.1}$$

where $a \geq 0$, $b \geq 0$, $0 \leq c \leq 1$, $0 \leq d \leq 1$, $a + c > 0$, $b + d > 0$, $0 < \eta, \xi < 1$, $L_k \geq 0$, and constants $M, N$ satisfy

$$M > 0, N \geq 0, \qquad \frac{M+N}{2} + \sum_{k=1}^{m} L_k \leq 1. \tag{2.2}$$

Then $u(t) \leq 0$ for $t \in J$.

However, the comparison principle above is not true.

*A Counter Example*

Let

$$u(t) = \begin{cases} \dfrac{3}{2}t^2 + 20, & t \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{5}{2}t^2 + 3, & t \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{2.3}$$

Then

$$u'(t)=\begin{cases}3t, & t\in\left[0,\dfrac{1}{2}\right],\\ 5t, & t\in\left(\dfrac{1}{2},1\right],\end{cases}$$
$$u''(t)=\begin{cases}3, & t\in\left[0,\dfrac{1}{2}\right],\\ 5, & t\in\left(\dfrac{1}{2},1\right].\end{cases} \tag{2.4}$$

And let $M=N=1/1000$, $a=b=c=d=1, m=1, t_1=1/2, L_1=1/1000, \theta(t)=(1/2)t$, $\eta=1/3$, and $\xi=1/6$. When $t\in[0,1/2)$, then

$$\frac{1}{1000}\left(\frac{3}{2}t^2+20\right)+\frac{1}{1000}\left(\frac{3}{2}\times\frac{t^2}{4}+20\right)\leq 3. \tag{2.5}$$

When $t\in(1/2,1]$, then

$$\frac{1}{1000}\left(\frac{5}{2}t^2+3\right)+\frac{1}{1000}\left(\frac{5}{2}\times\frac{t^2}{4}+3\right)\leq 5. \tag{2.6}$$

Hence $-u''(t)+Mu(t)+Nu(\theta(t))\leq 0$.

$$\Delta u'\left(\frac{1}{2}\right)=u'\left(\frac{1}{2}+\right)-u'\left(\frac{1}{2}\right)=5\times\frac{1}{2}-\left(3\times\frac{1}{2}\right)=1, \tag{2.7}$$

$$\frac{1}{1000}u\left(\frac{1}{2}\right)=\frac{1}{1000}\left(\frac{3}{2}\times\frac{1}{4}+20\right)=\frac{1}{1000}\times\frac{163}{8}. \tag{2.8}$$

Hence $\Delta u'(t_1)\geq L_1u(t_1)$.

$$u(0)-u'(0)=20,\qquad u\left(\frac{1}{3}\right)=\frac{3}{2}\times\frac{1}{9}+20. \tag{2.9}$$

Hence $u(0)-au'(0)\leq cu(1/3)$.

$$u(1)+u'(1)=\frac{5}{2}+3+5=\frac{21}{2},\qquad u\left(\frac{1}{6}\right)=\frac{3}{2}\times\frac{1}{36}+20. \tag{2.10}$$

Hence $u(1) + bu'(1) \leq du(1/6)$.

$$\frac{M+N}{2} + \sum_{k=1}^{m} L_k = \frac{2}{1000} < 1. \tag{2.11}$$

But we easily show that $u(t) > 0$, for all $t \in [0,1]$, which is a contradiction with (Theorem 2.1) in [1]. In fact, we can correct Theorem 2.1 in [1] as follows.

**Theorem 2.1.** *Suppose $u \in E \cap C(J)$ such that*

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) \leq 0 \quad t \neq t_k,\ t \in J = [0,1], \\ \Delta u'(t_k) \geq L_k u(t_k), \quad k = 1,2,\ldots,m, \\ u(0) - au'(0) \leq cu(\eta), \qquad u(1) + bu'(1) \leq du(\xi), \end{gathered} \tag{2.12}$$

*where $a \geq 0$, $b \geq 0$, $0 \leq c \leq 1$, $0 \leq d \leq 1$, $0 < \eta,\ \xi < 1$, $a + c > 0$, $b + d > 0$, $L_k > 0$, and constants $M, N$ satisfy*

$$M > 0, N \geq 0, \quad \frac{M+N}{2} + \sum_{k=1}^{m} L_k \leq 1. \tag{2.13}$$

*Then $u(t) \leq 0$ for $t \in J$.*

*Remark 2.2.* In this Theorem, we have to add $u \in C(J)$.

*Proof.* Suppose to contrary that there exist some $t \in J$, such that $u(t) > 0$.

If $u(1) = \max_{t\in J} u(t) > 0$, we have $u'(1) \geq 0$, $u(1) \geq u(\xi)$, and

$$du(\xi) \leq u(1) \leq u(1) + bu'(1) \leq du(\xi). \tag{2.14}$$

Therefore, $d = 1$ and $u(\xi)$ is maximum value.

If $u(0) = \max_{t\in J} u(t) > 0$, we have $u'(0) \leq 0$, $u(0) \geq u(\eta)$, and

$$cu(\eta) \leq u(0) \leq u(0) - au'(0) \leq cu(\eta). \tag{2.15}$$

Therefore, $c = 1$ and $u(\eta)$ is maximum value.

So there is a $\delta \in (0,1)$ such that

$$u(\delta) = \max_{t\in J} u(t) > 0, \quad \text{by} \quad \Delta u = 0, \quad \text{then } u'(\delta^+) \le 0, \quad u'(\delta^-) \ge 0. \tag{2.16}$$

It is obvious to see that $\delta \notin \{t_k, k = 1,2,\ldots,m\}$ by

$$\Delta u'(\delta) = u'(\delta^+) - u'(\delta) \ge L_k u(\delta) > 0 \tag{2.17}$$

which is a contradiction because of (2.16).

(i) Suppose that $u(t) \ge 0$ for $t \in [0,\delta]$.

By $u(\delta) = \max_{t\in J} u(t) > 0$, we get $\delta \in J^-$, $u''(\delta) \le 0$. On the other hand, by (2.12), we have

$$0 < Mu(\delta) + Nu(\theta(t)) \le u''(\delta) \tag{2.18}$$

which is a contradiction.

(ii) Suppose there exists $t_* \in [0,\delta]$ such that $u(t_*) = \min_{t\in[0,\delta)} u(t) < 0$. By (2.12), we get

$$\begin{gathered} u''(t) \ge (M+N)u(t_*), \quad t \in [0,\delta), t \ne t_k, \\ \Delta u(t_k) = 0, \\ \Delta u'(t_k) \ge L_k u(t_k), \quad k = 1,2,\ldots,m. \end{gathered} \tag{2.19}$$

Integrating from $s(t_* \le s \le \delta)$ to $\delta$, we get

$$\begin{aligned} u'(\delta) - u'(s) &\ge \int_s^\delta (M+N)u(t_*)ds + \sum_{s<t_k<\delta} L_k u(t_k) \\ &= (\delta - s)(M+N)u(t_*) + \sum_{s<t_k<\delta} L_k u(t_k) \\ &\ge (\delta - s)(M+N)u(t_*) + \sum_{k=1}^{m} L_k u(t_*). \end{aligned} \tag{2.20}$$

Hence

$$-u'(s) \ge (\delta - s)(M+N)u(t_*) + \sum_{k=1}^{m} L_k u(t_*), \quad t_* \le s \le \delta. \tag{2.21}$$

Then integrate from $t_*$ to $\delta$ to obtain

$$\begin{aligned}
-u(t_*) &< u(\delta) - u(t_*) \\
&\le \int_{t_*}^{\delta} (M+N)u(t_*)(s-\delta)ds - \sum_{k=1}^{m} L_k u(t_*)(\delta - t_*) \\
&= (M+N)u(t_*)\left[-\frac{(t_*-\delta)^2}{2}\right] - \sum_{k=1}^{m} L_k u(t_*)(\delta - t_*) \\
&\le -\left[\frac{M+N}{2}(\delta - t_*)^2 + \sum_{k=1}^{m} L_k\right]u(t_*) \\
&\le -\left(\frac{M+N}{2} + \sum_{k=1}^{m} L_k\right)u(t_*).
\end{aligned} \tag{2.22}$$

By (2.13), we get $u(t_*) > 0$ which is a contradiction. We complete the proof. □

This implies that in order to get the existence results of the multipoint BVPs [1], we have to require an additional continuity hypotheses on the function space.

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## References

[1] X. Yang, Z. He, and J. Shen, "Multipoint BVPs for second-order functional differential equations with impulses," *Mathematical Problems in Engineering*, vol. 2009, Article ID 258090, 16 pages, 2009.