*Research Article*

# Stochastic Finite Element for Structural Vibration

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This paper proposes a new method of calculating stochastic field. It is an improvement of the midpoint method of stochastic field. The vibration equation of a system is transformed to a static problem by using the Newmark method and the Taylor expansion is extended for the structural vibration analysis with uncertain factors. In order to develop computational efficiency and allow for efficient storage, the Conjugate Gradient method (CG) is also employed. An example is given, respectively, and calculated results are compared to validate the proposed methods.

## 1. Introduction

Material properties, geometry parameters, and applied loads of the structure are assumed to be stochastic. Although the finite element method analysis of complicated structures has become a generally widespread and accepted numerical method, regarding the given factors as constants cannot apparently correspond to the reality of a structure. In order to enhance computational accuracy, the influence of random factors must be considered.

Many physics parameters of material possess spatial variability, such as Young's modulus and Poisson's ratio, so we should regard them as stochastic fields. Stochastic field discretization is the problem that various stochastic finite element methods need to resolve, but discrete form of stochastic field plays the decisive influence on the calculation and computational accuracy of stochastic finite element. The simplest discretization is the midpoint method (MSF) [1]. The stochastic field is described by a single random variable replacing the value of the field at a central point of the mesh. The local average method of the stochastic field describes the stochastic field of an element in terms of the spatial average. The local average method of rectangle element is described by the mean, variance, and covariance [2]. It can be extended for 3D [3]. The stochastic field of nonrectangular element is described by the mean vector and covariance matrix using Gaussian quadrature [4]. The stochastic

field can be described by the shape function and nodal values, and it is necessary to know the related function [5]. Making use of the Karhunen-Loeve expansion, stochastic field is represented by series expansion [6]. When stiffness matrices are deduced, a weighted integral method is adopted to consider stochastic field of material parameters [7, 8]. When stochastic field is expressed by series expansion, the optimal linear-estimation method is applied to make the error of variance minimum [9].

The direct Monte Carlo simulation of the stochastic finite element method (DSFEM) requires a large number of samples, which requires much calculation time [10]. Monte Carlo simulation by applying the Neumann expansion (NSFEM) enhances computational efficiency and saves storage in such a way that the NSFEM combined with Monte Carlo simulation enhances the finite element model advantageously [11]. The preconditioned Conjugate Gradient method (PCG) applied in the calculation of stochastic finite elements can also enhance computational accuracy and efficiency [12]. According to first-order or second-order perturbation methods, calculation formulas can be obtained [5, 11, 14–16, 18–20, 22]. The result is called the PSFEM and has been adopted by many authors.

The PSFEM is often applied in dynamic analysis of structures and the second-order perturbation technique has been proved to be efficient [5]. Dynamic reliability of a frame is calculated by the SFEM and response sensitivity is formulated in the context of stiffness and mass matrix condensation [16]. When load actions are treated as stochastic processes, vibration of the structure is resolved by the PSFEM [17]. The PSFEM is an adequate tool for nonlinear structural dynamics. Nonlinearities due to material and geometrical effects have also been included [18]. By forming a new dynamic shape function matrix, dynamic analysis of the spatial frame structure is presented by the PSFEM [19]. The NSFEM is introduced in dynamic analysis within the framework of a Monte Carlo simulation [20]. The NSFEM is applied to the dynamic response of a random structure system and results are compared with those from the PSFEM and the DSFEM [21]. With the aid of the PSFEM, a stochastic formulation for nonlinear dynamic analysis of a structure is presented [22].

An improved midpoint method of stochastic field (IMSF) is presented. The IMSF is more accurate than the MSF. The Newmark method transforms differential equations into linear equations. The IMSF is used and the structural vibrations for a linear system are computed by the Taylor expansion method, the CG method, and the PCG method of stochastic finite element (TSFEM, CG, PCG). The TSFEM, the CG, and the PCG based on the MSF are called the MTSFEM, the MCG, and the MPCG. An example demonstrates the superiority of the proposed methods.

## 2. Improved Midpoint Method of Stochastic Field

When finite element method is used, structure is divided into small elements whose number is appropriate. In this paper, Young's modulus is assumed to be a Gaussian process. When element is appropriately small, Young's modulus of an element is described by a variable. The Young's modulus of structure is described by a group of variables. Without loss of generality, it is supposed that the structure is divided into elements of $m$ nodes and $n$ nodes. The Young's modulus of node within element $e$ of $m$ nodes is expressed by $a_{em1}, a_{em2}, \ldots, a_{emm}$. The Young's modulus of midpoint within element $e$ of $m$ nodes is expressed by $a_{eml}$. The Young's modulus of node within element $f$ of $n$ nodes is expressed by $a_{fn1}, a_{fn2}, \ldots, a_{fnn}$. The Young's modulus of midpoint within element $f$ of n nodes is expressed by $a_{fnl}$.

The Young's modulus within element $e$ is defined as

$$a_e = \frac{a_{em1} + a_{em2} + \cdots + a_{emm} + a_{eml}}{m+1}.$$
(2.1)

Its mean is

$$\mu_e = \mu\left(\frac{a_{em1} + a_{em2} + \cdots + a_{emm} + a_{eml}}{m+1}\right)$$

$$= \frac{\mu_{aem1} + \mu_{aem2} + \cdots + \mu_{aemm} + \mu_{aeml}}{m+1},$$
(2.2)

where the means of Young's modulus at the first node, the second node,..., the $m$th node of element $e$ are expressed by $\mu_{aem1}, \mu_{aem2}, \ldots, \mu_{aemm}$. The mean of Young's modulus at the midpoint of element $e$ is expressed by $\mu_{aeml}$.

The Young's modulus within element $f$ is defined as

$$a_f = \frac{a_{fn1} + a_{fn2} + \cdots + a_{fnn} + a_{fnl}}{n+1}.$$
(2.3)

Its mean is

$$\mu_f = \mu\left(\frac{a_{fn1} + a_{fn2} + \cdots + a_{fnn} + a_{fnl}}{n+1}\right)$$

$$= \frac{\mu_{afn1} + \mu_{afn2} + \cdots + \mu_{afnn} + \mu_{afnl}}{n+1},$$
(2.4)

where the means of Young's modulus at the first node, the second node,...,the $n$th node of element $f$ are expressed by $\mu_{afn1}, \mu_{afn2}, \ldots, \mu_{afnn}$. The mean of Young's modulus at the midpoint of element $f$ is expressed by $\mu_{afnl}$.

The covariance of Young's modulus between two elements that each has $m$ nodes is obtained by

$$
\begin{aligned}
\text{Cov}(a_e, a_{e'}) &= \text{Cov}\left(\frac{a_{em1} + a_{em2} + \cdots + a_{emm} + a_{eml}}{m+1}, \frac{a_{e'm1} + a_{e'm2} + \cdots + a_{e'mm} + a_{e'ml}}{m+1}\right) \\
&= \frac{1}{(m+1)^2}[\text{Cov}(a_{em1}, a_{e'm1} + a_{e'm2} + \cdots + a_{e'mm} + a_{e'ml}) \\
&\quad + \text{Cov}(a_{em2} + \cdots a_{emm} + a_{eml}, a_{e'm1} + a_{e'm2} + \cdots + a_{e'mm} + a_{e'ml})] \\
&= \frac{1}{(m+1)^2}[\text{Cov}(a_{em1}, a_{e'm1}) + \text{Cov}(a_{em1}, a_{e'm2}) \\
&\quad + \cdots + \text{Cov}(a_{em1}, a_{e'mm}) + \text{Cov}(a_{em1}, a_{e'ml})] \\
&\quad + \frac{1}{(m+1)^2}\text{Cov}(a_{em2} + \cdots + a_{emm} + a_{eml}, a_{e'm1} + a_{e'm2} + \cdots + a_{e'mm} + a_{e'ml}) \\
&= \frac{1}{(m+1)^2}\left(\sum_{g_1=1}^{m}\sum_{g_2=1}^{m}\text{Cov}(a_{emg_1}, a_{e'mg_2})\right) + \frac{1}{(m+1)^2}\left(\sum_{g_1=1}^{m}\text{Cov}(a_{emg_1}, a_{e'ml})\right) \\
&\quad + \frac{1}{(m+1)^2}\left(\sum_{g_2=1}^{m}\text{Cov}(a_{eml}, a_{e'mg_2}) + \text{Cov}(a_{eml}, a_{e'ml})\right),
\end{aligned}
$$

(2.5)

where $\text{Cov}(a_{emg_1}, a_{e'mg_2})$ = the covariance of Young's modulus between node $g_1(g_1 = 1, 2, \ldots, m)$ of element $e$ and node $g_2$ $(g_2 = 1, 2, \ldots, m)$ of element $e'$, $\text{Cov}(a_{emg_1}, a_{e'ml})$ = the covariance of Young's modulus between node $g_1$ of element $e$ and the midpoint of element $e'$, $\text{Cov}(a_{eml}, a_{e'mg_2})$ = the covariance of Young's modulus between the midpoint of element $e$ and node $g_2$ of element $e'$, and $\text{Cov}(a_{eml}, a_{e'ml})$ = the covariance of Young's modulus between the midpoint of element $e$ and the midpoint of element $e'$.

The covariance of Young's modulus between two elements is given in Appendix A.

Using covariance matrix, the correlation of Young's modulus between any two elements is given by

$$
C_{aa} = \begin{pmatrix}
\text{Cov}(a_1, a_1) & \text{Cov}(a_1, a_2) & \cdots & \text{Cov}(a_1, a_N) \\
\text{Cov}(a_2, a_1) & \text{Cov}(a_2, a_2) & \cdots & \text{Cov}(a_2, a_N) \\
\vdots & \vdots & & \vdots \\
\text{Cov}(a_N, a_1) & \text{Cov}(a_N, a_2) & \cdots & \text{Cov}(a_N, a_N)
\end{pmatrix}.
$$

(2.6)

A Gaussian vector $\vec{a} = [a_1, a_2, \ldots, a_N]^T$ is generated

$$
\vec{a} = LZ.
$$

(2.7)

$Z = [Z_1, Z_2, \ldots, Z_N]^T$ consists of $N$ Gaussian random variables with mean zero and unit standard deviation. The Cholesky matrix $L$ can be obtained through a decomposition of the covariance matrix; therefore,

$$\mu\left[ZZ^T\right] = I,$$
$$LL^T = C_{aa}.$$

(2.8)

$I$ is the identity matrix. The generation of vector $\vec{a}$ must satisfy the covariance matrix

$$\mu\left[\vec{a}\,\vec{a}^T\right] = \mu\left[LZ(LZ)^T\right]$$
$$= L\mu\left[ZZ^T\right]L^T = C_{aa}.$$

(2.9)

Once the decomposition has been completed, different samples of vector $\vec{a}$ can be acquired easily by (2.7). Thus, it is possible that Monte Carlo simulation resolves problem of stochastic finite element.

## 3. Dynamic Analysis of Finite Element

For a linear system, the dynamic equilibrium equation is given by

$$[M]\{\ddot{\delta}\} + [C]\{\dot{\delta}\} + [K]\{\delta\} = \{F\},$$

(3.1)

where $\{\ddot{\delta}\}, \{\dot{\delta}\}, \{\delta\}$ are the acceleration, velocity, and displacement vectors. $[M], [K]$, and $[C]$ are the global mass, stiffness, and damping matrices obtained by assembling the element variables in global coordinate system.

By using the Newmark method, (3.1) becomes

$$\{\delta_{t+\Delta t}\} = \left[\tilde{K}\right]^{-1}\left\{\tilde{F}_{t+\Delta t}\right\},$$

(3.2)

where $\{\delta_{t+\Delta t}\}[\tilde{K}]$ and $\{\tilde{F}_{t+\Delta t}\}$ indicate the displacement vector, stiffness matrix and load vector at time $t + \Delta t$. $[\tilde{K}]$ and $\{\tilde{F}_{t+\Delta t}\}$ are given in Appendix B.

## 4. Dynamic Analysis of Structure Based on CG

Equation (3.2) can be rewritten as

$$\left[\tilde{K}\right]\{\delta_{t+\Delta t}\} = \left\{\tilde{F}_{t+\Delta t}\right\}.$$

(4.1)

Using (2.7), (2.8), and (2.9), $N_1$ samples of vector $\vec{a}$ are produced. $N_1$ matrices $[\tilde{K}]$ and $N_1$ (4.1) are generated. For linear vibrations, (4.1) is a system of linear equations. The CG is

an adequate method for solving large systems of linear equations. It can be accomplished as follows.

first, select appropriate solution as initial values

$$\{\delta_{t+\Delta t}\}^{(0)} = \left(\delta^{(0)}{}_{(t+\Delta t)_1}, \delta^{(0)}{}_{(t+\Delta t)_2}, \ldots, \delta^{(0)}{}_{(t+\Delta t)_{N_1}}\right)^T; \tag{4.2}$$

calculate the first residual vector

$$r^{(0)} = \left\{\tilde{F}_{t+\Delta t}\right\} - \left[\tilde{K}\right]\{\delta_{t+\Delta t}\}^{(0)} \tag{4.3}$$

and vector

$$p^{(0)} = \left[\tilde{K}\right]^T r^{(0)} \tag{4.4}$$

where $[\tilde{K}]^T$ is the transposed matrix;

for $\tilde{i} = 0, 1, 2, \ldots, n_2 - 1$, iterate step by step as follows:

$$\alpha_{\tilde{i}} = \frac{\left(\left[\tilde{K}\right]p^{(\tilde{i})}, r^{(\tilde{i})}\right)}{\left(\left[\tilde{K}\right]p^{(\tilde{i})}, \left[\tilde{K}\right]p^{(\tilde{i})}\right)} = \frac{\left(p^{(\tilde{i})}, \left[\tilde{K}\right]^T r^{(\tilde{i})}\right)}{\left(\left[\tilde{K}\right]p^{(\tilde{i})}, \left[\tilde{K}\right]p^{(\tilde{i})}\right)}$$

$$= \frac{\left(\left[\tilde{K}\right]^T r^{(\tilde{i})}, \left[\tilde{K}\right]^T r^{(\tilde{i})}\right)}{\left(\left[\tilde{K}\right]p^{(\tilde{i})}, \left[\tilde{K}\right]p^{(\tilde{i})}\right)},$$

$$\{\delta_{t+\Delta t}\}^{(\tilde{i}+1)} = \{\delta_{t+\Delta t}\}^{(\tilde{i})} + \alpha_{\tilde{i}}p^{(\tilde{i})},$$

$$r^{(\tilde{i}+1)} = r^{(\tilde{i})} - \alpha_{\tilde{i}}\left[\tilde{K}\right]p^{(\tilde{i})}, \tag{4.5}$$

$$\beta_{\tilde{i}+1} = \frac{\left(\left[\tilde{K}\right]^T r^{(\tilde{i}+1)}, \left[\tilde{K}\right]^T r^{(\tilde{i}+1)}\right)}{\left(\left[\tilde{K}\right]^T r^{(\tilde{i})}, \left[\tilde{K}\right]^T r^{(\tilde{i})}\right)},$$

$$p^{(\tilde{i}+1)} = \left[\tilde{K}\right]^T r^{(\tilde{i}+1)} + \beta_{\tilde{i}+1}p^{(\tilde{i})}.$$

This process stops only if $r^{n_2}$ is small enough.

Vectors $\{\delta_{t+\Delta t}\}_1, \{\delta_{t+\Delta t}\}_2, \ldots, \{\delta_{t+\Delta t}\}_{N_1}$ are solutions of $N_1$ (4.1). The mean of $\{\delta_{t+\Delta t}\}$ is given by

$$\mu\{\delta_{t+\Delta t}\} = \frac{\{\delta_{t+\Delta t}\}_1 + \{\delta_{t+\Delta t}\}_2 + \cdots + \{\delta_{t+\Delta t}\}_{N_1}}{N_1}. \tag{4.6}$$

The variance of $\{\delta_{t+\Delta t}\}$ is given by

$$\mathrm{Var}\{\delta_{t+\Delta t}\} = \frac{1}{N_1 - 1}\sum_{i=1}^{N_1}\left(\{\delta_{t+\Delta t}\}_i - \mu\{\delta_{t+\Delta t}\}\right)^2. \tag{4.7}$$

Similarly, the mean and variance of vector $\{\delta_{t+i_1\Delta t}\}$ can be solved at time $t + i_1\Delta t$ step by step. At time $t' = t + i_2\Delta t$ $(i_2 = 1, 2, \ldots, n_1)$, the stress for element $d$ is given by

$$\{\sigma\} = [D][B]\left\{\delta_{t'}^d\right\}, \tag{4.8}$$

where $[D]$ = the material response matrix of element $d$, $[B]$ = the gradient matrix of element $d$ and $\{\delta_{t'}{}^d\}$ = the element $d$ nodal displacement vector at time $t'$.

Substituting $N_1$ samples of vector $\vec{a}$ into (4.8), vectors $\{\sigma\}_1, \{\sigma\}_2, \ldots, \{\sigma\}_{N_1}$ can be obtained.

The mean of $\{\sigma\}$ is given by

$$\mu\{\sigma\} = \frac{\{\sigma\}_1 + \{\sigma\}_2 + \cdots + \{\sigma\}_{N_1}}{N_1}. \tag{4.9}$$

The variance of $\{\sigma\}$ is given by

$$\mathrm{Var}\{\sigma\} = \frac{1}{N_1 - 1}\sum_{i=1}^{N_1}\left(\{\sigma\}_i - \mu\{\sigma\}\right)^2. \tag{4.10}$$

The CG belongs to methods of iteration that converge quickly. For practical purposes, PCG is applied to accelerate convergence.

## 5. Dynamic Analysis of Structure Based on the TSFEM

Young's modulus of the structure is given as function of $N$ random variables $a_1, a_2, \ldots, a_N$. The partial derivative of (4.1) with respect to $a_i$ is given by

$$\frac{\partial\{\delta_{t+\Delta t}\}}{\partial a_i} = \left[\tilde{K}\right]^{-1}\left(\frac{\partial\left\{\tilde{F}_{t+\Delta t}\right\}}{\partial a_i} - \frac{\partial\left[\tilde{K}\right]}{\partial a_i}\{\delta_{t+\Delta t}\}\right), \tag{5.1}$$

where

$$
\begin{aligned}
\frac{\partial \left\{ \tilde{F}_{t+\Delta t} \right\}}{\partial a_i} = {}& \frac{\partial \{F_{t+\Delta t}\}}{\partial a_i} + \frac{\partial [M]}{\partial a_i} \left( b_0 \{\delta_t\} + b_2 \{\dot{\delta}_t\} + b_3 \{\ddot{\delta}_t\} \right) \\
& + [M] \left( b_0 \frac{\partial \{\delta_t\}}{\partial a_i} + b_2 \frac{\partial \{\dot{\delta}_t\}}{\partial a_i} + b_3 \frac{\partial \{\ddot{\delta}_t\}}{\partial a_i} \right) \\
& + \frac{\partial [C]}{\partial a_i} \left( b_1 \{\delta_t\} + b_4 \{\dot{\delta}_t\} + b_5 \{\ddot{\delta}_t\} \right) \\
& + [C] \left( b_1 \frac{\partial \{\delta_t\}}{\partial a_i} + b_4 \frac{\partial \{\dot{\delta}_t\}}{\partial a_i} + b_5 \frac{\partial \{\ddot{\delta}_t\}}{\partial a_i} \right).
\end{aligned}
\tag{5.2}
$$

After $\partial \{\delta_t\}/\partial a_i = q_0$, $\partial \{\dot{\delta}_t\}/\partial a_i = \dot{q}_0$, and $\partial \{\ddot{\delta}_t\}/\partial a_i = \ddot{q}_0$ are given, (5.2) can be calculated.

The partial derivative of (5.1) with respect to $a_j$ is given by

$$
\frac{\partial^2 \{\delta_{t+\Delta t}\}}{\partial a_i \partial a_j} = \left[ \tilde{K} \right]^{-1} \left( \frac{\partial^2 \left\{ \tilde{F}_{t+\Delta t} \right\}}{\partial a_i \partial a_j} - \frac{\partial \left[ \tilde{K} \right]}{\partial a_i} \frac{\partial \{\delta_{t+\Delta t}\}}{\partial a_j} - \frac{\partial \left[ \tilde{K} \right]}{\partial a_j} \frac{\partial \{\delta_{t+\Delta t}\}}{\partial a_i} - \frac{\partial^2 \left[ \tilde{K} \right]}{\partial a_i \partial a_j} \{\delta_{t+\Delta t}\} \right), \tag{5.3}
$$

where

$$
\begin{aligned}
\frac{\partial^2 \left\{ \tilde{F}_{t+\Delta t} \right\}}{\partial a_i \partial a_j} = {}& \frac{\partial^2 \{F_{t+\Delta t}\}}{\partial a_i \partial a_j} + \frac{\partial^2 [M]}{\partial a_i \partial a_j} \left( b_0 \{\delta_t\} + b_2 \{\dot{\delta}_t\} + b_3 \{\ddot{\delta}_t\} \right) \\
& + \frac{\partial [M]}{\partial a_i} \left( b_0 \frac{\partial \{\delta_t\}}{\partial a_j} + b_2 \frac{\partial \{\dot{\delta}_t\}}{\partial a_j} + b_3 \frac{\partial \{\ddot{\delta}_t\}}{\partial a_j} \right) \\
& + \frac{\partial [M]}{\partial a_j} \left( b_0 \frac{\partial \{\delta_t\}}{\partial a_i} + b_2 \frac{\partial \{\dot{\delta}_t\}}{\partial a_i} + b_3 \frac{\partial \{\ddot{\delta}_t\}}{\partial a_i} \right) \\
& + [M] \left( b_0 \frac{\partial^2 \{\delta_t\}}{\partial a_i \partial a_j} + b_2 \frac{\partial^2 \{\dot{\delta}_t\}}{\partial a_i \partial a_j} + b_3 \frac{\partial^2 \{\ddot{\delta}_t\}}{\partial a_i \partial a_j} \right) \\
& + \frac{\partial^2 [C]}{\partial a_i \partial a_j} \left( b_1 \{\delta_t\} + b_4 \{\dot{\delta}_t\} + b_5 \{\ddot{\delta}_t\} \right) \\
& + \frac{\partial [C]}{\partial a_i} \left( b_1 \frac{\partial \{\delta_t\}}{\partial a_j} + b_4 \frac{\partial \{\dot{\delta}_t\}}{\partial a_j} + b_5 \frac{\partial \{\ddot{\delta}_t\}}{\partial a_j} \right) \\
& + \frac{\partial [C]}{\partial a_j} \left( b_1 \frac{\partial \{\delta_t\}}{\partial a_i} + b_4 \frac{\partial \{\dot{\delta}_t\}}{\partial a_i} + b_5 \frac{\partial \{\ddot{\delta}_t\}}{\partial a_i} \right) \\
& + [C] \left( b_1 \frac{\partial^2 \{\delta_t\}}{\partial a_i \partial a_j} + b_4 \frac{\partial^2 \{\dot{\delta}_t\}}{\partial a_i \partial a_j} + b_5 \frac{\partial^2 \{\ddot{\delta}_t\}}{\partial a_i \partial a_j} \right).
\end{aligned}
\tag{5.4}
$$

**Table 1:** Comparison of error.

| | The mean of vertical displacement at node 505 | The variance of vertical displacement at node 505 | The mean of horizontal stress at node 5 | The variance of horizontal stress at node 5 |
|---|---|---|---|---|
| PCG | 2.95% | 3.42% | 4.31% | 5.07% |
| CG | 3.11% | 3.84% | 4.57% | 5.24% |
| TSFEM | 5.12% | 6.01% | 6.28% | 7.17% |
| MPCG | 7.14% | 7.02% | 8.67% | 9.94% |
| MCG | 7.23% | 7.63% | 8.82% | 10.17% |
| MTSFEM | 12.76% | 13.27% | 14.47% | 16.53% |
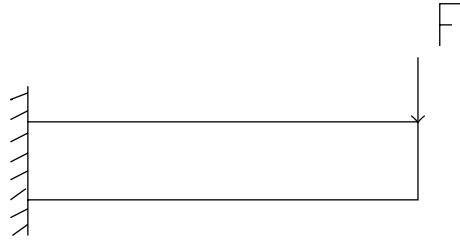


**Figure 1:** A cantilever beam.

After $\partial\{\delta_t\}/\partial a_j = q_1$, $\partial\{\dot\delta_t\}/\partial a_j = \dot q_1$, $\partial\{\ddot\delta_t\}/\partial a_j = \ddot q_1$, $\partial^2\{\delta_t\}/\partial a_i\partial a_j = r_0$, and $\partial^2\{\dot\delta_t\}/\partial a_i\partial a_j = \dot r_0$, $\partial^2\{\ddot\delta_t\}/\partial a_i\partial a_j = \ddot r_0$ are given, (5.4) can be calculated.

The displacement is expanded at the mean value point $\overline a = (\overline a_1, \overline a_2, \ldots, \overline a_i, \ldots, \overline a_{n_1})^T$ by means of a Taylor series. the mean of $\delta_{t+\triangle t}$ is obtained as

$$\mu\{\delta_{t+\triangle t}\} \approx \{\delta_{t+\triangle t}\}|_{a=\overline a} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2\{\delta_{t+\triangle t}\}}{\partial a_i\partial a_j}\bigg|_{a=\overline a}\mathrm{Cov}(a_i, a_j), \tag{5.5}$$

where $\mu\{\delta_{t+\triangle t}\}$ expresses mean value $\delta_{t+\triangle t}$ and $\mathrm{Cov}(a_i, a_j)$ is the covariance between $a_i$ and $a_j$.

The variance of $\delta_{t+\triangle t}$ is given by

$$\mathrm{Var}\{\delta_{t+\triangle t}\} \approx \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\{\delta_{t+\triangle t}\}}{\partial a_i}\bigg|_{a=\overline a} \cdot \frac{\partial\{\delta_{t+\triangle t}\}}{\partial a_j}\bigg|_{a=\overline a} \cdot \mathrm{Cov}(a_i, a_j). \tag{5.6}$$

The velocity vector $\{\dot\delta_{t+\triangle t}\}$ and the acceleration vector $\{\ddot\delta_{t+\triangle t}\}$ are given in Appendix B. The partial derivative of $\ddot\delta_{t+\triangle t}$ with respect to $a_i$ is given by

$$\frac{\partial\{\ddot\delta_{t+\triangle t}\}}{\partial a_i} = b_0\left(\frac{\partial\{\delta_{t+\triangle t}\}}{\partial a_i} - \frac{\partial\{\delta_t\}}{\partial a_i}\right) - b_2\frac{\partial\{\dot\delta_t\}}{\partial a_i} - b_3\frac{\partial\{\ddot\delta_t\}}{\partial a_i}. \tag{5.7}$$
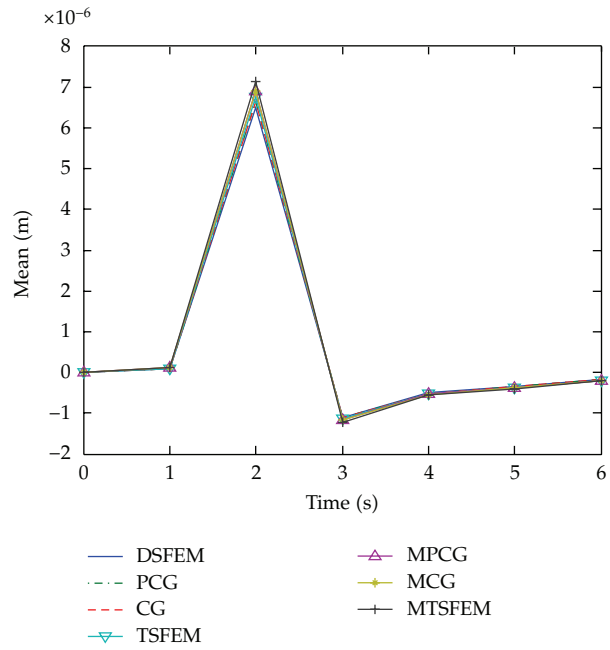
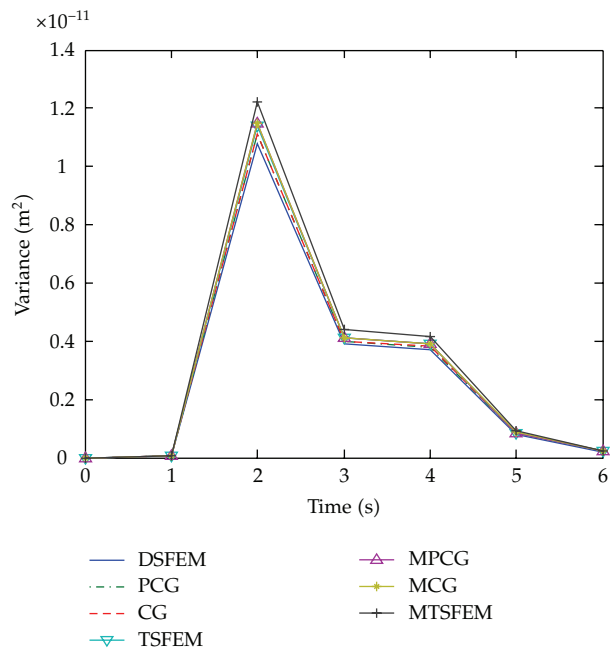**Figure 2:** The mean of vertical displacement at node 505.



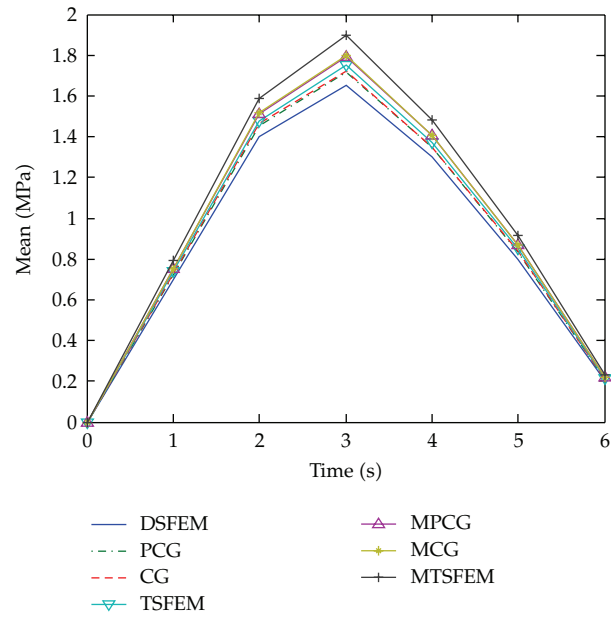**Figure 3:** The variance of vertical displacement at node 505.

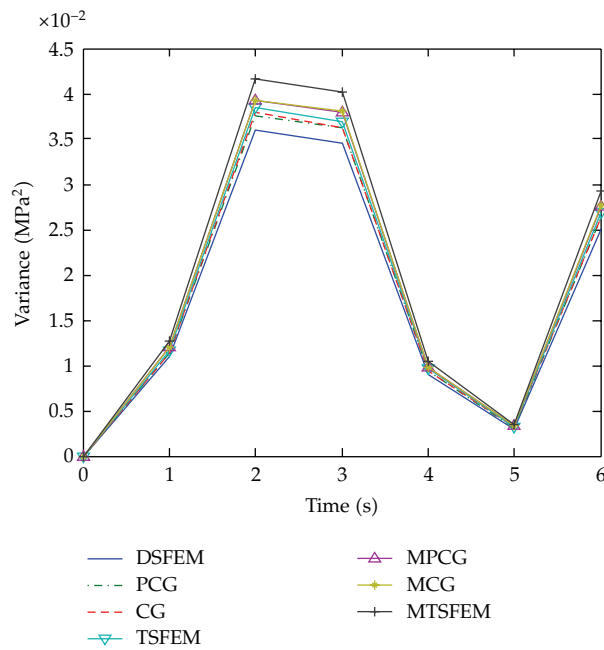**Figure 4:** The mean of horizontal stress at node 5.



**Figure 5:** The variance of horizontal stress at node 5.

**Table 2:** Comparison of CPU time.

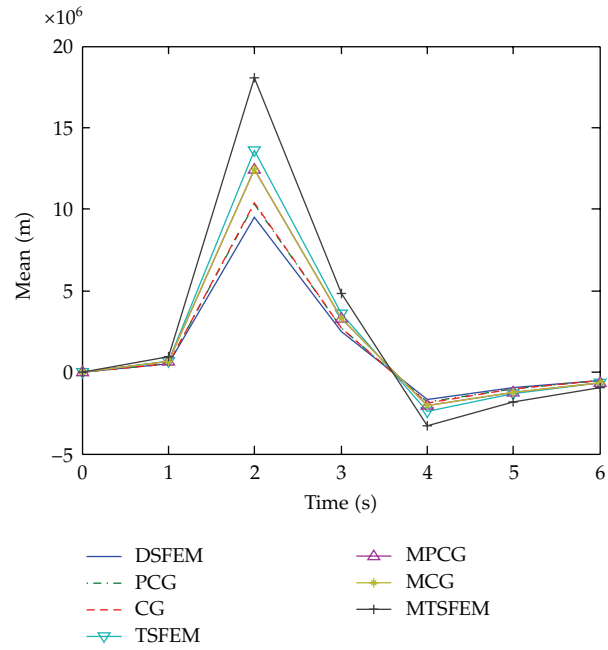| | DSFEM | PCG | CG | TSFEM | MPCG | MCG | MTSFEM |
|---|---|---|---|---|---|---|---|
| CPU time | 4 h 32 m 47 s | 1 h 15 m 43 s | 1 h 47 m 36 s | 10 h 45 m 34 s | 1 h 7 m 23 s | 1 h 36 m 31 s | 4 h 12 m 17 s |

h: hour; m: minute; s: second.

**Figure 6:** The mean of vertical displacement at node 505 for larger covariances.
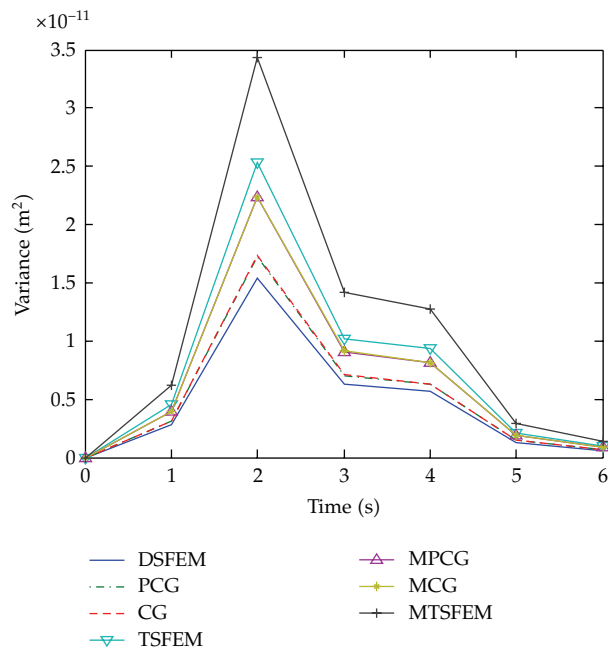


**Figure 7:** The variance of vertical displacement at node 505 for larger covariances.
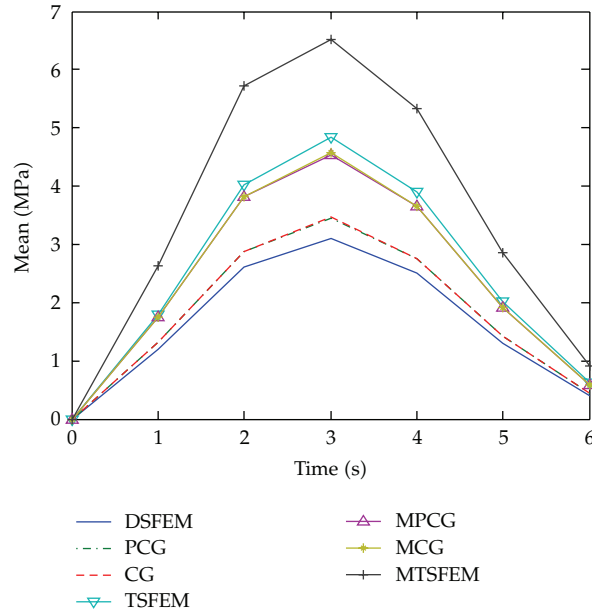
**Figure 8:** The mean of horizontal stress at node 5 for larger covariances.

**Table 3:** Comparison of error for larger covariances.

|  | The mean of vertical displacement at node 505 | The variance of vertical displacement at node 505 | The mean of horizontal stress at node 5 | The variance of horizontal stress at node 5 |
| --- | --- | --- | --- | --- |
| PCG | 9.27% | 12.15% | 11.43% | 14.16% |
| CG | 9.53% | 12.37% | 11.67% | 14.41% |
| TSFEM | 43.19% | 64.28% | 56.42% | 76.39% |
| MPCG | 31.42% | 45.13% | 47.16% | 63.17% |
| MCG | 31.75% | 45.41% | 47.43% | 63.42% |
| MTSFEM | 94.27% | 125.74% | 125.41% | 162.43% |

The partial derivative of $\dot{\delta}_{t+\triangle t}$ with respect to $a_i$ is given by

$$\frac{\partial\{\dot{\delta}_{t+\triangle t}\}}{\partial a_i} = \frac{\partial\{\dot{\delta}_t\}}{\partial a_i} + b_6\frac{\partial\{\ddot{\delta}_t\}}{\partial a_i} + b_7\frac{\partial\{\ddot{\delta}_{t+\triangle t}\}}{\partial a_i}. \tag{5.8}$$

The partial derivative of (5.7) with respect to $a_j$ is given by

$$\frac{\partial^2\{\ddot{\delta}_{t+\Delta t}\}}{\partial a_i\partial a_j} = b_0\left(\frac{\partial^2\{\delta_{t+\Delta t}\}}{\partial a_i\partial a_j} - \frac{\partial^2\{\delta_t\}}{\partial a_i\partial a_j}\right) - b_2\frac{\partial^2\{\dot{\delta}_t\}}{\partial a_i\partial a_j} - b_3\frac{\partial^2\{\ddot{\delta}_t\}}{\partial a_i\partial a_j}. \tag{5.9}$$
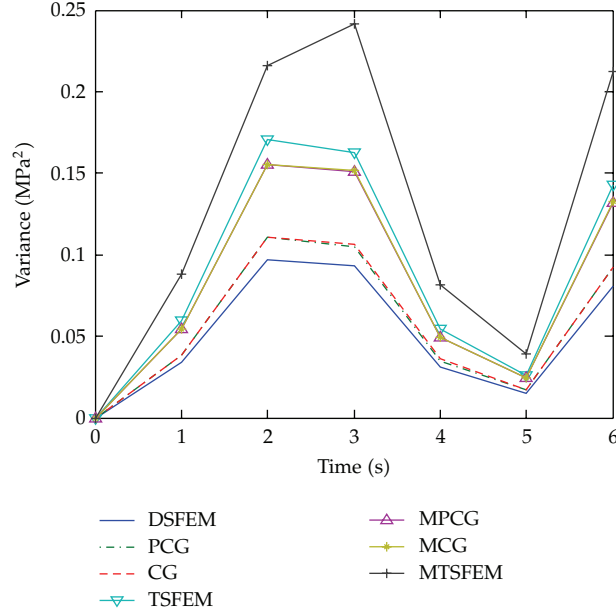
**Figure 9:** The variance of horizontal stress at node 5 for larger covariances.

The partial derivative of (5.8) with respect to $a_j$ is given by

$$\frac{\partial^2\{\dot{\delta}_{t+\Delta t}\}}{\partial a_i \partial a_j} = \frac{\partial^2\{\dot{\delta}_t\}}{\partial a_i \partial a_j} + b_6 \frac{\partial^2\{\ddot{\delta}_t\}}{\partial a_i \partial a_j} + b_7 \frac{\partial^2\{\ddot{\delta}_{t+\Delta t}\}}{\partial a_i \partial a_j}. \tag{5.10}$$

Equations (5.7), (5.8), (5.9) and (5.10), must be calculated for the following iteration.

Then, the mean and variance of displacement are obtained at time $t + i_1 \Delta t$ ($i_1 = 2, 3, \ldots, n_1$) step by step.

The partial derivative of (4.8) with respect to $a_i$ is given by

$$\frac{\partial\{\sigma\}}{\partial a_i} = \frac{\partial[D]}{\partial a_i}[B]\{\delta_{t'}^d\} + [D]\frac{\partial[B]}{\partial a_i}\{\delta_{t'}^d\} + [D][B]\frac{\partial\{\delta_{t'}^d\}}{\partial a_i}. \tag{5.11}$$

The partial derivative of (5.11) with respect to $a_j$ is given by

$$\begin{aligned}
\frac{\partial^2\{\sigma\}}{\partial a_i \partial a_j} &= \frac{\partial^2[D]}{\partial a_i \partial a_j}[B]\{\delta_{t'}^d\} + \frac{\partial[D]}{\partial a_i}\frac{\partial[B]}{\partial a_j}\{\delta_{t'}^d\} + \frac{\partial[D]}{\partial a_i}[B]\frac{\partial\{\delta_{t'}^d\}}{\partial a_j} \\[2mm]
&\quad + \frac{\partial[D]}{\partial a_j}\frac{\partial[B]}{\partial a_i}\{\delta_{t'}^d\} + [D]\frac{\partial^2[B]}{\partial a_i \partial a_j}\{\delta_{t'}^d\} + [D]\frac{\partial[B]}{\partial a_i}\frac{\partial\{\delta_{t'}^d\}}{\partial a_j} \\[2mm]
&\quad + \frac{\partial[D]}{\partial a_j}[B]\frac{\partial\{\delta_{t'}^d\}}{\partial a_i} + [D]\frac{\partial[B]}{\partial a_j}\frac{\partial\{\delta_{t'}^d\}}{\partial a_i} + [D][B]\frac{\partial^2\{\delta_{t'}^d\}}{\partial a_i \partial a_j}.
\end{aligned} \tag{5.12}$$

The stress is expanded at mean value point $\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_i, \ldots, \bar{a}_{n_1})^T$ by means of a Taylor series. By taking the expectation operator for two sides of the above (4.8), the mean of stress is obtained as

$$\mu\{\sigma\} \approx \{\sigma\}|_{a=\bar{a}} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \{\sigma\}}{\partial a_i \partial a_j}\bigg|_{a=\bar{a}} \mathrm{Cov}(a_i, a_j), \tag{5.13}$$

where $\mu\{\sigma\}$ expresses the mean of $\sigma$ and $\mathrm{Cov}(a_i, a_j)$ is the covariance between $a_i$ and $a_j$. The variance of $\sigma$ is given by

$$\mathrm{Var}\{\sigma\} \approx \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial\{\sigma\}}{\partial a_i}\bigg|_{a=\bar{a}} \cdot \frac{\partial\{\sigma\}}{\partial a_j}\bigg|_{a=\bar{a}} \cdot \mathrm{Cov}(a_i, a_j). \tag{5.14}$$

## 6. Numerical Example

Figure 1 shows a cantilever beam. The length is 1 m, the width is 0.2 m, and the height is 0.05 m. The load subjected to the cantilever beam is $100\sin(100t)$ N. Its material is the concrete. It is divided into 400 rectangle elements that have 505 nodes and 400 midpoints. Young's modulus is regarded as a stochastic process. For numerical calculation, the means of Young's modulus at each node and the midpoint within an element are $c_1(1.0 + \theta_1 x_i'/L)$. Horizontal coordinates of each node and the midpoint within an element are $x_i'$. The covariance of Young's modulus between any two nodes, between two midpoints and between each node and each midpoint are $c_2(1.0 + \theta_2 x_i/l)$. $c_1, c_2, \theta_1, \theta_2, l, L$ are constants. The distances between any two nodes, between two midpoints, and between each node and each midpoint are $x_i$. Figure 2 shows the mean of vertical displacement at node 505. the DSFEM simulates 100 samples. It is common knowledge that The DSFEM approaches the accurate solution gradually with the increase of the number of simulations. The DSFEM uses the Cholesky decomposition to solve linear equations and provides the reference solution. Figure 3 shows the variance of vertical displacement at node 505. Figure 4 shows the mean of horizontal stress at node 5. Figure 5 shows the variance of horizontal stress at node 5. Table 1 shows results obtained from the PCG, the CG, the TSFEM, the MPCG, the MCG and the MTSFEM compare with those of the DSFEM within six seconds. The PCG adopts the preconditioned Conjugate Gradient method to solve linear equations. The errors of the PCG, the CG and the TSFEM are smaller than those of the MPCG, the MCG, and the MTSFEM. The maximum error is obtained by the MTSFEM. The minimum error is produced by the PCG. Table 2 compares the CPU times of the above-mentioned methods when the cantilever beam has vibrated for six seconds. The PCG requires the least amount of CPU time. The MTSFEM requires the greatest amount of CPU time.

In order to test accuracy and computational efficiency of the above-mentioned methods, larger covariances of Young's modulus are selected. Figure 6 shows the mean of vertical displacement at node 505. Figure 7 shows the variance of vertical displacement at node 505. Figure 8 shows the mean of horizontal stress at node 5. Figure 9 shows the variance of horizontal stress at node 5. Table 3 indicates the errors of the above-mentioned methods compare to results of the DSFEM. The results produced by the PCG and the CG are close to those produced by the DSFEM. The TSFEM and the MTSFEM cannot achieve satisfactory results.

## 7. Conclusion

In this paper, improved midpoint method has the advantage of high accuracy. It can be conveniently applied to DSFEM, PSFEM, NSFEM, and CG. The mechanical vibration in a linear system is investigated by using the Taylor expansion. When Young's modulus is assumed to be a stochastic process, different samples of random variables are simulated. The combination of the CG method and Monte Carlo method makes this an effective method for analyzing a large vibration problem with the characteristics of high accuracy and quick convergence.

## Appendices

## A. Covariance of Young's Modulus between Two Elements

The covariance of Young's modulus between one element containing $m$ nodes and another element containing $n$ nodes is obtained by

$$
\begin{aligned}
\mathrm{Cov}\left(a_e, a_f\right) &= \mathrm{Cov}\left(\frac{a_{em1} + a_{em2} + \cdots + a_{emm} + a_{eml}}{m+1}, \frac{a_{fn1} + a_{fn2} + \cdots + a_{fnn} + a_{fnl}}{m+1}\right) \\
&= \frac{1}{(m+1)(n+1)}\left[\mathrm{Cov}\left(a_{em1}, a_{fn1} + a_{fn2} + \cdots a_{fnn} + a_{fnl}\right)\right. \\
&\qquad\left. + \mathrm{Cov}\left(a_{em2} + \cdots + a_{emm} + a_{eml}, a_{fn1} + a_{fn2} + \cdots a_{fnn} + a_{fnl}\right)\right] \\
&= \frac{1}{(m+1)(n+1)}\left[\mathrm{Cov}\left(a_{em1}, a_{fn1}\right) + \mathrm{Cov}\left(a_{em1}, a_{fn2}\right)\right. \\
&\qquad\left. + \cdots + \mathrm{Cov}\left(a_{em1}, a_{fnn}\right) + \mathrm{Cov}\left(a_{em1}, a_{fnl}\right)\right] \\
&\quad + \frac{1}{(m+1)(n+1)}\mathrm{Cov}\left(a_{em2} + \cdots + a_{emm} + a_{eml}, a_{fn1} + a_{fn2} + \cdots + a_{fnn} + a_{fnl}\right) \\
&= \frac{1}{(m+1)(n+1)}\left(\sum_{g_1=1}^{m}\sum_{g_3=1}^{n}\mathrm{Cov}\left(a_{emg_1}, a_{fng_3}\right)\right) \\
&\quad + \frac{1}{(m+1)(n+1)}\left(\sum_{g_1=1}^{m}\mathrm{Cov}\left(a_{emg_1}, a_{fnl}\right)\right) \\
&\quad + \frac{1}{(m+1)(n+1)}\left(\sum_{g_3=1}^{n}\mathrm{Cov}\left(a_{eml}, a_{fng_3}\right)\right) \\
&\quad + \frac{1}{(m+1)(n+1)}\mathrm{Cov}\left(a_{eml}, a_{fnl}\right),
\end{aligned}
$$

$$\tag{A.1}$$

where $\mathrm{Cov}\left(a_{emg_1}, a_{fng_3}\right)$ = the covariance of Young's modulus between node $g_1$ ($g_1 = 1, 2, \ldots, m$) of element $e$ and node $g_3$ ($g_3 = 1, 2, \ldots, n$) of element $f$, $\mathrm{Cov}\left(a_{emg_1}, a_{fnl}\right)$ = the

covariance of Young's modulus between node $g_1$ of element $e$ and the midpoint of element $f$, $\text{Cov}(a_{eml}, a_{fng_3})$ = the covariance of Young's modulus between the midpoint of element $e$ and node $g_3$ of element $f$, and $\text{Cov}(a_{eml}, a_{fnl})$ = the covariance of Young's modulus between the midpoint of element $e$ and the midpoint of element $f$.

The covariance of Young's modulus between two elements that each has $n$ nodes is given by

$$
\begin{aligned}
\text{Cov}(a_f, a_{f'}) = {} & \frac{1}{(n+1)^2}\left(\sum_{g_3=1}^{n}\sum_{g_4=1}^{n}\text{Cov}\left(a_{fng_3}, a_{f'ng_4}\right)\right) \\
& + \frac{1}{(n+1)^2}\left(\sum_{g_3=1}^{n}\text{Cov}\left(a_{fng_3}, a_{f'nl}\right)\right) \\
& + \frac{1}{(n+1)^2}\left(\sum_{g_4=1}^{n}\text{Cov}\left(a_{fnl}, a_{f'ng_4}\right) + Cov\left(a_{fnl}, a_{f'nl}\right)\right),
\end{aligned}
\tag{A.2}
$$

where $\text{Cov}(a_{fng_3}, a_{f'ng_4})$ = the covariance of Young's modulus between node $g_3$ ($g_3 = 1, 2, \ldots, n$) of element $f$ and node $g_4$ ($g_4 = 1, 2, \ldots, n$) of element $f'$, $\text{Cov}(a_{fng_3}, a_{f'nl})$ = the covariance of Young's modulus between node $g_3$ of element $f$ and the midpoint of element $f'$, $\text{Cov}(a_{fnl}, a_{f'ng_4})$ = the covariance of Young's modulus between the midpoint of element $f$ and node $g_4$ of element $f'$, and $\text{Cov}(a_{fnl}, a_{f'nl})$ = the covariance of Young's modulus between the midpoint of element $f$ and the midpoint of element $f'$.

## B. Newmark Method

For ease of programming, the comprehensive calculation steps of the Newmark method are as follows.

In the initial calculation the matrices $[K]$, $[M]$, and $[C]$ are formed. The initial values $\{\delta_t\}$, $\{\dot{\delta}_t\}$, $\{\ddot{\delta}_t\}$ are given. After selecting step $\Delta t$ and parameters $\gamma$, $\beta$, the following relevant parameters are calculated:

$$
\begin{aligned}
& \gamma \geq 0.50, \quad \beta \geq 0.25(0.5 + \gamma)^2, \\
& b_0 = \frac{1}{\beta(\Delta t)^2}, \quad b_1 = \frac{\gamma}{\beta\Delta t}, \quad b_2 = \frac{1}{\beta\Delta t}, \\
& b_3 = \frac{1}{2\beta} - 1, \quad b_4 = \frac{\gamma}{\beta} - 1, \quad b_5 = \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right), \\
& b_6 = \Delta t(1 - \gamma), \quad b_7 = \gamma\Delta t.
\end{aligned}
\tag{B.1}
$$

The stiffness matrix is defined as

$$
\left[\tilde{K}\right] = [K] + b_0[M] + b_1[C].
\tag{B.2}
$$

The stiffness matrix inversion $[\tilde{K}]^{-1}$ is solved.

Calculation of each step time At time $t + \triangle t$, the load vector is defined as

$$
\begin{aligned}
\left\{\tilde{F}_{t+\Delta t}\right\} &= \{F_{t+\Delta t}\} + [M]\left(b_0\{\delta_t\} + b_2\{\dot{\delta}_t\} + b_3\{\ddot{\delta}_t\}\right) \\
&\quad + [C]\left(b_1\{\delta_t\} + b_4\{\dot{\delta}_t\} + b_5\{\ddot{\delta}_t\}\right).
\end{aligned}
\tag{B.3}
$$

At time $t + \Delta t$, the displacement vector is given by

$$
\{\delta_{t+\Delta t}\} = \left[\tilde{K}\right]^{-1}\left\{\tilde{F}_{t+\Delta t}\right\}.
\tag{B.4}
$$

At time $t + \Delta t$, the velocity vector and acceleration vector are obtained as

$$
\begin{aligned}
\{\ddot{\delta}_{t+\triangle t}\} &= b_0(\{\delta_{t+\Delta t}\} - \{\delta_t\}) - b_2\{\dot{\delta}_t\} - b_3\{\ddot{\delta}_t\}, \\
\{\dot{\delta}_{t+\triangle t}\} &= \{\dot{\delta}_t\} + b_6\{\ddot{\delta}_t\} + b_7\{\ddot{\delta}_{t+\triangle t}\}.
\end{aligned}
\tag{B.5}
$$

Vectors $\{\delta_{t+i_1\Delta t}\}$, $\{\dot{\delta}_{t+i_1\triangle t}\}$, and $\{\ddot{\delta}_{t+i_1\triangle t}\}$ are solved at time $t + i_1\Delta t$ $(i_1 = 2, 3, \ldots, n_1)$ step by step.

## References

[1] A. Der Kiureghian and J.-B. Ke, "The stochastic finite element method in structural reliability," *Probabilistic Engineering Mechanics*, vol. 3, no. 2, pp. 83–91, 1988.

[2] E. Vanmarcke and M. Grigoriu, "Stochastic finite element analysis of simple beams," *Journal of Engineering Mechanics*, vol. 109, no. 5, pp. 1203–1214, 1983.

[3] S. Chakraborty and B. Bhattacharyya, "An efficient 3D stochastic finite element method," *International Journal of Solids and Structures*, vol. 39, no. 9, pp. 2465–2475, 2002.

[4] W. Q. Zhu, Y. J. Ren, and W. Q. Wu, "Stochastic FEM based on local averages of random vector fields," *Journal of Engineering Mechanics*, vol. 118, no. 3, pp. 496–511, 1992.

[5] W. K. Liu, T. Belytschko, and A. Mani, "Random field finite elements," *International Journal for Numerical Methods in Engineering*, vol. 23, no. 10, pp. 1831–1845, 1986.

[6] P. D. Spanos and R. Ghanem, "Stochastic finite element expansion for random media," *Journal of Engineering Mechanics*, vol. 115, no. 5, pp. 1035–1053, 1989.

[7] G. Deodatis, "Bounds on response variability of stochastic finite element systems," *Journal of Engineering Mechanics*, vol. 116, no. 3, pp. 565–585, 1990.

[8] M. Shinozuka and G. Deodatis, "Response variability of stochastic finite element systems," *Journal of Engineering Mechanics*, vol. 114, no. 3, pp. 499–519, 1988.

[9] C.-C. Li and A. Der Kiureghian, "Optimal discretization of random fields," *Journal of Engineering Mechanics*, vol. 119, no. 6, pp. 1136–1154, 1993.

[10] J. Astill, C. J. Nosseir, and M. Shinozuka, "Impact loading on structures with random properties," *Journal of Structural Mechanics*, vol. 1, no. 1, pp. 63–67, 1972.

[11] F. Yamazaki, M. Shinozuka, and G. Dasgupta, "Neumann expansion for stochastic finite element analysis," *Journal of Engineering Mechanics*, vol. 114, no. 8, pp. 1335–1354, 1988.

[12] M. Papadrakakis and V. Papadopoulos, "Robust and efficient methods for stochastic finite element analysis using Monte Carlo simulation," *Computer Methods in Applied Mechanics and Engineering*, vol. 134, no. 3-4, pp. 325–340, 1996.

[13] X. Q. Peng, L. Geng, W. Liyan, G. R. Liu, and K. Y. Lam, "A stochastic finite element method for fatigue reliability analysis of gear teeth subjected to bending," *Computational Mechanics*, vol. 21, no. 3, pp. 253–261, 1998.

[14] K. Handa and K. Andersson, "Application of finite element methods in the statistical analysis of structures," in *Proceedings of the 3rd International Conference on Structure Safety and Reliability*, Trondheim, Norway, 1981.

[15] T. Hisada and S. Nakagiri, "Role of the stochastic finite element method in structural safety and reliability," in *Proceedings of the 4th International Conference on Structure Safety and Reliability*, Kobe, Japan, 1985.

[16] S. Mahadevan and S. Mehta, "Dynamic reliability of large frames," *Computers and Structures*, vol. 47, no. 1, pp. 57–67, 1993.

[17] Q.-L. Zhang and U. Peil, "Random finite element analysis for stochastical responses of structures," *Computers and Structures*, vol. 62, no. 4, pp. 611–616, 1997.

[18] W. K. Liu, T. Belytschko, and A. Mani, "Probabilistic finite elements for nonlinear structural dynamics," *Computer Methods in Applied Mechanics and Engineering*, vol. 56, no. 1, pp. 61–81, 1986.

[19] Z. Lei and C. Qiu, "A dynamic stochastic finite element method based on dynamic constraint mode," *Computer Methods in Applied Mechanics and Engineering*, vol. 161, no. 3-4, pp. 245–255, 1998.

[20] S. Chakraborty and S. S. Dey, "A stochastic finite element dynamic analysis of structures with uncertain parameters," *International Journal of Mechanical Sciences*, vol. 40, no. 11, pp. 1071–1087, 1998.

[21] Z. Lei and C. Qiu, "Neumann dynamic stochastic finite element method of vibration for structures with stochastic parameters to random excitation," *Computers and Structures*, vol. 77, no. 6, pp. 651–657, 2000.

[22] Z. Lei and C. Qiu, "A stochastic variational formulation for nonlinear dynamic analysis of structure," *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 5–7, pp. 597–608, 2000.