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Research Article

On the Polyconvolution with the Weight Function for the Fourier Cosine, Fourier Sine, and the Kontorovich-Lebedev Integral Transforms

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The polyconvolution with the weight function γ of three functions f, g, and h for the integral transforms Fourier sine (F_s), Fourier cosine (F_c), and Kontorovich-Lebedev (K_{iy}), which is denoted by $\overset{\gamma}{*}(f,g,h)(x)$, has been constructed. This polyconvolution satisfies the following factorization property $Fc(\overset{\gamma}{*}(f,g,h))(y) = \sin y(F_s f)(y)\cdot(F_c g)(y)\cdot(K_{iy}h)(y)$, for all $y > 0$. The relation of this polyconvolution to the Fourier convolution and the Fourier cosine convolution has been obtained. Also, the relations between the polyconvolution product and others convolution product have been established. In application, we consider a class of integral equations with Toeplitz plus Hankel kernel whose solution in closed form can be obtained with the help of the new polyconvolution. An application on solving systems of integral equations is also obtained.

1. Introduction

The convolution of two functions f and g for the Fourier transform is well known [1]

$$\left(f \underset{F}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}. \tag{1.1}$$

This convolution has the factorization equality as below

$$F\left(f \underset{F}{*} g\right)(y) = (Ff)(y)(Fg)(y), \quad \forall y \in \mathbb{R}, \tag{1.2}$$

where F denotes the Fourier transform

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy} dx. \tag{1.3}$$

The convolution of f and g for the Kontorovich-Lebedev integral transform has been studied in [2]

$$\left(f \underset{K-L}{*} g\right)(x) = \frac{1}{2x} \iint_{0}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{xu}{v} + \frac{xv}{u} + \frac{uv}{x}\right)\right] f(u) g(v) du\, dv, \quad x > 0, \tag{1.4}$$

for which the following factorization identity holds:

$$K_{iy}\left(f \underset{K-L}{*} g\right) = (K_{iy} f) \cdot (K_{iy} g), \quad \forall y > 0. \tag{1.5}$$

Here K_{iy} is the Kontorovich-Lebedev transform [3]

$$K_{ix}[f] = \int_{0}^{\infty} K_{ix}(t) f(t) dt, \tag{1.6}$$

and $K_{ix}(t)$ is the Macdonald function [4].

The convolution of two functions f and g for the Fourier cosine is of the form [1]

$$\left(f \underset{1}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(y) [g(|x - y|) + g(x + y)] dy, \quad x > 0, \tag{1.7}$$

which satisfied the following factorization equality:

$$F_c\left(f \underset{1}{*} g\right)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0. \tag{1.8}$$

Here the Fourier cosine transform is of the form

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos yx \cdot f(x) dx, \quad y > 0. \tag{1.9}$$

The convolution with a weight function $\gamma(x) = \sin x$ of two functions f and g for the Fourier sine transform has been introduced in [5, 6]

$$\begin{aligned}\left(f \overset{\gamma}{*} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(y) [&\operatorname{sign}(x + y - 1) g(|x + y - 1|) + \operatorname{sign}(x - y + 1) g(|x - y + 1|) \\ &- g(x + y + 1) - \operatorname{sign}(x - y - 1) g(|x - y - 1|)] dy, \quad x > 0,\end{aligned} \tag{1.10}$$

and the following factorization identity holds:

$$F_s\left(f \overset{\gamma}{*} g\right)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y > 0. \tag{1.11}$$

Here the Fourier sine is of the form

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin yx \cdot f(x) dx, \quad y > 0. \tag{1.12}$$

The generalized convolution of two functions f and g for the Fourier sine and Fourier cosine transforms has been studied in [1]

$$\left(f \underset{2}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)\left[g(|x-u|) - g(x+u)\right] du, \quad x > 0, \tag{1.13}$$

and proved the following factorization identity [1]:

$$F_s\left(f \underset{2}{*} g\right)(y) = (F_s f)(y) \cdot (F_c g)(y), \quad \forall y > 0. \tag{1.14}$$

The generalized convolution of two functions f and g for the Fourier cosine and the Fourier sine transforms is defined by [7]

$$\left(f \underset{3}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)\left[\operatorname{sign}(u-x) g(|u-x|) + g(u+x)\right] du, \quad x > 0. \tag{1.15}$$

For this generalized convolution, the following factorization equality holds:

$$F_c\left(f \underset{3}{*} g\right)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0. \tag{1.16}$$

The generalized convolution with the weight function $\gamma(x) = \sin x$ for the Fourier cosine and the Fourier sine transforms of f and g has been introduced in [8]

$$\begin{aligned}\left(f \overset{\gamma}{\underset{1}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)\big[&g(|x+u-1|) + g(|x-u+1|) \\ &-g(x+u+1) - g(|x-u-1|)\big] du, \quad x > 0.\end{aligned} \tag{1.17}$$

It satisfies the factorization property

$$F_c\left(f \overset{\gamma}{\underset{1}{*}} g\right)(y) = \sin y (F_s f)(y)(F_c g)(y), \quad \forall y > 0. \tag{1.18}$$

The generalized convolution with the weight function $\gamma(x) = \sin x$ of f and g for the Fourier sine and Fourier cosine has been studied in [9]

$$\left(f \overset{\gamma}{\underset{2}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)\left[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u+1|)\right] du, \quad x > 0, \tag{1.19}$$

and satisfies the following factorization identity:

$$F_s\left(f \overset{\gamma}{\underset{2}{*}} g\right)(y) = \sin y (F_c f)(y)(F_c g)(y), \quad \forall y > 0. \tag{1.20}$$

Recently, the following generalized convolutions for Fourier cosine, Kontorovich-Lebedev and Fourier sine, Kontorovich-Lebedev are studied in [10] (f. 21)

$$(f * g)_{\{{c \atop s}\}}(x) = \frac{1}{2\pi x} \int_{\mathbb{R}_+^2} f(u) g(v) \left[e^{-x\cosh(u-v)} \pm e^{-x\cosh(u+v)}\right] du\, dv, \quad x > 0. \tag{1.21}$$

The respective factorization equalities are [10]

$$\left(F_{\{{c \atop s}\}}(f * g)_{\{{c \atop s}\}}\right)(y) = \left(F_{\{{c \atop s}\}} f\right)(y) K_{iy}[g], \quad x > 0,\ f \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+),\ g \in L_p^{0,\beta}, p > 1, \tag{1.22}$$

where

$$L_p^{0,\beta} = \left\{ f : \int_0^\infty |f(t)|^p K_0(\beta t) dt < \infty,\ 0 < \beta \leqslant 1 \right\}. \tag{1.23}$$

In 1997, Kakichev introduced a constructive method for defining a polyconvolution with a weight function γ of functions $f_1, f_2, \ldots, f_n$ for the integral transforms $K, K_1, K_2, \ldots, K_n$, which are denoted by $\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)(x)$, such that the following factorization property holds [11]:

$$K\left[\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)\right](y) = \gamma(y) \prod_{i=1}^{n} (K_i f_i)(y), \quad n \geqslant 3. \tag{1.24}$$

Polyconvolutions for the Hilbert, Stieltjes, Fourier cosine, and Fourier sine integral transforms have been studied in [12].

The polyconvolution of f, g, and h for the Fourier cosine and the Fourier sine transforms has the form [13]

$$*(f, g, h)(x) = \frac{1}{2\pi} \iint_0^\infty f(u) g(v) [h(|x+u-v|) + h(x-u+v\ |) - h(|x-u-v|) - h(x+u+v)] du\, dv, \quad x > 0, \tag{1.25}$$

which satisfies the following factorization property:

$$F_c(*(f,g,h))(y) = (F_s f)(y)\cdot(F_s g)(y)\cdot(F_c h)(y), \quad \forall y>0. \tag{1.26}$$

In recent years, many sciences were interested in the theory of convolution for the integral transforms and gave several interesting application (see [3, 14–21]), specially, the integral equations with the Toeplitz plus Hankel kernel [22–24]

$$f(x) + \int_0^\infty [k_1(x+y) + k_2(x-y)] f(y)\,dy = g(x), \quad x>0, \tag{1.27}$$

where k_1, k_2, and g are known functions, and f is an unknown function. Many partial cases of this equation can be solved in closed form with the help of the convolutions and generalized convolutions. In this paper, we construct and investigate the polyconvolution for the Fourier sine, Fourier cosine, and the Kontorovich-Lebedev transforms. Several properties of this new polyconvolution and its application on solving integral equation with Toeplitz plus Hankel equation and systems of integral equations are obtained.

2. Polyconvolution

Definition 2.1. The polyconvolution with the weight function $\gamma = \sin x$ of functions f, g, and h for the Fourier cosine, Fourier sine, and the Kontorovich-Lebedev integral transforms is defined as follows:

$$\overset{\gamma}{*}(f,g,h)(x) = \iiint_0^\infty \theta(x,u,v,w) f(u) g(v) h(w)\,du\,dv\,dw, \tag{2.1}$$

where

$$\begin{aligned}\theta(x,u,v,w) = \frac{1}{4\sqrt{2\pi}}\Big[&e^{-w\cosh(x+v+u-1)} + e^{-w\cosh(x+v-u+1)} + e^{-w\cosh(x-v+u-1)} + e^{-w\cosh(x-v-u+1)}\\ &-e^{-w\cosh(x+v+u+1)} - e^{-w\cosh(x+v-u-1)} - e^{-w\cosh(x-v+u+1)} - e^{-w\cosh(x-v-u-1)}\Big].\end{aligned} \tag{2.2}$$

Theorem 2.2. *Let f and g be functions in $L_1(\mathbb{R}_+)$, and let h be a function in $L_1(1/\sqrt{w},\mathbb{R}_+)$; then the polyconvolution* (2.1) *belongs to $L_1(\mathbb{R}_+)$ and satisfies the following factorization equality:*

$$F_c\Big(\overset{\gamma}{*}(f,g,h)\Big)(y) = \sin y (F_s f)(y)\cdot(F_c g)(y)\cdot(K_{iy} h), \quad \forall y>0. \tag{2.3}$$

Proof. Since $|e^{-w\cosh(x+u+v-1)} - e^{-w\cosh(x+u+v-1)}| \leqslant 1/\sqrt{w}$ for sufficient large $w > 0$, we have

$$\begin{aligned}\left|\overset{\gamma}{*}(f,g,h)(x)\right| &\leqslant \frac{1}{4\sqrt{2\pi}}\iiint_0^\infty |f(u)||g(v)||h(w)||\theta(x,u,v,w)|du\,dv\,dw \\ &\leqslant \frac{1}{\sqrt{2\pi}}\int_0^\infty |f(u)|du\cdot\int_0^\infty |g(v)|dv\cdot\int_0^\infty \frac{1}{\sqrt{w}}|h(w)|dw < +\infty.\end{aligned} \tag{2.4}$$

On the other hand, note that $\cosh(x+u+v-1) \geqslant (x+u+v-1)^2/2$; we have

$$e^{-w\cosh(x+u+v-1)} \leqslant e^{-w((x+u+v-1)^2/2)}, \quad \forall w > 0. \tag{2.5}$$

Using formula 3.321.3, page 321, in [4], we have

$$\begin{aligned}\int_0^\infty e^{-w\cosh(x+u+v-1)}dx &\leqslant \sqrt{\frac{2}{w}}\int_0^\infty e^{-(\sqrt{w/2}(x+u+v-1))^2} d\left(\sqrt{\frac{w}{2}}(x+u+v-1)\right) \\ &\leqslant 2\sqrt{\frac{2}{w}}\int_0^\infty e^{-s^2}ds = \sqrt{\frac{2\pi}{w}}.\end{aligned} \tag{2.6}$$

It shows that

$$\begin{aligned}&\iiiint_0^\infty e^{-w\cosh(x+u+v-1)}|f(u)||g(v)||h(w)|du\,dv\,dw\,dx \\ &\quad\leqslant \iiint_0^\infty \sqrt{\frac{2\pi}{w}}|h(w)||f(u)||g(v)|du\,dv\,dw \\ &\quad\leqslant \sqrt{2\pi}\int_0^\infty \frac{1}{\sqrt{w}}|h(w)|dw\cdot\int_0^\infty |f(u)|du\cdot\int_0^\infty |g(v)|dv < +\infty.\end{aligned} \tag{2.7}$$

By the same way, we obtain similar estimations for the 7 other terms. Therefore, from formulas (2.1), (2.2), and (2.7), we have

$$\int_0^\infty \left|\overset{\gamma}{*}(f,g,h)(x)\right|dx < +\infty. \tag{2.8}$$

It shows that the polyconvolution (2.1) belongs to $L_1(\mathbb{R}_+)$. We now prove the factorization equality (2.3). Indeed, we have

$$\begin{aligned}&\sin y\ (F_s f)(y)(F_c g)(y)(K_{iy}h) \\ &\quad= \frac{2}{\pi}\iiint_0^\infty \sin y\sin(yu)\cos(yv)K_{iy}(w)f(u)g(v)h(w)du\,dv\,dw.\end{aligned} \tag{2.9}$$

Using formula 2, page 130 in [4], we get

$$\begin{aligned}
&\sin y(F_s f)(y)(F_c g)(y)(K_{iy}h)\\
&\quad= \frac{2}{\pi}\iiiint_0^\infty \sin y \sin(yu)\cos(yv)\cos(y\alpha)e^{-w\cosh\alpha} f(u)g(v)h(w)du\,dv\,dw\,d\alpha\\
&\quad= \frac{1}{4\pi}\iiiint_0^\infty e^{-w\cosh\alpha}\left[\cos y(u-1+v+\alpha)+\cos y(u-1-v-\alpha)+\cos y(u-1+v-\alpha)\right.\\
&\qquad\qquad +\cos y(u-1-v+\alpha)-\cos y(u+1+v+\alpha)-\cos y(u+1-v-\alpha)\\
&\qquad\qquad \left.-\cos y(u+1+v-\alpha)-\cos y(u+1-v+\alpha)\right]\\
&\qquad \times f(u)g(v)h(w)du\,dv\,dw\,d\alpha.
\end{aligned} \tag{2.10}$$

Interchanging variables, we have

$$\begin{aligned}
&\int_0^\infty e^{-w\cosh\alpha}\left[\cos y(u-1+v+\alpha)-\cos y(u+1+v+\alpha)\right]d\alpha\\
&\qquad= \int_0^\infty \cos yx\left[e^{-w\cosh(x-u+1-v)}-e^{-w\cosh(x-u-1-v)}\right]dx.
\end{aligned} \tag{2.11}$$

Similarly,

$$\begin{aligned}
&\int_0^\infty e^{-w\cosh\alpha}\left[\cos y(u-1-v+\alpha)-\cos y(u+1-v+\alpha)\right]d\alpha\\
&\qquad= \int_0^\infty \cos yx\left[e^{-w\cosh(x-u+1+v)}-e^{-w\cosh(x-u-1+v)}\right]dx;\\
&\int_0^\infty e^{-w\cosh\alpha}\left[\cos y(u-1-v-\alpha)-\cos y(u+1-v-\alpha)\right]d\alpha\\
&\qquad= \int_0^\infty \cos yx\left[e^{-w\cosh(x+u-1-v)}-e^{-w\cosh(x+u+1-v)}\right]dx;\\
&\int_0^\infty e^{-w\cosh\alpha}\left[\cos y(u-1+v-\alpha)-\cos y(u+1+v-\alpha)\right]d\alpha\\
&\qquad= \int_0^\infty \cos yx\left[e^{-w\cosh(x+u-1+v)}-e^{-w\cosh(x+u+1+v)}\right]dx.
\end{aligned} \tag{2.12}$$

From fomulae (2.10)–(2.8), we have

$$\sin y(F_s f)(y)(F_c g)(y)(K_{iy}h) = F_c\left(\overset{\gamma}{*}(f,g,h)\right)(y). \tag{2.13}$$

The proof is complete. □

Definition 2.3. Let f be a function in $L_1(\mathbb{R}_+)$ and let h be a function in $L_1(\beta,\mathbb{R}_+)$; their norms are defined as follows:

$$\|f\|_{L_1(\mathbb{R}_+)} = \int_0^\infty |f(x)|dx, \qquad \|h\|_{L_1(\beta,\mathbb{R}_+)} = \int_0^\infty \beta(v)|h(v)|dv. \tag{2.14}$$

here $\beta(v) = 2/\sqrt{v}$.

Theorem 2.4. *Let f and g be functions in $L_1(\mathbb{R}_+)$, and let h be function in $L_1(\beta,\mathbb{R}_+)$; then the following estimation holds:*

$$\left\|\overset{\gamma}{*}(f,g,h)\right\|_{L_1(\mathbb{R}_+)} \leqslant \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\beta,\mathbb{R}_+)}. \tag{2.15}$$

Proof. From formulas (2.1), (2.2), and (2.7), we have

$$\int\left|\overset{\gamma}{*}(f,g,h)(x)\right|dx \leqslant 2\int_0^\infty \frac{1}{\sqrt{w}}|h(w)|dw \cdot \int_0^\infty |f(u)|du \cdot \int_0^\infty |g(v)|dv. \tag{2.16}$$

Therefore, by Definition 2.3,

$$\left\|\overset{\gamma}{*}(f,g,h)\right\|_{L_1(\mathbb{R}_+)} \leqslant \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\beta,\mathbb{R}_+)}. \tag{2.17}$$

□

Proposition 2.5. *Let $f,g \in L_1(\mathbb{R}_+)$, and let $h \in L_1(1/\sqrt{w},\mathbb{R}_+)$; then the following identity holds:*

$$\begin{aligned}\overset{\gamma}{*}(f,g,h) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\int_0^\infty h(w)\Bigg[&\left(\left(g \underset{1}{*} e^{-w\cosh t}\right)\underset{F}{*}(f(|t|)\operatorname{sign} t)\right)(x+1)\\ &-\left(\left(g \underset{1}{*} e^{-w\cosh t}\right)\underset{F}{*}(f(|t|)\operatorname{sign} t)\right)(x-1)\Bigg]dw.\end{aligned} \tag{2.18}$$

Proof. From the definition (2.1) of the polyconvolution and the convolution (1.7), we have

$$\begin{aligned}&\overset{\gamma}{*}(f,g,h)(x)\\ &= \frac{1}{4}\iint_0^\infty f(u)h(w)\Bigg[\left(g \underset{1}{*} e^{-w\cosh t}\right)(x-u+1) + \left(g \underset{1}{*} e^{-w\cosh t}\right)(x+u-1)\\ &\qquad -\left(g \underset{1}{*} e^{-w\cosh t}\right)(x+u+1) - \left(g \underset{1}{*} e^{-w\cosh t}\right)(x-u-1)\Bigg]du\,dw.\end{aligned} \tag{2.19}$$

From (2.19) and calculation, we obtain

$$\underset{1}{*}(f,g,h)(x) = \sqrt{\frac{\pi}{2}}\int_0^\infty h(w)\Bigg[\Bigg(\Big(g \underset{1}{*} e^{-w\cosh t}\Big) \underset{F}{*} f(|t|)\,\text{sign}\ t\Bigg)(x+1) - \Bigg(\Big(g \underset{1}{*} e^{-w\cosh t}\Big) \underset{F}{*} f(|t|)\,\text{sign}\ t\Bigg)(x-1)\Bigg]dw. \tag{2.20}$$

The proof is complete. □

Theorem 2.6. *Let* f, g, h *be functions in* $L_1(\mathbb{R}_+), \gamma(x) = \sin x$, *and let* l *and* k *be functions in* $L(1/\sqrt{w}, \mathbb{R})$; *then the following properties holds:*

(a) $\overset{\gamma}{*}(f, \overset{\gamma}{*}(g,h,k), l) = \overset{\gamma}{*}(g, \overset{\gamma}{*}(f,h,k), l)$;

(b) $\overset{\gamma}{*}(f \underset{2}{*} g, h, k) = \overset{\gamma}{*}(f, g \underset{1}{*} h, k)$;

(c) $\overset{\gamma}{*}(f \overset{\gamma}{*} g, h, k) = \overset{\gamma}{*}(f, g \overset{\gamma}{\underset{1}{*}} h, k)$;

(d) $\overset{\gamma}{*}(f \overset{\gamma}{\underset{2}{*}} g, h, k) = \overset{\gamma}{*}(f \overset{\gamma}{\underset{2}{*}} h, g, k)$;

(e) $\overset{\gamma}{*}(f, g \underset{3}{*} h, k) = \overset{\gamma}{*}(g, f \underset{3}{*} h, k)$.

Proof. First, we prove the assertion (c). From Theorem 2.2 and the convolutions (1.17), (1.10), we have

$$\begin{aligned} F_c\left(\overset{\gamma}{*}\left(f \overset{\gamma}{*} g, h, k\right)\right)(y) &= \sin y F_s\left(f \overset{\gamma}{*} g\right)(y) \cdot (F_c h)(y)(K_{iy}h) \\ &= \sin y \sin y (F_s f)(y)(F_s g)(y)(F_c h)(y)(K_{iy}h) \\ &= \sin y \cdot (F_s f)(y) \cdot F_c\left(g \overset{\gamma}{\underset{1}{*}} h\right)(y) \cdot (K_{iy}h) \\ &= F_c\left(\overset{\gamma}{*}\left(f, g \overset{\gamma}{\underset{1}{*}} h, k\right)\right). \end{aligned} \tag{2.21}$$

Therefore, the part (c) holds. Other parts can be proved in a similar way. □

3. Applications in Solving Integral Equations and Systems of Integral Equations

Consider the integral equation

$$f(x) + \iiint_0^\infty \theta(x,u,v,w) g(u) f(v) h(w)\, du\, dv\, dw + \int_0^\infty \theta_1(x,u) f(u) du + \int_0^\infty \theta_2(x,u) f(u) du = \varphi(x), \quad x > 0, \tag{3.1}$$

where g, h, k, l, and φ are known functions, f is an unknown function, $\theta(x, u, v, w)$ is given by the formula (2.2), and

$$\theta_1(x,u) = \frac{1}{2\sqrt{2\pi}}\big[k(x+u+1) - k(|x+u-1|)\operatorname{sign}(x+u-1) + k(|x-u+1|)\operatorname{sign}(x-u+1) - k(|x-u-1|)\operatorname{sign}(x-u-1)\big],$$
$$\theta_2(x,u) = \frac{1}{\sqrt{2\pi}}[l(x+u) + l(|x-u|)]. \quad (3.2)$$

Theorem 3.1. *Suppose that* $g, l, \varphi, k_1, k_2 \in L_1(\mathbb{R}_+)$, $h \in L_1(1/\sqrt{v}, \mathbb{R}_+)$, $k = k_1 \underset{2}{*} k_2$ *such that*

$$1 + \sin y (F_s g)(y)(K_{iy}h) + (F_s k_1)(y)(F_c k_2)(y) + (F_c l)(y) \neq 0, \quad (3.3)$$

then (3.1) *has a unique solution in* $L_1(\mathbb{R}_+)$ *whose closed form is*

$$f(x) = \varphi(x) - \left(\varphi \underset{1}{*} \xi\right)(x). \quad (3.4)$$

Here $\xi \in L_1(\mathbb{R}_+)$ *is defined uniquely by*

$$(F_c\xi)(y) = \frac{\sin y (F_s g)(y)(K_{iy}h) + \sin y (F_s k_1)(y)(F_c k_2)(y) + (F_c l)(y)}{1 + \sin y (F_s g)(y)(K_{iy}h) + \sin y (F_s k_1)(y)(F_c k_2)(y) + (F_c l)(y)}. \quad (3.5)$$

Proof. We obtain the following lemmas.

Lemma 3.2. *For* $f, k \in L_1(\mathbb{R}_+)$, *then the following operator also belongs to* $L_1(\mathbb{R}_+)$

$$\int_0^\infty f(u)\theta_1(x,u)\,du. \quad (3.6)$$

Moreover, the following factorization equality holds:

$$F_c\left(\int_0^\infty f(u)\theta_1(x,u)\,du\right)(y) = \sin y \cdot (F_s k)(y)(F_c f)(y), \quad \forall y > 0. \quad (3.7)$$

Lemma 3.3. *Let* $g \in L_1(\mathbb{R}_+)$, $h \in L_1(1/\sqrt{v}, \mathbb{R}_+)$; *then the generalized convolution* $(g \overset{\gamma}{\underset{3}{*}} h)(x)$ *belongs to* $L_1(\mathbb{R}_+)$ *and the respectively factorization equality is*

$$F_c\left(g \overset{\gamma}{\underset{3}{*}} h\right)(y) = \sin y (F_s g)(y)(K_{iy}h), \quad \forall y > 0, \quad (3.8)$$

where

$$\left(g \overset{\gamma}{\underset{3}{*}} h\right)(x) = \frac{1}{4}\iint_0^\infty \left[e^{-v\cosh(x+u-1)} + e^{-v\cosh(x-u+1)} - e^{-v\cosh(x+u+1)} - e^{-v\cosh(x-u-1)}\right] \times g(u)h(v)\,du\,dv, \quad x > 0. \tag{3.9}$$

We now prove Theorem 3.1 with the help of convolution (1.7), Lemmas 1, and 2. We have

$$(F_c f)(y) + \sin y (F_s g)(y) \cdot (F_c f)(y) \cdot (K_{iy} h) + (F_s k)(y) \cdot (F_c f)(y) \sin y + (F_c l)(y) \cdot (F_c f)(y) = (F_c \varphi)(y). \tag{3.10}$$

Therefore, by the given condition,

$$(F_c f)(y) = (F_c \varphi)(y)\left(1 - \frac{\sin y (F_s g)(y)(K_{iy} h) + \sin y (F_s k)(y) + (F_c l)(y)}{1 + \sin y (F_s g)(y)(K_{iy} h) + \sin y (F_s k)(y) + (F_c l)(y)}\right). \tag{3.11}$$

By the hypothesis $k = k_1 \underset{2}{*} k_2$, we see that $\sin y (F_s k)(y) = F_c(k_1 \overset{\gamma}{\underset{1}{*}} k_2)(y)$; using Lemma 3.3, we get

$$(F_c f)(y) = (F_c \varphi)(y)\left(1 - \frac{F_c\left(g \overset{\gamma}{\underset{4}{*}} h\right)(y) + F_c\left(k_1 \overset{\gamma}{\underset{1}{*}} k_2\right)(y) + (F_c l)(y)}{1 + F_c\left(g \overset{\gamma}{\underset{4}{*}} h\right)(y) + F_c\left(k_1 \overset{\gamma}{\underset{1}{*}} k_2\right)(y) + (F_c l)(y)}\right). \tag{3.12}$$

In virtue of the Wiener-Levy theorem [25], by the given condition, there exists a function $\xi \in L_1(\mathbb{R}_+)$ such that

$$(F_c \xi)(y) = \frac{\sin y (F_s g)(y)(K_{iy} h) + \sin y (F_s k_1)(y)(F_c k_2)(y) + (F_c l)(y)}{1 + \sin y (F_s g)(y)(K_{iy} h) + \sin y (F_s k_1)(y)(F_c k_2)(y) + (F_c l)(y)}. \tag{3.13}$$

From (3.12) and (3.13), we have

$$(F_c f)(y) = (F_c \varphi)(y)\left[1 - (F_c \xi)(y)\right]. \tag{3.14}$$

Then the solution in $L_1(\mathbb{R}_+)$ of (3.1) has the form

$$f(x) = \varphi(x) - \left(\varphi \underset{1}{*} \xi\right)(x). \tag{3.15}$$

The proof is complete. □

Remark 3.4. The integral equation (3.1) *is a special case of the integral equation with the Toeplitz plus Hankel kernel* (1.27) *for* $x > 0$ *and*

$$\begin{aligned}
&k_1(t)\\
&= \frac{1}{2\sqrt{2\pi}}\left[k(t+1) - k(|t-1|)\operatorname{sign}(t-1)\right] + \frac{1}{\sqrt{2\pi}}l(t)\\
&\quad + \frac{1}{4\sqrt{2\pi}}\iint_0^\infty g(u)h(w)\left[e^{-w\cosh(t+u-1)} + e^{-w\cosh(t-u+1)} - e^{-w\cosh(t+u+1)} - e^{-w\cosh(t-u-1)}\right]du\,dw\\
&k_2(t)\\
&= \frac{1}{2\sqrt{2\pi}}\left[k(|t+1|)\operatorname{sign}(t+1) - k(|t-1|)\operatorname{sign}(t-1)\right] + \frac{1}{\sqrt{2\pi}}\,l(|t|)\\
&\quad + \frac{1}{4\sqrt{2\pi}}\iint_0^\infty g(u)h(w)\left[e^{-w\cosh(t+u-1)} + e^{-w\cosh(t-u+1)} - e^{-w\cosh(t+u+1)} - e^{-w\cosh(t-u-1)}\right]du\,dw.
\end{aligned} \tag{3.16}$$

Next, we consider the following system of two integral equations:

$$\begin{aligned}
f(x) + \iiint_0^\infty \theta(x,u,v,w)g(u)h(v)k(w)du\,dv\,dw + \int_0^\infty \theta_3(x,u)g(u)du = p(x)\\
\int_0^\infty \theta_4(x,u)f(u)du + \int_0^\infty \theta_5(x,u)f(u)du + g(x) = q(x).
\end{aligned} \tag{3.17}$$

Here $\theta(x,u,v,w)$ is defined by (2.2), and

$$\begin{aligned}
\theta_3(x,u) &= \frac{1}{\sqrt{2\pi}}\left[l(x+u) - l(|x-u|)\operatorname{sign}(x-u)\right],\\
\theta_4(x,u) &= \frac{1}{\sqrt{2\pi}}\left[\xi(x+\mathrm{u}) + \xi(|x-u|)\operatorname{sign}(x-u)\right],\\
\theta_5(x,u) &= \frac{1}{2\sqrt{2\pi}}\left[\eta(|\,x+u-1) + \eta(|x-u-1|) - \eta(x+u+1) - \eta(x-u+1)\right],
\end{aligned} \tag{3.18}$$

h, k, l, ξ, η, p, q are known functions, and f and g are unknown functions.

Theorem 3.5. *Given that* $p, q, h, l, \xi, \eta_1, \eta_2 \in L_1(\mathbb{R}_+)$ *and* $k \in L_1(\beta, \mathbb{R}_+)$, $\eta = \eta_1 \underset{3}{*} \eta_2$ *such that* $1 - (F_c\varphi)(y) \neq 0$, *where*

$$\varphi(x) = \left(\overset{\gamma}{*}(\xi, h, k)\right)(x) - \left(\xi \underset{3}{*} l\right)(x) - *\left(\eta_1 \overset{\gamma}{*} \eta_2, h, k\right)(x) - \left(l \underset{1}{*} \eta\right)(x). \tag{3.19}$$

Then the system (3.17) *has a unique solution in* $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$*whose closed form is as follows*

$$
\begin{aligned}
f(x) &= p(x) - \left(\overset{\gamma}{*}(q,h,k)\right)(x) + \left(q \underset{3}{*} l\right) + \left(l \underset{1}{*} p\right)(x) \\
&\quad + \left(\overset{\gamma}{*}(q,h,k) \underset{1}{*} l\right)(x) + \left(\left(q \underset{3}{*} l\right) \underset{1}{*} l\right)(x), \\
g(x) &= q(x) \ - \left(\xi \underset{2}{*} p\right)(x) - \left(\eta \overset{\gamma}{\underset{2}{*}} p\right)(x) + \left(q \underset{2}{*} l\right)(x) - \left(\left(\xi \underset{2}{*} p\right) \underset{2}{*} l\right) - \left(\left(\eta \overset{\gamma}{\underset{2}{*}} p\right) \underset{2}{*} l\right)(x).
\end{aligned}
\tag{3.20}
$$

Here, $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$
(F_c l)(y) = \frac{(F_c\varphi)(y)}{1 - (F_c\varphi)(y)}. \tag{3.21}
$$

Proof. We need the following lemma.

Lemma 3.6. *Let* $\xi, f \in L_1(\mathbb{R}_+)$*; then*

$$
\begin{gathered}
\int_0^\infty [\xi(x+u) + \xi(|x-u|)\operatorname{sign}(x-u)] f(u)du \in L_1(\mathbb{R}_+), \\
F_s\left(\frac{1}{\sqrt{2\pi}}\int_0^\infty [\xi(x+u) + \xi(|x-u|)\operatorname{sign}(x-u)] f(u)du\right)(y) = (F_s\xi)(y)(F_c f)(y), \quad \forall y > 0.
\end{gathered}
\tag{3.22}
$$

Using Theorem 2.2, Lemma 3.6, and the generalized convolution (1.15), (1.19), we have

$$
\begin{gathered}
(F_c f)(y) + \sin y (F_s g)(y)(F_c h)(y)(K_{iy}k) + (F_s l)(y)(F_s g)(y) = (F_c p)(y), \\
(F_s\xi)(y)(F_c f)(y) + \sin y (F_c\eta)(y)(F_c f)(y) + (F_s g)(y) = (F_s q)(y).
\end{gathered}
\tag{3.23}
$$

On the other hand, from $\eta = \eta_1 \underset{3}{*} \eta_2$ we have $\sin y (F_c\eta)(y) = F_s(\eta_1 \overset{\gamma}{*} \eta_2)(y)$. Therefore, using Theorem 2.2 and the generalized convolution (1.15), (1.17), we have

$$
\begin{aligned}
\Delta &= \begin{vmatrix} 1 & \sin y (F_c h)(y) \cdot (K_{iy}k) + (F_s l)(y) \\ (F_s\xi)(y) + \sin y (F_c\eta)(y) & 1 \end{vmatrix} \\
&= 1 - F_c\left(\overset{\gamma}{*}(\xi,h,k)\right)(y) - F_c\left(\xi \underset{3}{*} l\right)(y) - F_c\left(*\left(\eta_1 \overset{\gamma}{*} \eta_2, h, k\right)\right)(y) - F_c\left(l \overset{\gamma}{\underset{1}{*}} \eta\right)(x).
\end{aligned}
\tag{3.24}
$$

Hence, in view of the Wiener-Levy theorem [25], by the given condition, there is a unique function $l \in L_1(\mathbb{R}_+)$ such that

$$\frac{1}{\Delta} = 1 + (F_c l)(y), \tag{3.25}$$

where

$$(F_c l) = \frac{F_c\left(\overset{\gamma}{*}(\xi, h, k)\right) + F_c\left(\xi \underset{3}{*} l\right) + F_c\left(*\left(\eta_1 \overset{\gamma}{*} \eta_2, h, k\right)\right) + F_c\left(l \overset{\gamma}{\underset{1}{*}} \eta\right)}{1 - F_c\left(\overset{\gamma}{*}(\xi, h, k)\right) - F_c\left(\xi \underset{3}{*} l\right) - F_c\left(*\left(\eta_1 \overset{\gamma}{*} \eta_2, h, k\right)\right) - F_c\left(l \overset{\gamma}{\underset{1}{*}} \eta\right)}. \tag{3.26}$$

On the other hand, using Theorem 2.2 and the generalized convolution (1.15), we have

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} (F_c p)(y) & \sin y (F_c h)(y) \cdot (K_{iy} k) + (F_s l)(y) \\ (F_s q)(y) & 1 \end{vmatrix} \\ &= (F_c p)(y) - F_c(*(q, h, k))(y) - F_c\left(q \underset{3}{*} l\right)(y). \end{aligned} \tag{3.27}$$

Hence, from (3.25), (3.27) we have

$$\begin{aligned} (F_s f)(y) &= [1 + (F_c l)(y)]\left[(F_c p)(y) - F_c(*(q, h, k))(y) - F_c\left(q \underset{3}{*} l\right)(y)\right] \\ &= (F_c p)(y) - F_c(*(q, h, k)) - F_c\left(q \underset{3}{*} l\right) + F_c\left(l \underset{1}{*} p\right)(y) - F_c\left(*(q, h, k) \underset{1}{*} l\right)(y) \\ &\quad - F_c\left(\left(q \underset{3}{*} l\right) \underset{1}{*} l\right)(y). \end{aligned} \tag{3.28}$$

It shows that

$$f(x) = p(x) - (*(q, h, k))(x) - \left(q \underset{3}{*} l\right) + \left(l \underset{1}{*} p\right)(x) - \left(*(q, h, k) \underset{1}{*} l\right)(x) - \left(\left(q \underset{3}{*} l\right) \underset{1}{*} l\right)(x). \tag{3.29}$$

Similarly, from the generalized convolutions (1.15), (1.19), we have

$$\begin{aligned} \Delta_2 &= \begin{vmatrix} 1 & (F_c p)(y) \\ (F_s \xi)(y) + \sin y (F_c \eta)(y) & (F_s q)(y) \end{vmatrix} \\ &= (F_s q)(y) - F_s\left(\xi \underset{2}{*} p\right)(y) - F_s\left(\eta \overset{\gamma}{\underset{2}{*}} p\right)(y). \end{aligned} \tag{3.30}$$

Using formulas (3.25), (3.30), we have

$$\begin{aligned}(F_s g)(y) &= [1+(F_c l)(y)]\left[(F_s q)(y) - F_s\left(\xi \underset{2}{*} p\right)(y) - F_s\left(\eta \overset{\gamma}{\underset{2}{*}} p\right)(y)\right]\\ &= (F_s q)(y) - F_s\left(\xi \underset{2}{*} p\right)(y) - F_s\left(\eta \overset{\gamma}{\underset{2}{*}} p\right)(y) + F_s\left(q \underset{2}{*} l\right) - F_s\left(\left(\xi \underset{2}{*} p\right) \underset{2}{*} l\right)(y)\\ &\quad - F_s\left(\left(\eta \overset{\gamma}{\underset{2}{*}} p\right) \underset{2}{*} l\right)(y).\end{aligned} \tag{3.31}$$

It shows that

$$g(x) = q(x) - \left(\xi \underset{2}{*} p\right)(x) - \left(\eta \overset{\gamma}{\underset{2}{*}} p\right)(x) + \left(q \underset{2}{*} l\right)(x) - \left(\left(\xi \underset{2}{*} p\right) \underset{2}{*} l\right) - \left(\left(\eta \overset{\gamma}{\underset{2}{*}} p\right) \underset{2}{*} l\right)(x). \tag{3.32}$$

Pair (f, g) defined by fomulae (3.29) and (3.32) is a solution in closed form in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$ of system (3.17). The proof is complete. □

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