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Research Article

Multimodeling Control via System Balancing

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A new approach in multimodeling strategy is proposed. Multimodel strategies in which control agents use different simplified models of the same system are being developed using balancing transformation and the corresponding order reduction concepts. Traditionally, the multimodeling concept was studied using the ideas of multitime scales (singular perturbations) and weak subsystem coupling. For all reduced-order models obtained, a Linear Quadratic Gaussian (LQG) control problem was solved. Different order reduction techniques were compared based on the values of the optimized criteria for the closed-loop case where the full-order balanced model utilizes regulators calculated to be the optimal for various reduced-order models. The results obtained were demonstrated on a real-world example: a multiarea power system consisting of two identical areas, that is, two identical power plants.

1. Introduction

Large-scale systems have been the subject of research work for several decades [1–20]. Some order reduction techniques were developed for the singularly perturbed class of systems, based on different mathematical procedures, such as graph metric [21]. The concept of multimodel strategies for large-scale systems is originated from [17]. According to that concept, a large-scale system may be controlled by several independent agents using several simplified models of the system. In the singular perturbation [17], methodology has been used to develop multimodel strategies by exploiting the nature of the system that has two fast subsystems mutually weakly coupled and both strongly connected to the slow subsystem. The basic contribution of [17] has been to establish a set of conditions under which a multimodel strategy is well posed in the sense that the performance of the control system is close to the performance that would have been obtained had the control strategy been designed by a single controller knowing the exact model of the overall system. Multimodel strategies [17] have been studied and employed in several papers, either within the context of singular perturbations or in different mathematical set-ups [14]. Conditions for a multimodel strategy to be well-posed are investigated in [14] for a linear quadratic Gaussian (LQG) optimal control problem [22].

The multimodeling structure used in the classical approach via singular perturbations and weak coupling is defined by a linear dynamic system that has one slow and two fast subsystems [11]. The fast subsystems are strongly connected to the slow subsystem and weakly connected (or not connected) among themselves. This structure describes well the dynamics of several real-world systems, for example, power systems [15, 17] and automobiles [2]. The corresponding multimodeling representation [11, 17] is defined by (1.1):

$$\begin{bmatrix} \dot{x}_0(t) \\ \varepsilon_1\dot{x}_1(t) \\ \varepsilon_2\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & \varepsilon_3 A_{12} \\ A_{20} & \varepsilon_3 A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{01} & B_{02} \\ B_{11} & \varepsilon_3 B_{12} \\ \varepsilon_3 B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{1.1}$$

$$y(t) = C \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} + D \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{1.2}$$

where $x_0 \in \Re^{n_0}$ are slow state variables, $x_1 \in \Re^{n_1}$, $x_2 \in \Re^{n_2}$ are fast state variables, $u_1 \in \Re^{m_1}$, $u_2 \in \Re^{m_2}$ are control inputs, $y \in \Re^{p}$ are outputs, ε_1, ε_2 are small positive singular perturbation parameters, and ε_3 is small weak-coupling parameter. (For $\varepsilon_3 = 0$, (1.1) describes multiparameter singularly perturbed system (MSPS) studied in detail in literature [5–11].)

For the purpose of deterministic optimal control of the above multimodeling structure, the quadratic performance criterion has to be minimized by the proper choice of the control variables $u_1(t)$ and $u_2(t)$. The performance criterion for the linear quadratic Gaussian optimization [22] is given by

$$J = \frac{1}{2}\int_0^{+\infty} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt, \quad Q = Q^T \geq 0,\ R = R^T > 0, \tag{1.3}$$

where

$$x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^T & Q_{11} & 0 \\ Q_{02}^T & 0 & Q_{22} \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix},$$
$$Q = q^T q = \begin{bmatrix} q_{01} & q_{11} & 0 \\ q_{02} & 0 & q_{22} \end{bmatrix}^T \begin{bmatrix} q_{01} & q_{11} & 0 \\ q_{02} & 0 & q_{22} \end{bmatrix} = \begin{bmatrix} q_{01}^T q_{01} + q_{02}^T q_{02} & q_{01}^T q_{11} & q_{02}^T q_{22} \\ q_{11}^T q_{01} & q_{11}^T q_{11} & 0 \\ q_{22}^T q_{02} & 0 & q_{22}^T q_{22} \end{bmatrix}. \tag{1.4}$$

In the general multimodeling case, all zero-elements in matrices R and Q can be replaced by $O(\varepsilon)$ elements. ($O(\varepsilon^i)$ is defined by $O(\varepsilon^i) < c\varepsilon^i$, where c is a bounded constant, i is a real number, and $\varepsilon = \|[\varepsilon_1\ \varepsilon_2]\|$.)

To solve the multimodeling problem one proceeds with constructing two different models of (1.1), obtained by setting $\varepsilon_1 = 0$, which leads to the first model for the first controller, and by setting $\varepsilon_2 = 0$, which produces the second model for the second controller. In order to simplify equations, without loss of generality, small coupling parameter ε_3 is set to zero.

The fast dynamics of the other subsystem is approximated by an algebraic equation (the corresponding ε_i is set to zero).

Two approximations of the original model derived from (1.1) are

$$\begin{bmatrix} \dot{x}_0(t) \\ \varepsilon_1 \dot{x}_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{01} & B_{02} \\ B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$
$$\begin{bmatrix} \dot{x}_0(t) \\ 0 \\ \varepsilon_2 \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{01} & B_{02} \\ B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \tag{1.5}$$

The above equations can be rewritten as:

$$\begin{bmatrix} \dot{x}_0(t) \\ \varepsilon_1 \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} \left(A_{00} - A_{02}A_{22}^{-1}A_{20}\right) & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} B_{01} & \left(B_{02} - A_{02}A_{22}^{-1}B_{22}\right) \\ B_{11} & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$
$$\begin{bmatrix} \dot{x}_0(t) \\ \varepsilon_2 \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \left(A_{00} - A_{01}A_{11}^{-1}A_{10}\right) & A_{02} \\ A_{20} & A_{22} \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \left(B_{01} - A_{01}A_{11}^{-1}B_{11}\right) & B_{02} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{1.6}$$

leading to two different models of the original system. The algebraic equations defined in (1.5) are used in (1.3) by each controller to form their own performance criterion. Such simplified criteria are optimized by each controller via the corresponding reduced-order model and the obtained control strategies form the multimodeling strategy. The multimodeling strategy is well posed if it is $O(\varepsilon)$ close to the global optimal control strategy obtained by performing direct optimization on the original system and the original performance criterion, as shown in [14].

2. The Use of Balancing Transformation for the System Order Reduction

Robust order reduction based on the use of balancing transformation has been described in [20]. Concisely, consideration is given to a linear, time invariant system as in (2.1):

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x(t_0) = x_0,$$
$$y(t) = Cx(t) + Du(t), \tag{2.1}$$

where $x(t)$ is an n-dimensional state vector, $u(t)$ is an m-dimensional input vector, and $y(t)$ is a p-dimensional output vector.

For a linear, time invariant system (2.1) the corresponding transfer function for the open-loop system is given by (2.2):

$$G(s) = C(sI - A)^{-1}B + D. \tag{2.2}$$

It is assumed that the system (2.1) is asymptotically stable and that a $G(s)$ is of minimal realization.

Assumption 1. A system is asymptotically stable, a pair (A, B) is controllable, and a pair (A, C) is observable.

The controllability and observability Gramians of the original system (2.1) satisfy the algebraic equations of Lyapunov [15, 16]:

$$\begin{aligned} PA^T + AP + BB^T &= 0, \\ QA + A^TQ + C^TC &= 0. \end{aligned} \tag{2.3}$$

For a system that is controllable and observable, both controllability Gramian and observability Gramian are positive definite matrices, $P > 0$, $Q > 0$.

The balancing transformation is applied on the space vector in order to achieve that the controllability and the observability Gramians are identical and diagonal, that is,

$$\begin{gathered} x_b(t) = Tx(t), \qquad \det(T) \neq 0 \Longrightarrow \\ \frac{dx_b(t)}{dt} = A_b x_b(t) + B_b u(t), \\ y_b(t) = C_b x_b(t) + D_b u(t) = y(t), \end{gathered} \tag{2.4}$$

$$A_b = TAT^{-1}, \qquad B_b = TB, \qquad C_b = CT^{-1}, \qquad D_b = D, \tag{2.5}$$

$$P_b = Q_b = \Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0, \tag{2.6}$$

where σ_i are known as the Hankel singular values (HSVs).

Assuming that the original system is controllable and observable, a balanced system will also be both controllable and observable, since the similarity transformation preserves controllability and observability of the system that was transformed [15, 16]. Hence all σ_i are positive. Furthermore, both original and balanced systems are of minimal realization. The transfer function of the balanced system given by

$$G_b(s) = C_b(sI - A_b)^{-1}B_b + D = G(s) \tag{2.7}$$

stays unchanged thanks to a coordinate change through a nonsingular transformation. The balanced controllability and observability Gramians satisfy the following algebraic Lyapunov equations:

$$\begin{aligned}\Sigma A_b^T + A_b\Sigma + B_bB_b^T = 0 &\Longleftrightarrow P_bA_b^T + A_bP_b + B_bB_b^T = 0,\\ \Sigma A_b + A_b^T\Sigma + C_b^TC_b = 0 &\Longleftrightarrow Q_bA_b + A_b^TQ_b + C_b^TC_b = 0.\end{aligned} \tag{2.8}$$

The idea of the order reduction through balancing transformation can be linked with the canonical system decomposition. It was shown that the system's modes that were either uncontrollable or unobservable did not appear in the system transfer function. In [15, 16] it is shown that the system modes that are both weakly controllable and weakly observable have little influence on the system dynamics; so they can be neglected.

However, it was noticed that those modes which are weakly controllable and well observable cannot be neglected as can neither be neglected those modes that are well controllable and weakly observable. Let us assume that the balanced system (2.4)–(2.6) is partitioned in the following way:

$$\begin{gathered}A_b = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}, \qquad B_b = \begin{bmatrix} B_{11} \\ B_{22}\end{bmatrix}, \qquad C_b = \begin{bmatrix} C_{11} & C_{22}\end{bmatrix}, \qquad D_b = D,\\ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2\end{bmatrix}, \qquad \Sigma_1 = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}, \qquad \Sigma_2 = \operatorname{diag}\{\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n\}.\end{gathered} \tag{2.9}$$

Assuming that $\sigma_r > \sigma_{r+1}$, *balanced truncation* produces a system of lower order, r, defined by

$$\begin{aligned}\frac{dx_1(t)}{dt} &= A_{11}x_1(t) + B_{11}u(t),\\ y(t) &= C_{11}x_1(t) + Du(t),\end{aligned} \tag{2.10}$$

and the corresponding transfer function of the reduced-order system is

$$G_{11}(s) = C_{11}(sI - A_{11})^{-1}B_{11} + D. \tag{2.11}$$

The reduced-order system attained in this way is both controllable and observable since all corresponding HSVs are positive. Furthermore, the reduced-order system is balanced and asymptotically stable. It was shown in literature, for example, [20] that the H_∞ norm for the reduced-order system, obtained through the truncation procedure given above, satisfies the condition:

$$\|G(s) - G_{11}(s)\|_\infty \le 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_n). \tag{2.12}$$

It was noticed that the reduced-order system obtained through the balanced truncation procedure gives very good approximation of the original system in the case of pulse input for both control signals (good spectra approximation on higher frequencies) but shows a considerable, steady-state error in the case of step input (poor spectra approximation on lower frequencies). This error is due to the fact that the original system and the reduced order system have different DC gains. Actually, after the above-described truncation through balancing transformation, most of the spectra on lower frequencies are kept and also some of the spectra on higher frequencies, but some of the spectra on lower frequencies are lost as well as most of the spectra on higher frequencies. By eliminating part of the spectra on lower frequencies (which occur in the neglected part of the system—state variables $x_2(t)$) we have produced gain that differs from the gain of the original system that was balanced. This discrepancy was eliminated in [16] where a technique of *balanced residualization* was proposed that produced an accurate (exact) DC gain and a very good spectra approximation on lower frequencies and sometimes even on middle frequencies. It should be noted that in [20] a residualisation technique was also used. An improved truncation method that preserves the exact DC gain value as in the original system is given in [16] and applied in [19].

3. Multimodeling via System Balancing

Two multimodeling order reduction concepts are mentioned above: the first one uses the small-parameter idea and the second one is based on balancing transformation, presented through two specific methods—balanced truncation and balanced residualization. Here, we will present the idea of multimodeling via system balancing, which is more general than multimodeling via singular perturbation since the latter requires special structures of the original model. Multimodeling via singular perturbation is performed on the assumption that two fast weakly connected subsystems, whose system matrices are invertible, are both strongly coupled to the slow subsystem while no such assumption is necessary for the multimodeling via system balancing.

Let us consider a time invariant linear system represented by (3.1):

$$\begin{bmatrix}\dot{x}_0(t)\\ \dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = A\begin{bmatrix}x_0(t)\\ x_1(t)\\ x_2(t)\end{bmatrix} + B\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix},$$
$$y(t) = C\begin{bmatrix}x_0(t)\\ x_1(t)\\ x_2(t)\end{bmatrix} + D\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}, \tag{3.1}$$

where

$$A = \begin{bmatrix}A_{00} & A_{01} & A_{02}\\ A_{10} & A_{11} & A_{12}\\ A_{20} & A_{21} & A_{22}\end{bmatrix}, \qquad B = \begin{bmatrix}B_{01} & B_{02}\\ B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}. \tag{3.2}$$

The balanced model is derived as in (2.4)-(2.5), that is,

$$\begin{gathered}x_b(t) = Tx(t), \qquad \det(T) \neq 0 \Longrightarrow \\ \frac{dx_b(t)}{dt} = A_b x_b(t) + B_b u(t), \\ y_b(t) = C_b x_b(t) + D_b u(t) = y(t), \\ A_b = TAT^{-1}, \qquad B_b = TB, \qquad C_b = CT^{-1}, \qquad D_b = D, \\ P_b = Q_b = \Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0.\end{gathered} \tag{3.3}$$

The HSVs are the same as the HSVs of the original model (1.1) and are sorted in a descending order so that they can help in determining how to partition the model: the first few state space variables for the slow subsystem and the following two for the two fast parts, as in (3.4).

The balanced model will be partitioned in the following manner (3.4)-(3.5):

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b1}(t) \\ \dot{x}_{b2}(t) \end{bmatrix} = \begin{bmatrix} A_{b00} & A_{b01} & A_{b02} \\ A_{b10} & A_{b11} & A_{b12} \\ A_{b20} & A_{b21} & A_{b22} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \\ x_{b2}(t) \end{bmatrix} + \begin{bmatrix} B_{b01} & B_{b02} \\ B_{b11} & B_{b12} \\ B_{b21} & B_{b22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{3.4}$$

$$y_b(t) = \begin{bmatrix} C_{b0} & C_{b1} & C_{b2} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \\ x_{b2}(t) \end{bmatrix} + \begin{bmatrix} D_{b1} & D_{b2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = y(t), \tag{3.5}$$

where $x_{b0} \in \Re^{n_0}$ are slow or the common-core state variables of the balanced model; $x_{b1} \in \Re^{n_1}$, $x_{b2} \in \Re^{n_2}$ are state variables of the balanced model, having a small amount of total energy $(n_0 + n_1 + n_2 = n)$; $u_1 \in \Re^{m_1}$, $u_2 \in \Re^{m_2}$ are control inputs of the balanced and original model $(m_1 + m_2 = m)$; $y_b \in \Re^p$, $y \in \Re^p$ are outputs of the balanced and original model; respectively, and A_{bij}, B_{bik}, C_{bj}, and D_{bk} $(i, j = 0, 1, 2$ and $k = 1, 2)$ are submatrices of the balanced system matrices A_b, B_b, C_b, and D_b, having the corresponding dimensions.

Instead of zero-submatrices in [12] a more general case is considered here, where submatrices A_{b12}, A_{b21}, B_{b12}, and B_{b21} are nonzero matrices, as it is usually the case in the real-world systems.

The corresponding quadratic performance criterion which has to be minimized is

$$J_b = \frac{1}{2} \int_0^{+\infty} \left[x_b^T(t) Q_b x_b(t) + u^T(t) R_b u(t) \right] dt, \quad Q_b = Q_b^T \geq 0, \ R_b = R_b^T > 0, \tag{3.6}$$

where, without loss of generality, we assume that Q_b and R_b are block-diagonal matrices:

$$Q_b = \begin{bmatrix} Q_{b00} & 0 & 0 \\ 0 & Q_{b11} & 0 \\ 0 & 0 & Q_{b22} \end{bmatrix}, \qquad R_b = \begin{bmatrix} R_{b11} & 0 \\ 0 & R_{b22} \end{bmatrix}. \tag{3.7}$$

Now the decomposition of the balanced model (3.4) is performed into two reduced-order models, both of them composed of the common core—a subsystem corresponding to slow modes of the balanced model, and one of the two different remaining subsystems, corresponding to the modes having a small amount of total energy, namely, *the first* reduced-order model and *the second* reduced-order model. Cross-coupling matrices A_{b12}, A_{b21} will be neglected though they have nonzero values. *The first* reduced-order model obtained is

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b1}(t) \end{bmatrix} = A_{1r} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + B_{1r} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad y_{b1}(t) = \begin{bmatrix} C_{b0} & C_{b1} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + D_{b1} u_1(t), \tag{3.8}$$

$$A_{1r} = \begin{bmatrix} A_{b00} & A_{b01} \\ A_{b10} & A_{b11} \end{bmatrix}, \qquad B_{1r} = \begin{bmatrix} B_{b01} & B_{b02} \\ B_{b11} & B_{b12} \end{bmatrix}. \tag{3.9}$$

The second reduced-order model is

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b2}(t) \end{bmatrix} = A_{2r} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + B_{2r} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad y_{b2}(t) = \begin{bmatrix} C_{b0} & C_{b2} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + D_{b2} u_2(t), \tag{3.10}$$

where

$$A_{2r} = \begin{bmatrix} A_{b00} & A_{b02} \\ A_{b20} & A_{b22} \end{bmatrix}, \qquad B_{2r} = \begin{bmatrix} B_{b01} & B_{b02} \\ B_{b12} & B_{b22} \end{bmatrix}. \tag{3.11}$$

It is interesting to note that *the first* and *the second* reduced-order models in (3.8) and (3.10) both have HSVs corresponding to the common core modes of the original system and two complement sets of the HSVs corresponding to the modes of the original system that have a small amount of total energy, if we keep all parts of the B_b matrix.

If, however, coupling submatrices B_{b12} and B_{b21} are set to zero-matrices of the corresponding dimensions, as suggested in [11], the HSVs obtained for the first and the second subsystems will differ from those of the original full-order model, and the reduced-order models will be as in (3.12)–(3.13) and (3.14). These two types of reduced-order models would still contain all control signals of the original and balanced models.

The *first* reduced-order model is obtained as in [11, 17]:

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b1}(t) \end{bmatrix} = A_{1r} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + B_{1r} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad y_{b1}(t) = \begin{bmatrix} C_{b0} & C_{b1} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + D_{b1} u_1(t), \tag{3.12}$$

where

$$A_{1r} = \begin{bmatrix} A_{b00} & A_{b01} \\ A_{b10} & A_{b11} \end{bmatrix}, \qquad B_{1r} = \begin{bmatrix} B_{b01} & B_{b02} \\ B_{b11} & 0 \end{bmatrix}. \tag{3.13}$$

The *second* reduced-order model is obtained as in [11, 17]:

$$\begin{gathered} \begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b2}(t) \end{bmatrix} = A_{2r} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + B_{2r} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ y_{b2}(t) = \begin{bmatrix} C_{b0} & C_{b2} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + D_{b2} u_2(t). \\ A_{2r} = \begin{bmatrix} A_{b00} & A_{b02} \\ A_{b20} & A_{b22} \end{bmatrix}, \qquad B_{2r} = \begin{bmatrix} B_{b01} & B_{b02} \\ 0 & b_{b22} \end{bmatrix}. \end{gathered} \tag{3.14}$$

The same approximation is done for the performance criterion (1.3); hence two performance criteria are needed, which leads to multicriteria optimization problem. Depending on the actual problem setup, very often described by differential games, the two controllers find their own optimal strategies and apply such strategies to the global system defined by (1.1). In such a way, the multimodeling strategy is well posed if the performance criterion under the multimodeling strategy is $O(\sigma_{r+1}/\sigma_r)$ close to the global optimal control strategy obtained by performing direct optimization of the original performance criterion for the original system.

The following two criteria to be optimized now are

$$\begin{gathered} J_1 = \frac{1}{2} \int_0^{+\infty} \left\{ \begin{bmatrix} x_{b0}(t) & x_{b1}(t) \end{bmatrix} Q_1 \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} R_1 \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\} dt, \\ Q_1 = Q_1^T \geq 0, \quad R_1 = R_1^T > 0, \end{gathered} \tag{3.15}$$

$$\begin{gathered} J_2 = \frac{1}{2} \int_0^{+\infty} \left\{ \begin{bmatrix} x_{b0}(t) & x_{b2}(t) \end{bmatrix} Q_2 \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} R_2 \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\} dt, \\ Q_2 = Q_2^T \geq 0, \quad R_2 = R_2^T > 0. \end{gathered} \tag{3.16}$$

Q_1 and Q_2 are block-diagonal matrices and R_1 and R_2 are submatrices of R_b having corresponding dimensions:

$$\begin{gathered} Q_1 = Q_1^T = \begin{bmatrix} Q_{b00} & 0 \\ 0 & Q_{b11} \end{bmatrix} \geq 0, \quad R_1 = R_1^T = R_{b11} > 0, \\ Q_2 = Q_2^T = \begin{bmatrix} Q_{b00} & 0 \\ 0 & Q_{b22} \end{bmatrix} \geq 0, \quad R_2 = R_2^T = R_{b22} > 0. \end{gathered} \tag{3.17}$$

The optimal control problem we refer to, following the suggestions in [2, 14, 17], is to minimize a convex sum of J_1 and J_2 given in (3.15) and (3.16), that is, for some γ_1 and γ_2

$$J_{MMC} = \gamma_1 J_1 + \gamma_2 J_2, \quad \gamma_1 + \gamma_2 = 1, \ \gamma_1 > 0, \ \gamma_2 > 0. \tag{3.18}$$

This corresponds to a *Pareto optimal cooperative strategy* [6].

In Pareto optimal strategy a situation is considered in which decision makers should decide on their strategies through mutual cooperation [14, 17]. The essence of this is that no variation from a Pareto optimal strategy can decrease the costs of either of the decision makers. Let each decision maker have a quadratic cost functional as in (3.15) and (3.16). A Pareto solution is a pair $u_{\text{opt1}}(t), u_{\text{opt2}}(t)$ which minimizes (3.18) for some γ_1 and γ_2 [6]. The optimal feedback solution to (3.15) and (3.16) is given by (3.19):

$$u_{\text{opt}}(t) = \begin{bmatrix} u_{\text{opt1}}(t) \\ u_{\text{opt2}}(t) \end{bmatrix} = -R_b^{-1} B_b^T P_b x_b(t). \tag{3.19}$$

Here P_b is the positive semidefinite stabilizing solution of the algebraic Riccati equation:

$$A_b^T P_b + P_b A_b + Q_b - P_b S_b P_b = 0, \quad S_b = B_b R_b^{-1} B_b^T. \tag{3.20}$$

The optimal state regulator is a special case of this problem where the decision makers agree on a choice of γ_1 and γ_2 as weighting factors. Without prejudice to the generality, we chose $\gamma_1 = \gamma_2 = 0.5$. We expect that the approximation of the optimization criteria would be

$$\frac{1}{2}(J_1 + J_2) \approx J_b, \tag{3.21}$$

and according to [17] it is to be expected that:

$$J_{MMC} = \frac{1}{2} J_1 + \frac{1}{2} J_2 + O\left(\frac{\sigma_{r+1}}{\sigma_r}\right) \approx J. \tag{3.22}$$

The original result is obtained in [9], where instead of $O(\sigma_{r+1}/\sigma_r)$, $O(\|\mu\|^2)$ was used, where μ stood for a norm of $[\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3]$, in specific $\|\mu\| = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}$. Still adhering to this idea but including criteria for balancing truncation, instead of $O(\|\mu\|^2)$ we can use $O(\sigma_{r+1}/\sigma_r)$ having in mind that the order r is chosen where σ_{r+1} is of order of percent of the σ_r. In [6] a recursive algorithm is developed for solving multiparameter Riccati equations with the rate of convergence $O(\|\mu\|^i)$ and the rate of accuracy for the near-optimal strategy is $O(\|\mu\|^{i+1})$ where i is the iteration number.

The required solution of the algebraic Riccati equation (3.20) is based on the standard assumption [5, 12].

Assumption 2. The triples $(A_{1r}, B_{1r}, \text{chol}(Q_1))$ and $(A_{2r}, B_{2r}, \text{chol}(Q_2))$ are stabilizable-detectable. (Here $\text{chol}(Q)$ is the Cholesky decomposition of a matrix Q.)

Assumption 3. The pairs $(A_{1r}, \text{chol}(Q_1))$ and $(A_{2r}, \text{chol}(Q_2))$ are detectable-observable.

The matrices (A_{ir}, B_{ir}), $i = 1, 2$ are given by (3.23):

$$A_{1r} = \begin{bmatrix} A_{b00} & A_{b01} \\ A_{b10} & A_{b11} \end{bmatrix}, \quad B_{1r} = \begin{bmatrix} B_{b01} & B_{b02} \\ B_{b11} & 0 \end{bmatrix}, \quad A_{2r} = \begin{bmatrix} A_{b00} & A_{b02} \\ A_{b20} & A_{b22} \end{bmatrix}, \quad B_{2r} = \begin{bmatrix} B_{b01} & B_{b02} \\ 0 & B_{b22} \end{bmatrix}. \tag{3.23}$$

The optimal feedback solution to (3.12)–(3.13) and (3.15) is given by

$$u_{\text{the first}_{\text{opt}}}(t) = \begin{bmatrix} u_{\text{first}_{\text{opt1}}}(t) \\ u_{\text{first}_{\text{opt2}}}(t) \end{bmatrix} = -R_1^{-1} B_{1r}^T P_1 \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix}, \tag{3.24}$$

where P_1 is the positive semidefinite stabilizing solution of the corresponding algebraic Riccati equation (3.25):

$$A_{1r}^T P_1 + P_1 A_{1r} + Q_1 - P_1 S_1 P_1 = 0, \quad S_1 = B_{1r} R_1^{-1} B_{1r}^T. \tag{3.25}$$

The optimal feedback solution to (3.14) and (3.16) is given by

$$u_{\text{the second}_{\text{opt}}}(t) = \begin{bmatrix} u_{\text{second}_{\text{opt1}}}(t) \\ u_{\text{second}_{\text{opt2}}}(t) \end{bmatrix} = -R_2^{-1} B_{2r}^T P_2 \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix}, \tag{3.26}$$

where P_2 is the positive semidefinite stabilizing solution of the corresponding algebraic Riccati equation:

$$A_{2r}^T P_2 + P_2 A_{2r} + Q_2 - P_2 S_2 P_2 = 0, \quad S_2 = B_{2r} R_2^{-1} B_{2r}^T. \tag{3.27}$$

It is intuitively clear that a further reduction could be made for some models. If the control signals in the original model are weakly coupled, then it is possible to form two subsystems as described above, however *the first* one having only inputs u_1 and *the second* subsystem having only inputs u_2. This would mean that the neglected modes having a small amount of total energy are considered to have reached their steady-state values, rather than as changing variables. Models of the type (1.1) are sound candidates for this type of reduction. This would be as if the submatrices B_{b02} in (3.13) and B_{b01} in (3.14) were set to zero-matrices of the corresponding dimensions, resulting in (3.28)–(3.31) and (3.32)–(3.35), respectively. *The first* reduced-order model obtained by neglecting the control variable u_2 in (3.12)–(3.13) is

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b1}(t) \end{bmatrix} = \begin{bmatrix} A_{b00} & A_{b01} \\ A_{b10} & A_{b11} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + \begin{bmatrix} B_{b01} & 0 \\ B_{b11} & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{3.28}$$

or

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b1}(t) \end{bmatrix} = A_{1r} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + B_{1r} u_1(t), \tag{3.29}$$

$$y_{b1}(t) = \begin{bmatrix} C_{b0} & C_{b1} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + D_{b1} u_1(t), \tag{3.30}$$

$$A_{1r} = \begin{bmatrix} A_{b00} & A_{b01} \\ A_{b10} & A_{b11} \end{bmatrix}, \qquad B_{1r} = \begin{bmatrix} B_{b01} \\ B_{b11} \end{bmatrix}. \tag{3.31}$$

The second reduced-order model obtained by neglecting the control variables u_1 in (3.14) is presented in (3.32)–(3.35):

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b2}(t) \end{bmatrix} = \begin{bmatrix} A_{b00} & A_{b02} \\ A_{b20} & A_{b22} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + \begin{bmatrix} 0 & B_{b02} \\ 0 & B_{b22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{3.32}$$

or

$$\begin{bmatrix} \dot{x}_{b0}(t) \\ \dot{x}_{b2}(t) \end{bmatrix} = A_{2r} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + B_{2r} u_2(t), \tag{3.33}$$

$$y_{b2}(t) = \begin{bmatrix} C_{b0} & C_{b2} \end{bmatrix} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} + D_{b2} u_2(t), \tag{3.34}$$

$$A_{2r} = \begin{bmatrix} A_{b00} & A_{b02} \\ A_{b20} & A_{b22} \end{bmatrix}, \qquad B_{2r} = \begin{bmatrix} B_{b02} \\ B_{b22} \end{bmatrix}. \tag{3.35}$$

The same approximation is done for the performance criterion (3.15)-(3.16); hence two performance criteria are obtained, as in (3.36)-(3.37) and (3.38)-(3.39), each of them having its own control signal. Taking into consideration multicriteria optimization, both of these criteria would be regarded as having equal importance. Again, the multimodeling strategy will be considered to be well posed if the performance criterion under the multimodeling strategy is $O(\sigma_{r+1}/\sigma_r)$ close to the global optimal control strategy obtained by performing direct optimization on the original system and the original performance criterion.

The two criteria to be optimized now are

$$J_{\text{the first}} = \frac{1}{2} \int_0^{+\infty} \left\{ \begin{bmatrix} x_{b0}(t) & x_{b1}(t) \end{bmatrix} Q_1 \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} + u_1(t)^T R_1 u_1(t) \right\} dt, \tag{3.36}$$

$$Q_1 = Q_1^T = \begin{bmatrix} Q_{b00} & 0 \\ 0 & Q_{b11} \end{bmatrix} \geq 0, \quad R_1 = R_1^T = R_{b11} > 0, \tag{3.37}$$

$$J_{\text{the second}} = \frac{1}{2}\int_0^{+\infty}\left\{\begin{bmatrix} x_{b0}(t) & x_{b2}(t)\end{bmatrix} Q_2 \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t)\end{bmatrix} + u_2(t)^T R_2 u_2(t)\right\} dt, \tag{3.38}$$

$$Q_2 = Q_2^T = \begin{bmatrix} Q_{b00} & 0 \\ 0 & Q_{b22}\end{bmatrix} \geq 0, \quad R_2 = R_2^T = R_{b22} > 0. \tag{3.39}$$

In the simulation example it has been chosen that Q_1 and Q_2 are unity matrices of the corresponding dimensions, while R_1 and R_2 are unity matrices of the corresponding dimensions multiplied by some weighing factor which is in accordance with the suggestions made in [17]:

$$Q_1 = Q_1^T = I_{n_0+n_1} \geq 0, \qquad R_1 = R_1^T = \text{Const} \cdot I_{m_1} > 0, \tag{3.40}$$

$$Q_2 = Q_2^T = I_{n_0+n_2} \geq 0, \qquad R_2 = R_2^T = \text{Const} \cdot I_{m_2} > 0. \tag{3.41}$$

The optimal feedback solution to (3.36)-(3.37) and (3.40) is given by (3.42):

$$u_{\text{the first}_{\text{opt}}}(t) = -R_1^{-1} B_{1r}^T P_{\text{the first}} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t)\end{bmatrix}, \quad B_{1r} = \begin{bmatrix} B_{b01} \\ B_{b11}\end{bmatrix}, \tag{3.42}$$

where $P_{\text{the first}}$ is the positive semidefinite stabilizing solution of the algebraic Riccati equation:

$$A_{1r}^T P_{\text{the first}} + P_{\text{the first}} A_{1r} + Q_1 - P_{\text{the first}} S_{\text{the first}} P_{\text{the first}} = 0, \quad S_{\text{the first}} = B_{1r} R_1^{-1} B_{1r}^T, \tag{3.43}$$

while the optimal feedback solution to (3.38)-(3.39) and (3.41) is given by:

$$u_{\text{the second}_{\text{opt}}}(t) = -R_2^{-1} B_{2r}^T P_{\text{the second}} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t)\end{bmatrix}, \quad B_{2r} = \begin{bmatrix} B_{b02} \\ B_{b22}\end{bmatrix}. \tag{3.44}$$

Here $P_{\text{the second}}$ is the positive semidefinite stabilizing solution of the algebraic Riccati equation:

$$A_{2r}^T P_{\text{the second}} + P_{\text{the second}} A_{2r} + Q_2 - P_{\text{the second}} S_{\text{the second}} P_{\text{the second}} = 0, \quad S_{\text{the second}} = B_{2r} R_2^{-1} B_{2r}^T. \tag{3.45}$$

It could be explored at a later stage what the sufficient and necessary conditions should be for the existence of solutions to Riccati equations (3.43) and (3.45). Some useful ideas and efficient iterative solutions with the existence conditions could be found in the work of Mukaidani et al. [5–11].

The next step is to implement optimal control from (3.42) and (3.44) to the full-order model. Optimal regulators (3.42) and (3.44) are easier to design than an optimal regulator for the full-order model (3.6). Since both control agents are of equal importance to the full-order model, approximation of the optimal control strategy could be

$$u_{\text{approx}_{\text{opt}}}(t) = \begin{bmatrix} u_{\text{the first}_{\text{opt}}}(t) \\ u_{\text{the second}_{\text{opt}}}(t) \end{bmatrix} = \begin{bmatrix} -R_1^{-1} B_{1r}^T P_{\text{the first}} \begin{bmatrix} x_{b0}(t) \\ x_{b1}(t) \end{bmatrix} \\ -R_2^{-1} B_{2r}^T P_{\text{the second}} \begin{bmatrix} x_{b0}(t) \\ x_{b2}(t) \end{bmatrix} \end{bmatrix}, \tag{3.46}$$

as was shown in the simulation example.

The optimal control problem we refer to, following the suggestions in [2, 14, 17], is to minimize a convex sum of $J_{\text{the first}}$ and $J_{\text{the second}}$ given in (3.36) and (3.38) for some γ_1 and γ_2:

$$J_{\text{approximated}} = \gamma_1 J_{\text{the first}} + \gamma_2 J_{\text{the second}}, \quad \gamma_1 + \gamma_2 = 1, \ \gamma_1 > 0, \ \gamma_2 > 0. \tag{3.47}$$

This corresponds to a *Pareto optimal cooperative strategy* [6].

In Pareto optimal strategy we have considered a situation in which decision makers decide on their strategies through mutual cooperation [14, 17]. The essence of this approach is that no variation from a Pareto optimal strategy can decrease the costs of any of the decision makers. Let each decision maker have a quadratic cost functional as in (3.36) and (3.38). A Pareto solution is a pair $u_{\text{the first}_{\text{opt}}}(t), u_{\text{the second}_{\text{opt}}}(t)$ which minimized (3.47) for some γ_1 and γ_2 [6]. The optimal state regulator is a special aspect of this problem where the decision makers agree on the choice of γ_1 and γ_2 as weighting factors.

Without prejudice to the generality of the above considerations, we chose $\gamma_1 = \gamma_2 = 0.5$. According to [17] it is to be expected that the approximation of the optimization criteria would be

$$J_{\text{approximated}} = J_{\text{bal. I and II}} = \frac{1}{2}\left(J_{\text{the first}} + J_{\text{the second}}\right) \approx J_b. \tag{3.48}$$

4. Example

The methods for the order reduction displayed here were tested on an example known from literature and taken from [17] with some modification. Such example is a state space model of a power system consisting of two interconnected identical areas, where each area consists of one plant. The model behavior was simulated in the open-loop as well as in the closed-loop.

For the model example a system was chosen having two inputs and one output. Based on HSVs a decision was made on what should be taken for the reduced order while the order of the slow or common-core subsystem and of the two fast parts (i.e., parts having a small amount of total energy) was chosen as suggested in [17]. Several reduced-order models were produced, using the methods as mentioned and described above.

The efficiency of these approximations was compared first in the open-loop case for typical input functions: impulse, step, ramp, and sine. In the open-loop case a comparison was also made with the balanced model for all frequency characteristics (both magnitude and phase spectra) of all available transfer functions. Control strategy in this paper was different from the one used in [17]. Optimal linear quadratic regulators (LQRs) were designed for original and balanced models as well as for the ones of reduced order. Gain matrices of the reduced-order models were expanded by zeros to obtain the full order, and these reduced-order regulators were used to close the loop on the balanced model. Different approximations were compared with respect to the values of optimized LQG criteria.

The model in the state-space representation is of the order $n = 9$ with the astatism of the second order therefore being unstable and is described by means of the matrices:

$$A = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.5 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.05 & 0 & -0.1 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.05 & 0.1 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 32.7 & -32.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & -40 & 0 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & -40 & 0 & 0 & 0 & 0 & -10 \end{bmatrix},$$

$$B^T = \begin{bmatrix} 0 & 0 & 0.1 & 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}. \tag{4.1}$$

Inspired by [1, page 227, Case 3] and [17] we have chosen

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{4.2}$$

For the model chosen for simulation, state-space variables as in [17] are as follows: x_1 and v_1 the integral of the area control error for the area 1, x_2 and v_2 the integral of the area control error for the area 2, x_3 and Δf_1 frequency variation for the area 1, x_4 and Δf_2 frequency variation for the area 2, x_5 and ΔP_{12} tie-line power flow variation, x_6 and ΔP_{G1} turbine output variation for the plant 1 or the area 1, x_7 and Δa_1 turbine valve position variation for the plant 1 or the area 1, x_8 and ΔP_{G2} turbine output variation for the plant 2 or the area 2, and x_9 and Δa_2 turbine valve position variation for the plant 2 or the area 2, and control signals are as follows: u_1 and ΔP_{c1} speed changer variation for the plant 1 or the area 1, and u_2 and ΔP_{c1} speed changer variation for the plant 2 or the area 2.

It is significant to note that the astatism is not inherent to the system—it is induced by including the integrals of the area control error (ACE) v_1 and v_2 into state vector. So the simulation could be performed with the modification of this model where these two variables would be omitted from the state space vector but retained in the output or outputs, resulting in the model having full-order $n = 7$.

Here we have chosen another modification. To make the model in the simulation example stabilized, matrix A is changed according to the *prescribed degree of stability*:

$$A = A + \alpha I_9, \tag{4.3}$$

where α is chosen to be −10 and I_9 denotes unity matrix of dimension 9. Other system matrices remained unchanged. Matrix A which is changed to have the prescribed degree of stability is

$$A = \begin{bmatrix} -10 & 0 & 4.5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 4.5 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.05 & 0 & -0.1 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10.05 & 0.1 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 32.7 & -32.7 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15 & 5 & 0 & 0 \\ 0 & 0 & -40 & 0 & 0 & 0 & -20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 & 5 \\ 0 & 0 & 0 & -40 & 0 & 0 & 0 & 0 & -20 \end{bmatrix}. \tag{4.4}$$

However, the system obtained in this way is still unobservable—only four of its modes are observable. From the matrix dimensions in (4.4) it is obvious that $n = 9$ is the order of the system.

Table 1 shows HSVs of the original and balanced models. On the basis of the HSVs a decision was made that the reduced order could be (and so it was chosen to be) $r = 2$, since the third Hankel singular value is more than 400 (410.4444) times smaller than the second one. For the reduced-order model of order $r = 2$ good quality approximation will be achieved, both in the open- and in closed-loop cases. However, in this manner we would omit state space variables that are important to be retained.

Multimodeling allows us to reduce the order of the model while keeping all state space variables accounted for, only decoupled into two intersecting subsets.

As for the multimodeling, the reduced-order system which consisted of a common core subsystem and subsystem corresponding to modes having a small amount of total energy was called *the first* while *the second* referred to the one which consisted of a common core subsystem and subsystem corresponding to the remaining modes, having a smaller amount of total energy.

According to the suggestions made in [17] the slow or common core subsystem order was chosen to be of order $n_0 = 5$, and the orders of the two fast subsystems, that is, subsystems corresponding to modes with a small amount of total energy were $n_1 = 2$, and $n_2 = 2$, respectively; so *the first* and *the second* subsystems were of the same order, $n_0 + n_1 = n_0 + n_2 = 7$. For *the first* and *the second* reduced-order models a multimodeling was performed through *balanced truncation*. In this way two subsystems were obtained having one input, with u_1 being input to *the first subsystem*, as in (3.28)–(3.31), and u_2 being input to *the second one*, as in (3.32)–(3.35).

Table 1 contains also the HSVs for *the first* and *the second* reduced-order models (ROMs) obtained through multimodeling using *balanced truncation*. It seems that the largest four HSVs for *the first* subsystem and *the second* one are identical. However, the difference between them is of the order 10^{-15} or less.

Each of the model approximations as well as the original one was tested on the open-loop case for typical input functions: impulse, step, ramp, and sine. In the time domain all of the approximations produce good and similar performances, except for both the *first* and the *second* subsystems obtained through multimodeling, that exhibit a considerable degree of difference in behavior when compared to models in the cases of *ramp* and *sine* input signals. However, the behavior of the *first model* and the *second one* is almost identical.

Table 1: Hankel singular values.

For the original and balanced models	For the *first* ROM obtained through multimodeling	For the *second* ROM obtained through multimodeling
0.14006	0.099037	0.099037
0.023921	0.016915	0.016915
$0.58281 \cdot 10^{-4}$	$0.41211 \cdot 10^{-4}$	$0.41211 \cdot 10^{-4}$
$0.41813 \cdot 10^{-5}$	$0.29566 \cdot 10^{-5}$	$0.29566 \cdot 10^{-5}$
$0.25878 \cdot 10^{-8}$	$0.17367 \cdot 10^{-21}$	$0.13413 \cdot 10^{-20}$
$0.61066 \cdot 10^{-9}$	$0.1615 \cdot 10^{-23}$	$0.19797 \cdot 10^{-21}$
$0.52878 \cdot 10^{-10}$	$1.4047 \cdot 10^{-28}$	$5.6210 \cdot 10^{-26}$
$0.1103 \cdot 10^{-10}$		
$0.58426 \cdot 10^{-13}$		

The optimal gain matrices were subsequently calculated for the original model, its balanced equivalent, and the four reduced-order models: one obtained through *balanced truncation*, one obtained through *balanced residualization*, whereas two reduced-order models were comprised of the slow subsystem and one fast subsystem—called the *first* and the *second*—consisted of the slow subsystem and the rest of the fast subsystem. Matrices needed for optimization criteria, chosen in accordance with those from [17], and used for the original and balanced nonreduced model and for various reduced-order models are in (4.5):

$$\begin{aligned} Q &= I_9, & R &= 20I_2, \\ Q_{\text{truncated}} = Q_{\text{rezidualized}} &= I_2, & R_{\text{truncated}} &= R_{\text{rezidualized}} = 20I_2, \\ Q_{\text{the first}} = Q_{\text{the second}} &= I_7, & R_{\text{the first}} &= R_{\text{the second}} = 20. \end{aligned} \tag{4.5}$$

Here I_k denotes unity matrix of dimension k.

Simulations were carried out using MATLAB. Proportional regulator gain matrices were computed for the original and the balanced models of the full-order and the truncated and residualized reduced-order models of order 2 as well as for the first and the second reduced models of order 7.

Corresponding matrices for the first and the second subsystems, $K_{\text{the first}}$ and $K_{\text{the second}}$, are close to submatrices of K_{balanced} and of the necessary dimensions:

$$\begin{gathered} K_{\text{balanced}} = \\ \begin{bmatrix} -3.28 \cdot 10^{-3} & -1.561 \cdot 10^{-3} & 0.34 \cdot 10^{-4} & -0.15 \cdot 10^{-4} & -0.33 \cdot 10^{-3} & 0.89 \cdot 10^{-3} & -0.32 \cdot 10^{-5} & -0.111 \cdot 10^{-3} & -2.65 \cdot 10^{-3} \\ -3.28 \cdot 10^{-3} & -1.561 \cdot 10^{-3} & 0.34 \cdot 10^{-4} & -0.15 \cdot 10^{-4} & 0.586 \cdot 10^{-3} & -0.89 \cdot 10^{-3} & 0.32 \cdot 10^{-5} & 0.111 \cdot 10^{-3} & 2.65 \cdot 10^{-3} \end{bmatrix}, \\ K_{\text{truncated}} = \begin{bmatrix} -3.2778 \cdot 10^{-3} & -1.5597 \cdot 10^{-3} \\ -3.2778 \cdot 10^{-3} & -1.5597 \cdot 10^{-3} \end{bmatrix} \\ K_{\text{rezidualized}} = \begin{bmatrix} -3.2785 \cdot 10^{-3} & -1.5635 \cdot 10^{-3} \\ -3.2785 \cdot 10^{-3} & -1.5635 \cdot 10^{-3} \end{bmatrix}, \\ K_{\text{the first}} = \begin{bmatrix} -3.28 \cdot 10^{-3} & -1.561 \cdot 10^{-3} & 0.34 \cdot 10^{-4} & -0.15 \cdot 10^{-4} & -0.397 \cdot 10^{-3} & -0.78 \cdot 10^{-4} & -0.247 \cdot 10^{-3} \end{bmatrix}, \\ K_{\text{the second}} = \begin{bmatrix} -3.280 \cdot 10^{-3} & -1.561 \cdot 10^{-3} & 0.34 \cdot 10^{-4} & -0.15 \cdot 10^{-4} & 0.586 \cdot 10^{-3} & -0.111 \cdot 10^{-3} & 2.234 \cdot 10^{-3} \end{bmatrix}. \end{gathered} \tag{4.6}$$

Finally, a regulator is composed of the $K_{\text{the first}}$ and $K_{\text{the second}}$ as in

$$K_{\text{bal. I and II}} = \begin{bmatrix} -3.28\cdot10^{-3} & -1.561\cdot10^{-3} & 0.34\cdot10^{-4} & -0.15\cdot10^{-4} & -0.397\cdot10^{-3} & -0.78\cdot10^{-4} & -0.247\cdot10^{-3} & 0 & 0 \\ -3.28\cdot10^{-3} & -1.561\cdot10^{-3} & 0.34\cdot10^{-4} & -0.15\cdot10^{-4} & 0.586\cdot10^{-3} & 0 & 0 & -0.111\cdot10^{-3} & 2.234\cdot10^{-3} \end{bmatrix}. \quad (4.7)$$

The closed-loop systems were also tested for various input functions: impulse, step, ramp and sine. Regulators, calculated on reduced order models were subsequently used to form the closed loop on the balanced full-order model.

The corresponding Lyapunov equations were solved and the traces of their solutions were compared to derive which of the closed-loop models obtained in this manner is the closest to the optimal case with a regulator having gain matrix K_{balanced}. The values of the optimal criteria for the balanced model with its optimal P regulator and the values of suboptimal criteria for the balanced model with an optimal regulator for the various reduced-order models, are seemingly identical, as in Criteria B:

$$\text{Criteria } B = [0.4584 \;\; 0.4584 \;\; 0.4584 \;\; 0.4584 \;\; 0.4584 \;\; 0.4584]. \quad (4.8)$$

From the values of Criteria B it appears that we have reached the desired approximation of the optimization criteria:

$$J_{\text{approximated}} = J_{\text{bal. I and II}} = \frac{1}{2}J_{\text{the first}} + \frac{1}{2}J_{\text{the second}} \approx 0.4584. \quad (4.9)$$

Absolute errors with respect to balanced full-order optimal case are

$$\text{Criteria Error } B = \left[0 \;\; 0.768\cdot10^{-5} \;\; 0.772\cdot10^{-5} \;\; 0.1402\cdot10^{-4} \;\; 0.1030\cdot10^{-4} \;\; 0.469\cdot10^{-5}\right]. \quad (4.10)$$

However, the difference in the optimal criteria value is in the Criteria Error B and for the balanced model with optimal regulator for the truncated model it is $0.768\cdot10^{-5}$.

For the balanced model with optimal regulator for the residualized the difference in the optimal criteria value is $0.772\cdot10^{-5}$. The difference in the optimal criteria value for the balanced model with optimal regulator for the first subsystem is $0.1402\cdot10^{-4}$. For the balanced model with optimal regulator for the second subsystem the difference is $0.1030\cdot10^{-4}$.

For the regulator composed of the $K_{\text{the first}}$ and $K_{\text{the second}}$ gain matrices as described above, the difference between the corresponding suboptimal criteria and the optimal one is the smallest of all tested reduced-order models: $0.469\cdot10^{-5}$. This concurs with the quality approximation expected from the results of [17].

The corresponding relative errors with respect to the optimal criteria value for the balanced system are

$$\text{Criteria Relative Error } B = [0\% \;\; 0.0017\% \;\; 0.0017\% \;\; 0.0031\% \;\; 0.0022\% \;\; 0.0010\%]. \quad (4.11)$$

Apparently, a combination of the first and the second optimal gain matrices produces performance of the closed-loop model that is the closest to the balanced full-order model. The relative difference in the optimal criteria value is in the Criteria Relative Error *B* and, respectively, for the balanced model using regulator optimal for the truncated model it is 0.0017%. As for the balanced model with optimal regulator for the residualized, it is identical to the above value, that is, 0.0017%. For the balanced model using regulator optimal for *the first subsystem* relative criteria error is 0.0031%, and finally, for the balanced model with optimal regulator for *the second subsystem* it is 0.0022%.

For the regulator composed of the $K_{\text{the first}}$ and $K_{\text{the second}}$ as described above, the relative difference between the corresponding suboptimal criteria and the optimal one is the smallest of all tested reduced-order models: 0.0010%. This result confirms that the approximation achieved in this way is of a good quality.

5. Conclusion

Many order reduction techniques have been developed throughout the past few decades, and this problem will stay under consideration for as long as the engineering practice keeps developing. Here are performed and combined two already established techniques for order reduction by means of multimodeling and balancing. The novel approach has given very good results in the open-loop as well as in the closed-loop model approximation for real engineering purposes. In those cases where physical interpretation of the state space variables requires that not a single variable should be lost, multimodeling provides a way to reduce order and simplify the regulator design. The cases of this kind are the real-world models of power systems and cars, for example. The choice of order reduction technique can be made more appropriate provided that all kinds of information about the original system and the related constraints are incorporated.

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