Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 457697, 9 pages
doi:10.1155/2011/457697

Research Article

Lie Symmetry Analysis of Kudryashov-Sinelshchikov Equation

Mehdi Nadjafikhah and Vahid Shirvani-Sh

Department of Mathematics, Islamic Azad University, Karaj Branch, P.O. Box 31485-313, Karaj, Iran

Correspondence should be addressed to Mehdi Nadjafikhah, m_nadjafikhah@iust.ac.ir

Received 4 May 2011; Accepted 11 June 2011

Academic Editor: Yuri Vladimirovich Mikhlin

The Lie symmetry method is performed for the fifth-order nonlinear evolution Kudryashov-Sinelshchikov equation. We will find ones and two-dimensional optimal systems of Lie subalgebras. Furthermore, preliminary classification of its group-invariant solutions is investigated.

1. Introduction

The theory of Lie symmetry groups of differential equations was developed by Lie [1]. Such Lie groups are invertible point transformations of both the dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and are of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential equations were discussed in the literature, and the most important ones are reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions, and the detection of linearizing transformations (for many other applications of Lie symmetries, see [2–4]).

In the present paper, we study the following fifth-order nonlinear evolution equation:

$$u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} = eu_{x^2}, \tag{1.1}$$

where a, b, c, d, and e are positive constants. This equation was introduced recently by Kudryashov and Sinelshchikov [5], which is the generalization of the famous Kawahara equation. By using the reductive perturbation method, they obtained (1.1). Kudryashov and Demina found some exact solutions of the generalized nonlinear evolution equations such as (1.1); see [6]. The study of nonlinear wave processes in viscoelastic tube is the important problem in such tubes similar to large arteries (see [7–9]).

In this paper, by using the Lie point symmetry method, we will investigate (1.1), looking at the representation of the obtained symmetry group on its Lie algebra. We will find the preliminary classification of group-invariant solutions, and then we can reduce (1.1) to an ordinary differential equation.

This work is organized as follows. In Section 2, we recall some results needed to construct Lie point symmetries of a given system of differential equations. In Section 3, we give the general form of an infinitesimal generator admitted by (1.1) and find transformed solutions. Section 4 is devoted to the construction of the group-invariant solutions and its classification which provides in each case reduced forms of (1.1).

2. Method of Lie Symmetries

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations (see [2, 3, 10]). To begin, let us consider that the general case of a nonlinear system of partial differential equations of order nth in p-independent and q-dependent variables is given as a system of equations

$$\Delta_\nu\left(x, u^{(n)}\right) = 0, \quad \nu = 1, \ldots, l, \tag{2.1}$$

involving $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system (2.1),

$$\begin{aligned} \tilde{x}^i &= x^i + s\xi^i(x,u) + O\left(s^2\right), \ i = 1, \ldots, p, \\ \tilde{u}^j &= u^j + s\eta^j(x,u) + O\left(s^2\right), \ j = 1, \ldots, q, \end{aligned} \tag{2.2}$$

where s is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator $\mathbf{v}$ associated with the above group of transformations can be written as

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{j=1}^{q} \eta^j(x,u)\partial_{u^j}. \tag{2.3}$$

A symmetry of a differential equation is a transformation, which maps solutions of the equation to other solutions. The invariance of the system (2.1) under the infinitesimal transformations leads to the invariance conditions (Theorem 2.36 of [2])

$$\mathrm{Pr}^{(n)}\mathbf{v}\left[\Delta_\nu\left(x, u^{(n)}\right)\right] = 0, \quad \nu = 1, \ldots, l, \ \text{whenever} \ \Delta_\nu\left(x, u^{(n)}\right) = 0, \tag{2.4}$$

where $\mathrm{Pr}^{(n)}$ is called the nth-order prolongation of the infinitesimal generator given by

$$\mathrm{Pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J}\varphi_{\alpha}^{J}\left(x, u^{(n)}\right)\partial_{u_J^{\alpha}}, \tag{2.5}$$

where $J = (j_1, \ldots, j_k)$, $1 \le j_k \le p$, $1 \le k \le n$, and the sum is over all $J^{'}$s of order $0 < \#J \le n$. If $\#J = k$, the coefficient φ_J^{α} of $\partial_{u_J^{\alpha}}$ will only depend on kth and lower-order derivatives of u,

$$\varphi_{\alpha}^{J}\left(x, u^{(n)}\right) = D_J\left(\varphi_{\alpha} - \sum_{i=1}^{p}\xi^{i}u_i^{\alpha}\right) + \sum_{i=1}^{p}\xi^{i}u_{J,i}^{\alpha}, \tag{2.6}$$

where $u_i^{\alpha} := \partial u^{\alpha}/\partial x^{i}$ and $u_{J,i}^{\alpha} := \partial u_J^{\alpha}/\partial x^{i}$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket.

3. Lie Symmetries of (1.1)

We consider the one-parameter Lie group of infinitesimal transformations on $(x^1 = x,\ x^2 = t,\ u^1 = u)$,

$$\begin{aligned}\tilde{x} &= x + s\xi(x,t,u) + O\left(s^2\right),\\ \tilde{t} &= x + s\eta(x,t,u) + O\left(s^2\right),\\ \tilde{u} &= x + s\varphi(x,t,u) + O\left(s^2\right),\end{aligned} \tag{3.1}$$

where s is the group parameter, and $\xi^1 = \xi$ and $\xi^2 = \eta$, and $\varphi^1 = \varphi$ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form

$$\mathbf{v} = \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_t + \varphi(x,t,u)\partial_u, \tag{3.2}$$

By (2.5), its fifth prolongation is

$$\begin{aligned}\mathrm{Pr}^{(5)}\mathbf{v} = \mathbf{v} &+ \varphi^{x}\partial_{u_x} + \varphi^{t}\partial_{u_t} + \varphi^{x^2}\partial_{u_{x^2}} + \varphi^{xt}\partial_{u_{xt}}\\ &+ \cdots + \varphi^{t^2}\partial_{u_{t^2}} + \varphi^{xt^4}\partial_{u_{xt^4}} + \varphi^{t^5}\partial_{u_{t^5}},\end{aligned} \tag{3.3}$$

Table 1: The commutator table.

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	0
$\mathbf{v}_2$	0	0	$\mathbf{v}_1$
$\mathbf{v}_3$	0	$-\mathbf{v}_1$	0

where, for instance, by (2.6) we have

$$\begin{aligned} \varphi^{x} &= D_x(\varphi - \xi u_x - \eta u_t) + \xi u_{x^2} + \eta u_{xt}, \\ \varphi^{t} &= D_t(\varphi - \xi u_x - \eta u_t) + \xi u_{xt} + \eta u_{t^2}, \\ &\vdots \\ \varphi^{t^5} &= D_x^5(\varphi - \xi u_x - \eta u_t) + \xi u_{x^5 t} + \eta u_{t^5}, \end{aligned} \tag{3.4}$$

where D_x and D_t are the total derivatives with respect to x and t respectively. By (2.4), the vector field $\mathbf{v}$ generates a one-parameter symmetry group of (1.1) if and only if

$$\begin{aligned} &\mathrm{Pr}^{(5)}\mathbf{v}[u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} - eu_{x^2}] = 0, \\ &\text{whenever } u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} - eu_{x^2} = 0. \end{aligned} \tag{3.5}$$

The condition (3.5) is equivalent to

$$\begin{aligned} &au_x\varphi + au\varphi^{x} + \varphi^{t} - e\varphi^{x^2} + b\varphi^{x^3} + c\varphi^{x^4} + d\varphi^{x^5} = 0, \\ &\text{whenever } u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} - eu_{x^2} = 0. \end{aligned} \tag{3.6}$$

Substituting (3.4) into (3.6), and equating the coefficients of the various monomials in partial derivatives with respect to x and various power of u, we can find the determining equations for the symmetry group of (1.1). Solving this equation, we get the following forms of the coefficient functions:

$$\xi = c_2 at + c_3, \quad \eta = c_1, \quad \varphi = c_2, \tag{3.7}$$

where c_1, c_2, and c_3 are arbitrary constants. Thus, the Lie algebra of infinitesimal symmetry of (1.1) is spanned by the three vector fields:

$$\mathbf{v}_1 = \partial_x, \ \mathbf{v}_2 = \partial_t, \ \mathbf{v}_3 = t\partial_x + \frac{1}{a}\partial_u. \tag{3.8}$$

The commutation relations between these vector fields are given in Table 1.

Theorem 3.1. *The Lie algebra $\pounds_3$ spanned by v_1, v_2, and v_3 is second Bianchi class type and it is solvable and Nilpotent [11].*

To obtain the group transformation which is generated by the infinitesimal generators $\mathbf{v}_i$ for $i = 1, 2, 3$, we need to solve the three systems of first-order ordinary differential equations

$$\begin{aligned}
\frac{d\tilde{x}(s)}{ds} &= \xi_i\left(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)\right), \quad \tilde{x}(0) = x, \\
\frac{d\tilde{t}(s)}{ds} &= \eta_i\left(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)\right), \quad \tilde{t}(0) = t, \qquad i = 1, 2, 3. \\
\frac{d\tilde{u}(s)}{ds} &= \varphi_i\left(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)\right), \quad \tilde{u}(0) = u,
\end{aligned} \tag{3.9}$$

Exponentiating the infinitesimal symmetries of (1.1), we get the one-parameter groups $G_i(s)$ generated by $\mathbf{v}_i$ for $i = 1, 2, 3$,

$$\begin{aligned}
G_1 &: (t, x, u) \longmapsto (x + s, t, u), \\
G_2 &: (t, x, u) \longmapsto (x, t + s, u), \\
G_3 &: (t, x, u) \longmapsto \left(x + ts, t, u + \frac{s}{a}\right).
\end{aligned} \tag{3.10}$$

Consequently, we have the following.

Theorem 3.2. *If $u = f(x, t)$ is a solution of* (1.1), *so are the functions*

$$\begin{aligned}
G_1(s) \cdot f(x, t) &= f(x - s, t), \\
G_2(s) \cdot f(x, t) &= f(x, t - s), \\
G_3(s) \cdot f(x, t) &= f(x - ts, t) + \frac{s}{a}.
\end{aligned} \tag{3.11}$$

4. Optimal System and Invariant Solution of (1.1)

In this section, we obtain the optimal system and reduced forms of (1.1) by using symmetry group properties obtained in the previous section. Since the original partial differential equation has two independent variables, then this partial differential equation transforms into the ordinary differential equation after reduction.

Definition 4.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subalgebras with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter subalgebras form an optimal system if every s-parameter subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation $\overline{\mathfrak{h}} = \mathrm{Ad}(g(\mathfrak{h}))$ [2].

Table 2: Adjoint representation table of the infinitesimal generators $\mathbf{v}_i$.

$\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}_j$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - \varepsilon\mathbf{v}_1$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2 + \varepsilon\mathbf{v}_1$	$\mathbf{v}_3$

Theorem 4.2. *Let H and $\overline{H}$ be connected s-dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathfrak{h}$ and $\overline{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of G, then $\overline{H} = gHg^{-1}$ are conjugate subgroups if and only if $\overline{\mathfrak{h}} = \mathrm{Ad}(g(\mathfrak{h}))$ are conjugate subalgebras [2].*

By Theorem 4.2, the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by nonzero vector in $\mathfrak{g}$. This problem is attacked by the naïve approach of taking a general element $\mathbf{V}$ in $\mathfrak{g}$ and subjecting it to various adjoint transformation so as to "simplify" it as much as possible. Thus, we will deal with th construction of the optimal system of subalgebras of $\mathfrak{g}$.

To compute the adjoint representation, we use the Lie series

$$\mathrm{Ad}\left(\exp(\varepsilon\mathbf{v}_i)\mathbf{v}_j\right) = \mathbf{v}_j - \varepsilon\left[\mathbf{v}_i, \mathbf{v}_j\right] + \frac{\varepsilon^2}{2}\left[\mathbf{v}_i, \left[\mathbf{v}_i, \mathbf{v}_j\right]\right] - \cdots, \tag{4.1}$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, 2, 3$. Then, we have Table 2.

Theorem 4.3. *An optimal system of one-dimensional Lie algebras of* (1.1) *is provided by* (1) $\mathbf{v}_2$, *and* (2) $\mathbf{v}_3 + \alpha\mathbf{v}_2$.

Proof. Consider the symmetry algebra $\mathfrak{g}$ of (1.1) whose adjoint representation was determined in Table 2, and

$$\mathbf{V} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \tag{4.2}$$

is a nonzero vector field in $\mathfrak{g}$. We will simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to $\mathbf{V}$. Suppose first that $a_3 \neq 0$. Scaling $\mathbf{V}$ if necessary, we can assume that $a_3 = 1$. Referring to Table 2, if we act on such a $\mathbf{V}$ by $\mathrm{Ad}(\exp(a_1\mathbf{v}_2))$, we can make the coefficient of $\mathbf{v}_1$ vanish, and the vector field $\mathbf{V}$ takes the form

$$\mathbf{V}' = \mathrm{Ad}\left(\exp(a_1\mathbf{v}_2)\right)\mathbf{V} = a_2'\mathbf{v}_2 + \mathbf{v}_3 \tag{4.3}$$

for certain scalar a_2'. So, depending on the sign of a_2', we can make the coefficient of $\mathbf{v}_2$ either $+1$, -1, or 0. In other words, every one-dimensional subalgebra generated by a $\mathbf{V}$ with $a_3 \neq 0$ is equivalent to one spanned by either $\mathbf{v}_3 + \mathbf{v}_2$, $\mathbf{v}_3 - \mathbf{v}_2$, or $\mathbf{v}_3$.

The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_3 = 0$. If $a_2 \neq 0$, we scale to make $a_2 = 1$, and then the vector field $\mathbf{V}$ takes the form

$$\mathbf{V}'' = a_1''\mathbf{v}_1 + \mathbf{v}_2, \tag{4.4}$$

for certain scalar a_1''. Similarly, we can vanish a_1'', so every one-dimensional subalgebra generated by a $\mathbf{V}$ with $a_3 = 0$ is equivalent to the subalgebra spanned by $\mathbf{v}_2$. □

Theorem 4.4. *An optimal system of two-dimensional Lie algebras of* (1.1) *is provided by*

$$\langle \alpha\mathbf{v}_2 + \mathbf{v}_3, \beta\mathbf{v}_1 + \gamma\mathbf{v}_3 \rangle. \tag{4.5}$$

Symmetry group method will be applied to (1.1) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

Equation (1.1) is expressed in the coordinates (x, t, u), so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (χ, ζ) corresponding to the infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to get the reduced equation.

In what follows, we begin the reduction process of (1.1); note that the reduced form does not allow us to have the self-similar solutions.

4.1. Galilean-Invariant Solutions

First, consider $\mathbf{v}_3 = t\partial_x + (1/a)\partial_u$. To determine independent invariants I, we need to solve the first partial differential equations $\mathbf{v}_i(I) = 0$, that is, invariants ζ and χ can be found by integrating the corresponding characteristic system, which is

$$\frac{dt}{0} = \frac{dx}{t} = \frac{adu}{1}. \tag{4.6}$$

The obtained solution is given by

$$\chi = t, \qquad \zeta = u - \frac{x}{at}. \tag{4.7}$$

Therefore, a solution of our equation in this case is

$$u = f(x, \chi, \zeta) = \zeta + \frac{x}{a\, t}. \tag{4.8}$$

The derivatives of u are given in terms of ζ and χ as

$$u_x = \frac{1}{at}, \; u_{x^2} = u_{x^3} = u_{x^4} = u_{x^5} = 0, \; u_t = \zeta_\chi - \frac{1}{at^2}x. \tag{4.9}$$

Substituting (4.9) into (1.1), we obtain the order ordinary differential equation

$$\zeta_{\chi} + \frac{1}{\chi}\zeta = 0. \tag{4.10}$$

The solution of this equation is $\zeta = c_1/\chi$. Consequently, we obtain that

$$u(x,t) = \frac{x + ac_1}{at}. \tag{4.11}$$

4.2. Travelling Wave Solutions

The invariants of $\mathbf{v}_2 + c_0\mathbf{v}_1 = c_0\partial_x + \partial_t$ are $\chi = x - c_0 t$ and $\zeta = u$, so the reduced form of (1.1) is

$$-c_0\zeta_{\chi} + a\zeta\zeta_{\chi} + b\zeta_{\chi^3} + c\zeta_{\chi^4} + d\zeta_{\chi^5} - e\zeta_{\chi^2} = 0. \tag{4.12}$$

The family of the periodic solution for (4.12) when $a = 1$ takes the following form (see [5]):

$$\zeta = a_0 + A\, sn^4\{m\chi, k\} + Bsn\{m\chi, k\}\frac{d}{d\chi}sn\{m\chi, k\}, \tag{4.13}$$

where $sn\{m\chi, k\}$ is Jacobi elliptic function.

The invariants of $\mathbf{v}_3 + \beta\mathbf{v}_2 = t\partial_x + \beta\partial_t + (1/a)\partial_u$ are $\chi = x - t^2/2\beta$ and $\zeta = u - t/a\beta$, so the reduced form of (1.1) is

$$\frac{1}{a\beta} - \frac{t}{\beta}\zeta_{\chi} + a\zeta\zeta_{\chi} + b\zeta_{\chi^3} + c\zeta_{\chi^4} + d\zeta_{\chi^5} - e\zeta_{\chi^2} = 0. \tag{4.14}$$

The invariants of $\mathbf{v}_2 = \partial_t$ are $\chi = x$ and $\zeta = u$, then the reduced form of (1.1) is

$$a\zeta\zeta_{\chi} + b\zeta_{\chi^3} + c\zeta_{\chi^4} + d\zeta_{\chi^5} - e\zeta_{\chi^2} = 0. \tag{4.15}$$

The invariants of $\mathbf{v}_1 = \partial_x$ are $\chi = t$ and $\zeta = u$ then the reduced form of (1.1) is $\zeta_{\chi} = 0$, then the solution of this equation is $u(x,t) = cte$.

Acknowledgments

This research was supported by Islamic Azad University of Karaj branch. The authors are grateful to the referee for careful reading and useful suggestions.

References

[1] S. Lie, *Theories der Tranformationgruppen*, Dritter und Letzter Abschnitt, Teubner, Leipzig, Germany, 1893.

[2] P. J. Olver, *Applications of Lie Groups to Differential Equations*, vol. 107, Springer, New York, NY, USA, 2nd edition, 1993.

[3] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, vol. 81, Springer, New York, NY, USA, 1989.

[4] G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations*, Springer, New York, NY, USA, 1974.

[5] N. A. Kudryashov and D. I. Sinelshchikov, "Nonlinear evolution equation for describing waves in a viscoelastic tube," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 6, pp. 2390–2396, 2011.

[6] N. A. Kudryashov and M. V. Demina, "Traveling wave solutions of the generalized nonlinear evolution equations," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 551–557, 2009.

[7] Y. C. Fung, *Biomechanics: Mechanical Properties of Living Tissues*, Springer, New York, NY, USA, 1993.

[8] N. A. Kudryashov and I. L. Chernyavskii, "Nonlinear waves in fluid flow through a viscoelastic tube," *Fluid Dynamics*, vol. 41, no. 1, pp. 49–62, 2006.

[9] C. G. Caro, T. J. Pedly, R. C. Schroter, and W. A. Seed, *The Mechanics of the Circulation*, Oxford University Press, Oxford, UK, 1978.

[10] M. Nadjafikhah, "Lie symmetries of inviscid Burgers' equation," *Advances in Applied Clifford Algebras*, vol. 19, no. 1, pp. 101–112, 2009.

[11] S. V. Khabirov, "Classification of three-dimensional Lie algebras in R^3 and their second-order differential invariants," *Lobachevskii Journal of Mathematics*, vol. 31, no. 2, pp. 152–156, 2010.