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Research Article

A Note on the Inversion of Sylvester Matrices in Control Systems

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We give a sufficient condition (the solvability of two standard equations) of Sylvester matrix by using the displacement structure of the Sylvester matrix, and, according to the sufficient condition, we derive a new fast algorithm for the inversion of a Sylvester matrix, which can be denoted as a sum of products of two triangular Toeplitz matrices. The stability of the inversion formula for a Sylvester matrix is also considered. The Sylvester matrix is numerically forward stable if it is nonsingular and well conditioned.

1. Introduction

Let $R[x]$ be the space of polynomials over the real numbers. Given univariate polynomials $f(x), g(x) \in R[x]$, $a_1 \neq 0$, where

$$f(x) = a_1 x^n + a_2 x^{n-1} + \cdots + a_{n+1}, \quad a_1 \neq 0, \qquad g(x) = b_1 x^m + b_2 x^{m-1} + \cdots + b_{m+1}, \quad b_1 \neq 0. \tag{1.1}$$

Let S denote the Sylvester matrix of $f(x)$ and $g(x)$:

$$S = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_{n+1} & & & \\ & a_1 & a_2 & \cdots & \cdots & a_{n+1} & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & a_1 & a_2 & \cdots & \cdots & a_{n+1} \\ b_1 & b_2 & \cdots & b_m & b_{m+1} & & & \\ & b_1 & b_2 & \cdots & b_m & b_{m+1} & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ & & & b_1 & b_2 & \cdots & b_m & b_{m+1} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \text{ row} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ row} \end{matrix} \tag{1.2}$$

Sylvester matrix is applied in many science and technology fields. The solutions of Sylvester matrix equations and matrix inequations play an important role in the analysis and design of control systems. In determining the greatest common divisor of two polynomials, the Sylvester matrix plays a vital role, and the magnitude of the inverse of the Sylvester matrix is important in determining the distance to the closest polynomials which have a common root. Assuming that all principal submatrices of the matrix are nonsingular, in [1], Jing Yang et al. have given a fast algorithm for the inverse of Sylvester matrix by using the displacement structure of $m+n$-order Sylvester matrix.

By using the displacement structure of the Sylvester matrix, in this paper, we give a sufficient condition (the solvability of two standard equations) of Toeplitz matrix, and, according to the sufficient condition, we derive a new fast algorithm for the inversion of a Sylvester matrix, which can be denoted as a sum of products of two triangular Toeplitz matrices. At last, the stability of the inversion formula for a Sylvester matrix is also considered. The Sylvester matrix is numerically forward stable if it is nonsingular and well conditioned.

In this paper, $\|\cdot\|_2$ always denotes the Euclidean or spectral norm and $\|\cdot\|_F$ the Frobenius norm.

2. Preliminary Notes

In this section, we present a lemma that is important to our main results.

Lemma 2.1 (see [2, Section 2.4.8]). *Let $A, B \in \mathbb{C}^{n,n}$ and $\alpha \in \mathbb{C}$. Then for any floating-point arithmetic with machine precision ε, one has that*

$$\begin{aligned} fl(\alpha A) &= \alpha A + E, \qquad \|E\|_F \le \varepsilon|\alpha|\|A\|_F \le \varepsilon\sqrt{n}|\alpha|\|A\|_2, \\ fl(A+B) &= A + B + E, \qquad \|E\|_F \le \varepsilon\|A+B\|_F \le \varepsilon\sqrt{n}\|A+B\|_2, \\ fl(AB) &= AB + E, \qquad \|E\|_F \le \varepsilon n\|A\|_F\|B\|_F. \end{aligned} \tag{2.1}$$

As usual, one neglects the errors of $O(\varepsilon^2)$, $O(\tilde{\varepsilon}^2)$, and $O(\varepsilon\tilde{\varepsilon})$.

3. Sylvester Inversion Formula

In this section, we present our main results.

Theorem 3.1. *Let matrix S be a Sylvester matrix; then it satisfies the formula*

$$KS - SK = e_m f^T - e_{m+n} g^T, \tag{3.1}$$

where $K = (0 \;\; e_1 \;\; e_2 \;\; \cdots \;\; e_{m+n-1})$ is a $(m+n) \times (m+n)$ shift matrix,

$$f = \begin{pmatrix} b_1 & \cdots & b_m & b_{m+1} - a_1 & \cdots & -a_n \end{pmatrix}^T, \qquad g = \begin{pmatrix} 0 & \cdots & 0 & b_1 & \cdots & b_m \end{pmatrix}^T. \tag{3.2}$$

Proof. We have that

$$KS - SK = \begin{pmatrix} 0 & a_1 & \cdots & a_{n+1} & 0 & \cdots & 0 \\ & & a_1 & \cdots & a_{n+1} & \cdots & 0 \\ & & & \vdots & & & \\ b_1 & \cdots & b_{m+1} & 0 & \cdots & & 0 \\ & b_1 & \cdots & b_{m+1} & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & & & \cdots & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & a_1 & \cdots & a_{n+1} & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & \cdots & a_1 & & \cdots & & a_n \\ & b_1 & \cdots & b_{m+1} & & & \\ & & & \vdots & & & \\ 0 & \cdots & 0 & b_1 & & \cdots & b_m \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & \cdots & & & 0 \\ 0 & & & \cdots & & & 0 \\ & & & \vdots & & & \\ b_1 & \cdots & b_m & b_{m+1} - a_1 & -a_2 & \cdots & -a_n \\ 0 & & & \cdots & & & 0 \\ 0 & & & \vdots & & & 0 \\ 0 & \cdots & -b_1 & \cdots & & & -b_m \end{pmatrix}$$

$$e_m f^T - e_{m+n} g^T = \begin{pmatrix} 0 & & & \cdots & & & 0 \\ 0 & & & \cdots & & & 0 \\ & & & \vdots & & & \\ b_1 & \cdots & b_m & b_{m+1} - a_1 & -a_2 & \cdots & -a_n \\ 0 & & & \cdots & & & 0 \\ 0 & & & \vdots & & & 0 \\ 0 & \cdots & -b_1 & \cdots & & & -b_m \end{pmatrix}. \tag{3.3}$$

So $KS - SK = e_m f^T - e_{m+n} g^T$. □

Theorem 3.2. *Let matrix S be a Sylvester matrix and $x = (x_1 \; x_2 \; \cdots \; x_{m+n})^T$, $y = (y_1 \; y_2 \; \cdots \; y_{m+n})^T$, $\mu = (\mu_1 \; \mu_2 \; \cdots \; \mu_{m+n})^T$, and $V = (V_1 \; V_2 \; \cdots \; V_{m+n})^T$ the solutions of the systems of equations $Sx = e_m$, $Sy = e_{m+n}$, $S^T\mu = f$, and $S^T V = g$, respectively, where e_m and e_{m+n} are both $(m+n) \times 1$ vectors; then*

(a) *S is invertible, and the column vector w_j $(j = 1, 2, \ldots, m+n)$ of S^{-1} satisfies the recurrence relation*

$$\begin{aligned} w_{m+n} &= y, \\ w_{i-1} &= K w_i + \mu_i x - V_i y, \quad i = m+n, \ldots, 3, 2, \end{aligned} \tag{3.4}$$

(b)

$$S^{-1} = S_1U_1 + S_2U_2$$
$$= \begin{pmatrix} y_{m+n} & y_{m+n-1} & \cdots & y_2 & y_1 \\ & y_{m+n} & \ddots & y_3 & y_2 \\ & & \ddots & \ddots & \vdots \\ & & & y_{m+n} & y_{m+n-1} \\ & & & & y_{m+n} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ -V_{m+n} & 1 & & & \\ \vdots & \vdots & \ddots & & \\ -V_3 & -V_4 & \cdots & 1 & \\ -V_2 & -V_3 & \cdots & -V_{m+n} & 1 \end{pmatrix}$$
$$+ \begin{pmatrix} x_{m+n} & x_{m+n-1} & \cdots & x_2 & x_1 \\ & x_{m+n} & \ddots & x_3 & x_2 \\ & & \ddots & \ddots & \vdots \\ & & & x_{m+n} & x_{m+n-1} \\ & & & & x_{m+n} \end{pmatrix} \begin{pmatrix} 0 & & & & \\ \mu_{m+n} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ \mu_3 & \mu_4 & \cdots & 0 & \\ \mu_2 & \mu_3 & \cdots & \mu_{m+n} & 0 \end{pmatrix}. \tag{3.5}$$

Proof. By Theorem 3.1 $Sx = e_m$ and $Sy = e_{m+n}$, we have that

$$KS = SK + e_m f^T - e_{m+n} g^T = SK + Sxf^T - Syg^T = S\left[K + xf^T - yg^T\right], \tag{3.6}$$

so

$$\begin{aligned} K^iS &= K^{i-1}S\left[K + xf^T - yg^T\right] \\ &= K^{i-2}KS\left[K + xf^T - yg^T\right] \\ &= K^{i-2}S\left[K + xf^T - yg^T\right]^2 \\ &\ \vdots \\ &= S\left[K + xf^T - yg^T\right]^i. \end{aligned} \tag{3.7}$$

Hence, we have that

$$K^i e_{m+n} = K^i Sy = S\left[K + xf^T - yg^T\right]^i y, \quad i = 0, 1, \ldots, \ m + n - 1. \tag{3.8}$$

Let

$$w_{m+n-i} = \left[K + xf^T - yg^T\right]^i y, \quad i = 0, 1, \ldots, \ m + n - 1. \tag{3.9}$$

It is easy to see that $K^i e_{m+n} = e_{m+n-i}$, and by (3.8)

$$Sw_{m+n-i} = K^i e_{m+n} = e_{m+n-i}, \quad i = 0, 1, \ldots, \ m+n-1. \tag{3.10}$$

From $Sy = e_{m+n}$, we have that $w_{m+n} = y$. Let $X = (w_1 \ \ w_2 \ \ \cdots \ \ w_{m+n})$; then

$$[SX = S(w_1 \ \ w_2 \ \ \cdots \ \ w_{m+n}) \ = (Sw_1 \ \ Sw_2 \ \ \cdots \ \ Sw_{m+n}) = I_{m+n}], \tag{3.11}$$

so the matrix S is invertible and the inverse of S is the matrix X.

From (3.1), we have that

$$S^{-1}(KS - SK)S^{-1} = S^{-1}\left(e_m f^T - e_{m+n} g^T\right)S^{-1}, \tag{3.12}$$

and thus

$$S^{-1}K - KS^{-1} = x\mu^T - yV^T. \tag{3.13}$$

So

$$\left(S^{-1}K - KS^{-1}\right)e_i = \left(x\mu^T - yV^T\right)e_i. \tag{3.14}$$

Since $Ke_i = e_{i-1}$, we have that

$$w_{i-1} = Ks_i + \mu_i x - V_i y, \quad i = m+n, \ldots, 3, 2, \tag{3.15}$$

and hence

$$\begin{gathered} w_{m+n} = y, \\ w_{i-1} = Kw_i + \mu_i x - V_i y, \quad i = m+n, \ldots, 3, 2. \end{gathered} \tag{3.16}$$

For (b), by(3.4)

$$\begin{gathered} w_{m+n} = y, \\ w_{m+n-1} = Ky + \mu_{m+n} x - V_{m+n} y, \\ w_{m+n-2} = K^2 y + K\mu_{m+n} x - KV_{m+n} y + \mu_{m+n-1} x - V_{m+n-1} y, \\ \cdots \\ w_1 = K^{m+n-1} y + K^{m+n-2}\mu_{m+n} x - K^{m+n-2} V_{m+n} y + \cdots + \mu_2 x - V_2 y. \end{gathered} \tag{3.17}$$

So

$$
\begin{aligned}
S^{-1} &= \begin{pmatrix} w_1 & w_2 & \cdots & w_{m+n} \end{pmatrix} \\
&= \begin{pmatrix} K^{m+n-1}y & K^{m+n-2}y & \cdots & Ky & y \end{pmatrix}
\begin{pmatrix} 1 & & & & \\ -V_{m+n} & 1 & & & \\ \vdots & \vdots & \ddots & & \\ -V_3 & -V_4 & \cdots & 1 & \\ -V_2 & -V_3 & \cdots & -V_{m+n} & 1 \end{pmatrix} \\
&\quad + \begin{pmatrix} K^{m+n-1}x & K^{m+n-2}x & \cdots & Kx & x \end{pmatrix}
\begin{pmatrix} 0 & & & & \\ \mu_{m+n} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ \mu_3 & \mu_4 & \cdots & 0 & \\ \mu_2 & \mu_3 & \cdots & \mu_{m+n} & 0 \end{pmatrix} \\
&= \begin{pmatrix} y_{m+n} & y_{m+n-1} & \cdots & y_2 & y_1 \\ & y_{m+n} & \ddots & y_3 & y_2 \\ & & \ddots & \ddots & \vdots \\ & & & y_{m+n} & y_{m+n-1} \\ & & & & y_{m+n} \end{pmatrix}
\begin{pmatrix} 1 & & & & \\ -V_{m+n} & 1 & & & \\ \vdots & \vdots & \ddots & & \\ -V_3 & -V_4 & \cdots & 1 & \\ -V_2 & -V_3 & \cdots & -V_{m+n} & 1 \end{pmatrix} \\
&\quad + \begin{pmatrix} x_{m+n} & x_{m+n-1} & \cdots & x_2 & x_1 \\ & x_{m+n} & \ddots & x_3 & x_2 \\ & & \ddots & \ddots & \vdots \\ & & & x_{m+n} & x_{m+n-1} \\ & & & & x_{m+n} \end{pmatrix}
\begin{pmatrix} 0 & & & & \\ \mu_{m+n} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ \mu_3 & \mu_4 & \cdots & 0 & \\ \mu_2 & \mu_3 & \cdots & \mu_{m+n} & 0 \end{pmatrix}.
\end{aligned}
\tag{3.18}
$$

This completes the proof. □

4. Stability Analysis

In this section, we will show that the Sylvester inversion formula presented in this paper is evaluation forward stable.

If, for all well conditioned problems, the computed solution $\tilde{x}$ is close to the true solution x, in the sense that the relative error $\|x - \tilde{x}\|_2 / \|x\|_2$ is small, then we call the algorithm forward stable (the author called this weakly in [3]). Round-off errors will occur in the matrix computation.

Theorem 4.1. *Let matrix S be a nonsingular Sylvester matrix and well conditioned; then the formula in Theorem 3.2 is forward stable.*

Proof. Assume that we have computed the solutions $\tilde{x}$, $\tilde{y}$, $\tilde{\mu}$, and $\tilde{V}$ of $Sx = e_m$, $Sy = e_{m+n}$, $S^T\mu = f$, and $S^T V = g$ in Theorem 3.2 which are perturbed by the normwise relative errors bounded by $\tilde{\varepsilon}$,

$$\|\tilde{x}\|_2 \le \|x\|_2(1+\tilde{\varepsilon}), \qquad \|\tilde{y}\|_2 \le \|y\|_2(1+\tilde{\varepsilon}), \qquad \|\tilde{\mu}\|_2 \le \|\mu\|_2(1+\tilde{\varepsilon}), \qquad \left\|\tilde{V}\right\|_2 \le \|V\|_2(1+\tilde{\varepsilon}). \tag{4.1}$$

Thus, we have that

$$\begin{aligned} &\|S_1\|_F \le \sqrt{m+n}\|y\|_2, \qquad \|S_2\|_F \le \sqrt{m+n}\|x\|_2, \\ &\|U_1\|_F \le \sqrt{m+n}\sqrt{1+\|V\|_2^2}, \qquad \|U_2\|_F \le \sqrt{m+n}\|\mu\|_2 \,. \end{aligned} \tag{4.2}$$

The inversion formula in Theorem 3.2, using the perturbed solutions $\tilde{x}$, $\tilde{y}$, $\tilde{\mu}$, and $\tilde{V}$, can be expressed as

$$\begin{aligned} \tilde{S}^{-1} &= fl\left(\tilde{S}_1\tilde{U}_1 + \tilde{S}_2\tilde{U}_2\right) \\ &= fl((S_1+\Delta S_1)(U_1+\Delta U_1) + (S_2+\Delta S_2)(U_2+\Delta U_2)) \\ &= S^{-1} + \Delta S_1 U_1 + S_1\Delta U_1 + \Delta S_2 U_2 + S_2 \Delta U_2 + E + F. \end{aligned} \tag{4.3}$$

Here, and in the sequel, E is the matrix containing the error which results from computing the matrix products and F contains the error from subtracting the matrices. For the error matrices ΔS_1, ΔU_1, ΔS_2, and ΔU_2, we have that

$$\begin{aligned} &\|\Delta S_1\|_F \le \tilde{\varepsilon}\|S_1\|_F \le \tilde{\varepsilon}\sqrt{m+n}\|y\|_2, \qquad \|\Delta S_2\|_F \le \tilde{\varepsilon}\|S_2\|_F \le \tilde{\varepsilon}\sqrt{m+n}\|x\|_2, \\ &\|\Delta U_1\|_F \le \tilde{\varepsilon}\|U_1\|_F \le \tilde{\varepsilon}\sqrt{m+n}\sqrt{1+\|V\|_2^2}, \qquad \|\Delta U_2\|_F \le \tilde{\varepsilon}\|U_2\|_F \le \tilde{\varepsilon}\sqrt{m+n}\|\mu\|_2. \end{aligned} \tag{4.4}$$

By Lemma 2.1, we have the following bounds on E and F:

$$\begin{aligned} \|E\|_2 &\le \|E\|_F \\ &\le \varepsilon(m+n)(\|S_1\|_F\|U_1\|_F + \|S_2\|_F\|U_2\|_F) \\ &\le \varepsilon(m+n)^2\left(\|y\|_2\sqrt{1+\|V\|_2^2} + \|x\|_2\|\mu\|_2\right) \\ &\le \varepsilon(m+n)^2\left(\|y\|_2\left(1+\|V\|_2^2\right) + \|x\|_2\|\mu\|_2\right) \\ \|F\|_2 &\le \sqrt{m+n}\varepsilon\left\|S^{-1}\right\|_2. \end{aligned} \tag{4.5}$$

Consequently, adding all these error bounds, by (4.3), we have that

$$\left\|\widetilde{S}^{-1}-S^{-1}\right\|_2 \le (m+n)(2\widetilde{\varepsilon}+(m+n)\varepsilon)\left(\|y\|_2\left(1+\|V\|_2^2\right)+\|x\|_2\|\mu\|_2\right)+\sqrt{m+n}\varepsilon\left\|S^{-1}\right\|_2. \tag{4.6}$$

From the equations $Sx = e_m$, $Sy = e_{m+n}S^T\mu = f$, and $S^TV = g$ in Theorem 3.2, we have that

$$\|y\|_2 \le \left\|S^{-1}\right\|_2, \quad \|V\|_2 \le \left\|S^{-1}\right\|_2\|g\|_2, \quad \|x\|_2 \le \left\|S^{-1}\right\|_2, \quad \|\mu\|_2 \le \left\|S^{-1}\right\|_2\|f\|_2. \tag{4.7}$$

Thus, the relative error is

$$\begin{aligned}\frac{\left\|\widetilde{S}^{-1}-S^{-1}\right\|_2}{\|S^{-1}\|_2} &\le \frac{(m+n)(2\widetilde{\varepsilon}+(m+n)\varepsilon)\left(\|y\|_2\left(1+\|V\|_2^2\right)+\|x\|_2\|\mu\|_2\right)_2}{\|S^{-1}\|_2}+\varepsilon\sqrt{m+n} \\ &\le (m+n)(2\widetilde{\varepsilon}+(m+n)\varepsilon)\left(1+\left\|S^{-1}\right\|_2(\|g\|_2+\|f\|_2)\right)+\varepsilon\sqrt{m+n}.\end{aligned} \tag{4.8}$$

Since S is well conditioned, $\|S^{-1}\|_2$ is finite. It is easy to see that $\|g\|_2, \|f\|_2$ are finite. Therefore, the formula presented in Theorem 3.2 is forward stable.

This completes the proof. □

5. Numerical Example

This section gives an example to illustrate our results. All the following tests are performed by MATLAB 7.0.

Example 5.1. Given $f(x) = x+1$, $g(x) = x^2+x+1$, that is, $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $n = 1$, $m = 2$, $f = (1,1,0)^{\mathrm{T}}$, $g = (0,1,1)^T$, and $S = \left(\begin{smallmatrix}1&1&0\\0&1&1\\1&1&1\end{smallmatrix}\right)$.

So

$$\begin{aligned}KS-SK &= \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\1&1&1\end{pmatrix}-\begin{pmatrix}1&1&0\\0&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} \\ &= \begin{pmatrix}0&1&1\\1&1&1\\0&0&0\end{pmatrix}-\begin{pmatrix}0&1&1\\0&0&1\\0&1&1\end{pmatrix}=\begin{pmatrix}0&0&0\\1&1&0\\0&-1&-1\end{pmatrix},\end{aligned} \tag{5.1}$$

$$e_mf^T-e_{m+n}g^T = \begin{pmatrix}0\\1\\0\end{pmatrix}(1\ \ 1\ \ 0)-\begin{pmatrix}0\\0\\1\end{pmatrix}(0\ \ 1\ \ 1)=\begin{pmatrix}0&0&0\\1&1&0\\0&-1&-1\end{pmatrix}. \tag{5.2}$$

Therefore, $KS-SK = e_mf^T-e_{m+n}g^T$.

By the condition of Theorem 3.2, we can get

$$x = (-1, 1, 0)^T, \qquad y = (1, -1, 1)^T, \qquad \mu = (1, 0, 0)^T, \qquad V = (0, 1, 0)^T. \tag{5.3}$$

Obviously, S is invertible and $S^{-1} = \left(\begin{smallmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{smallmatrix}\right)$. And it is easy to see that $w_3 = y$,

$$\begin{aligned}
w_1 &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\
&= Sw_2 + \mu_2 x - V_2 y, \\
w_2 &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\
&= Sw_3 + \mu_3 x - V_3 y, \\
S^{-1} &= \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 & 1 \\ & 1 & -1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 0 & 0 \end{pmatrix} \\
&= S_1 U_1 + S_2 U_2.
\end{aligned} \tag{5.4}$$

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References

[1] J. Yang, Z. Xu, and Q. Lu, "A fast algorithm for the inverse of Sylvester matrices," *Journal on Numerical Methods and Computer Applications*, vol. 31, no. 2, pp. 92–98, 2010.

[2] G. H. Golub and C. F. Van Loan, *Matrix Computations*, vol. 3 of *Johns Hopkins Series in the Mathematical Sciences*, Johns Hopkins University Press, Baltimore, Md, USA, 2nd edition, 1989.

[3] J. R. Bunch, "The weak and strong stability of algorithms in numerical linear algebra," *Linear Algebra and Its Applications*, vol. 88/89, pp. 49–66, 1987.


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