Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 810324, 18 pages
doi:10.1155/2011/810324

Research Article

Solution of Higher-Order ODEs Using Backward Difference Method

Mohamed Bin Suleiman, Zarina Bibi Binti Ibrahim, and Ahmad Fadly Nurullah Bin Rasedee

Department of Mathematics, Faculty of Science, UPM, Selangor Darul Ehsan, 43400 Serdang, Malaysia

Correspondence should be addressed to Mohamed Bin Suleiman, mohamed@math.upm.edu.my

Received 9 November 2010; Revised 25 March 2011; Accepted 13 May 2011

Academic Editor: Francesco Pellicano

The current numerical technique for solving a system of higher-order ordinary differential equations (ODEs) is to reduce it to a system of first-order equations then solving it using first-order ODE methods. Here, we propose a method to solve higher-order ODEs directly. The formulae will be derived in terms of backward difference in a constant stepsize formulation. The method developed will be validated by solving some higher-order ODEs directly with constant stepsize. To simplify the evaluations of the integration coefficients, we find the relationship between various orders. The result presented confirmed our hypothesis.

1. Introduction

Differential equations constantly arise in various branches of science and engineering. Many of these problems are in the form of higher-order ordinary differential equations (ODEs). A few examples where these problems can be found are, in the motion of projectiles, the bending of a thin clamped beam and population growth.

The popular practice for solving a system of higher-order ODEs is by reducing it to a system of first-order equations and then solving with first-order methods. These methods worked, so that methods for solving higher-order ODEs have been disregarded as robust codes. However, the work by Krogh [1], Suleiman [2], Majid and Suleiman [3], and Omar and Suleiman [4] has revived the interest in solving higher-order ODEs directly and the theoretical development of the methods.

Related works for solving higher-order ODEs can be found in Collatz [5], Gear [6], Krogh [1, 7], and Suleiman [2]. Krogh [7] proposed the direct integration (DI) method for nonstiff problems using modified divided difference while Suleiman [2] proposed the DI method using the standard divided difference. In this paper, we will derive the constant stepsize backward difference formulae of solving higher-order ODEs up to third order. The main reason for developing the constant stepsize formulae is that, in developing the theory on convergence and stability, the approach is through constant stepsize formulation. Another reason is that it is possible to use this formula in conjunction with other similar formulae as in Majid and Suleiman [3] to develop a code for variable stepsize and order.

The advantage of such a code is that the integration or differentiation constants are calculated only once at the start of the first step of integration, whereas other formulations calculate the constants at every step.

In this paper, we will focus only on nonstiff ODEs of the form

$$y^{(d)} = f\left(x, \tilde{Y}\right), \tag{1.1}$$

$$\begin{gathered} \tilde{Y}(x) = f\left(x, y, y', y'', \dots, y^{(d-1)}\right), \\ \tilde{\eta} = \left(\eta, \eta', \eta'', \dots, \eta^{(d-1)}\right), \end{gathered} \tag{1.2}$$

where $\tilde{Y}(a) = \tilde{\eta}$ in the interval $a \leq x \leq b$ and d is the order of the ODE.

Without loss of generality, we will be considering the scalar equation in (1.1).

This paper will be organized as follows. First, the integration coefficients of the explicit constant stepsize backward difference formulation of the DI method will be derived. Then, the coefficients of the implicit method are formulated and their relationship with the explicit coefficients is shown. We start the derivation with the coefficients of the first-order system, which is given in Henrici [8]. Next, the second-order coefficients are derived and their relationship with the corresponding first-order coefficients is given, likewise the relationship of the coefficients for the second- and third-order systems. Finally, the method developed using backward difference will be validated numerically.

2. The Formulation of the Predict-Evaluate-Correct-Evaluate (PECE) Multistep Method in Its Backward Difference Form (MSBD) for Nonstiff Higher-Order ODEs

The code developed will be using the PECE mode. The predictor and corrector will have the following form:

predictor:

$${}^{pr}y_{n+1}^{(d-t)} = \sum_{i=0}^{t-1} \frac{h^i}{i!} y_n^{(d-t+1)} + h^d \sum_{i=0}^{k-1} \gamma_{(d-t),i} \nabla^i f_n, \quad t = 1, 2, \dots, d, \tag{2.1}$$

corrector:

$$y_{n+1}^{(d-t)} = \sum_{i=0}^{t-1} \frac{h^i}{i!} y_n^{(d-t+1)} + h^d \sum_{i=0}^{k-1} \gamma^*_{(d-t),i} \nabla^i f_{n+1}, \quad t = 1, 2, \dots, d. \tag{2.2}$$

The corrector will be reformulated, so that it will be in terms of the predictor. The reformulated corrector can be written as

$$y_{n+1}^{(d-t)} = {}^{pr}y_{n+1}^{(d-t)} + \gamma_{(d-t),k}\nabla^{kpr}\left(f_{n+1}\right), \quad t = 1,2,\ldots,d, \tag{2.3}$$

where ${}^{pr}f_{n+1}$ indicates f_{n+1} evaluated using predicted values. The integration coefficient $\gamma_{(d-t),i}$ and $\gamma^{*}_{(d-t),i}$ will be derived using the method of generating function. Finally, the constant stepsize algorithm will be constructed and validated with some test problems and examples from physical situations.

3. Derivation up to Third-Order Explicit Integration Coefficients

Integrating (1.1) once yields

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f\left(x, y, y', y''\right) dx. \tag{3.1}$$

Let $P_n(x)$ be the interpolating polynomial which interpolates the k values (x_n, f_n), $(x_{n-1}, f_{n-1}), \ldots, (x_{n-k+1}, f_{n-k+1})$, then

$$P_n(x) = \sum_{i=0}^{k-1}(-1)^i\binom{-s}{i}\nabla^i f_n. \tag{3.2}$$

Next, approximating f in (3.1) with $P_n(x)$ and letting

$$x = x_n + sh \quad \text{or} \quad s = \frac{x - x_n}{h} \tag{3.3}$$

gives us

$$y(x_{n+1}) = y(x_n) + \int_0^1 \sum_{i=0}^{k-1}(-1)^i\binom{-s}{i}\nabla^i f_n\, ds, \tag{3.4}$$

or

$$y(x_{n+1}) = y(x_n) + h\sum_{i=0}^{k-1}\gamma_{1,i}\nabla^i f_n, \tag{3.5}$$

where

$$\gamma_{1,i} = (-1)^i\int_0^1\binom{-s}{i} ds. \tag{3.6}$$

The generating function $G_1(t)$ for the coefficients $\gamma_{1,i}$ is defined as follows:

$$G_1(t) = \sum_{i=0}^{\infty} \gamma_{1,i} t^i. \tag{3.7}$$

Substituting $\gamma_{1,i}$ in (3.6) in $G_1(t)$ gives

$$\begin{aligned} G_1(t) &= \sum_{i=0}^{\infty} (-t)^i \int_0^1 \binom{-s}{i} ds, \\ G_1(t) &= \int_0^1 (1-t)^{(-s)} ds, \\ G_1(t) &= \int_0^1 e^{-s\ \log(1-t)}\, ds \end{aligned} \tag{3.8}$$

which leads to

$$G_1(t) = -\left[\frac{(1-t)^{-1}}{\log(1-t)} - \frac{1}{\log(1-t)}\right]. \tag{3.9}$$

Equation (3.9) can be written as

$$-\left(\sum_{i=0}^{\infty} \gamma_{1,i} t^i\right) \log(1-t) = \left[\frac{t}{(1-t)}\right] \tag{3.10}$$

or

$$\left(\gamma_{1,0} + \gamma_{1,1} t + \gamma_{1,2} t^2 + \gamma_{1,3} t^3 + \cdots\right)\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots\right) = t\left(1 + t + t^2 + t^3 + \cdots\right). \tag{3.11}$$

Hence, the coefficients of $\gamma_{1,k}$ are given by

$$\begin{aligned} &\sum_{i=0}^{k}\left(\frac{\gamma_{1,i}}{k-i+1}\right) = 1, \\ &\gamma_{1,k} = 1 - \sum_{i=0}^{k-1}\left(\frac{\gamma_{1,i}}{k-i+1}\right) \quad k = 1,2,\ldots,\ \gamma_{1,0} = 1. \end{aligned} \tag{3.12}$$

4. Second-Order ODE Formulae

Integrate (1.1) twice for second-order ODEs where $d = 2$. Integrating once leads to the same coefficients as given in (3.6). Integrating twice yields

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + h^2 \sum_{i=0}^{k} \gamma_{2,i} \nabla^i f_n. \tag{4.1}$$

Substituting x with s gives

$$\gamma_{2,i} = (-1)^i \int_0^1 \frac{(1-s)}{1!} \binom{-s}{i} ds. \tag{4.2}$$

The generating function $G_2(t)$ of the coefficients $\gamma_{2,i}$ is defined as follows

$$G_2(t) = \sum_{i=0}^{\infty} \gamma_{2,i} t^i. \tag{4.3}$$

Substituting (4.2) into $G_2(t)$ above gives

$$G_2(t) = \int_0^1 \frac{(1-s)}{1!} e^{-s\log(1-t)} ds. \tag{4.4}$$

Substituting $G_1(t)$ into (4.4) yields

$$G_2(t) = \frac{1}{1!}\left[\frac{1}{\log(1-t)} - \frac{1!G_1(t)}{\log(1-t)}\right]. \tag{4.5}$$

Equation (4.5) can be written as

$$\left(\sum_{i=0}^{\infty} \gamma_{2,i} t^i\right) \log(1-t) = \frac{1}{1!}[1 - 1!G_1(t)] \tag{4.6}$$

or

$$\left(\gamma_{2,0} + \gamma_{2,1}t + \gamma_{2,2}t^2 + \cdots\right)\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots\right) = \frac{1}{1!}\left[-1 + 1!\left(\gamma_{1,0} + \gamma_{1,1}t + \gamma_{1,2}t^2 + \cdots\right)\right]. \tag{4.7}$$

Hence the coefficients of $\gamma_{2,k}$ in relation to coefficients of the previous order $\gamma_{1,k}$ are given by

$$\sum_{i=0}^{k} \frac{\gamma_{2,i}}{k-i+1} = \gamma_{1,k+1},$$
$$\gamma_{2,0} = \gamma_{1,1}, \quad \gamma_{2,k} = \gamma_{1,k+1} - \sum_{i=0}^{k} \frac{\gamma_{2,i}}{k-i+1}, \quad k = 1, 2, \ldots. \tag{4.8}$$

5. Third-Order Formulae

Next, the case of the third-order ODE where $d = 3$ will be considered. In the case of y'', y', the corresponding coefficients are $\gamma_{1,i}, \gamma_{2,i}$ as in (3.6) and (4.2). For the solution $y(x)$, integrating three times yields

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \int_{x_n}^{x_{n+1}} \frac{(x_{n+1}-x)^{(2)}}{(2)!} f(x, y, y', y'')\,dx \tag{5.1}$$

or in the backward difference formulation, given by

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + h^3 \sum_{i=0}^{k} \gamma_{3,i} \nabla^i f_n, \tag{5.2}$$

where

$$\gamma_{3,i} = (-1)^i \int_0^1 \frac{(1-s)^2}{2!} \binom{-s}{i} ds. \tag{5.3}$$

The generating function $G_3(t)$ of the coefficients $\gamma_{3,i}$ is defined as follows:

$$G_3(t) = \sum_{i=0}^{\infty} \gamma_{3,i} t^i. \tag{5.4}$$

Substituting (5.3) into $G_3(t)$ above yields

$$G_3(t) = \int_0^1 \frac{(1-s)^2}{2!} e^{-s\log(1-t)} ds. \tag{5.5}$$

As in (4.4), we now substitute $G_2(t)$ in (5.5) which gives

$$G_3(t) = \frac{1}{2!}\left[\frac{1}{\log(1-t)} - \frac{2!G_2(t)}{\log(1-t)}\right]. \tag{5.6}$$

Equation (5.6) can be written as

$$\left(\sum_{i=0}^{\infty}\gamma_{3,i}t^i\right)\log(1-t) = \frac{1}{2!}[1-2!G_2(t)] \tag{5.7}$$

or

$$\left(\gamma_{3,0}+\gamma_{3,1}t+\gamma_{3,2}t^2+\cdots\right)\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots\right) = \frac{1}{2!}\left[-1+2!\left(\gamma_{2,0}+\gamma_{2,1}t+\gamma_{2,2}t^2+\cdots\right)\right]. \tag{5.8}$$

Hence, the coefficients of $\gamma_{3,k}$ in relation to coefficients of the previous order $\gamma_{2,k}$ are given by

$$\begin{gathered}\sum_{i=0}^{k}\frac{\gamma_{3,i}}{k-i+1} = \gamma_{2,k+1}, \\ \gamma_{3,0} = \gamma_{2,1}, \quad \gamma_{3,k} = \gamma_{2,k+1} - \sum_{i=0}^{k}\frac{\gamma_{3,i}}{k-i+1}, \quad k = 1,2,\ldots.\end{gathered} \tag{5.9}$$

6. Derivation up to the Third-Order Implicit Integration Coefficients

Integrating (1.1) once yields

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f\left(x,y,y',y''\right)dx. \tag{6.1}$$

Let $P_n(x)$ be the interpolating polynomial which interpolates the k values (x_n, f_n), $(x_{n-1}, f_{n-1}),\ldots,(x_{n-k+1}, f_{n-k+1})$:

$$P_n(x) = \sum_{i=0}^{k}(-1)^i\binom{-s}{i}\nabla^i f_{n+1}. \tag{6.2}$$

As in the previous derivation, we choose

$$x = x_{n+1} + sh, \quad \text{or} \quad s = \frac{x-x_{n+1}}{h}. \tag{6.3}$$

Replacing x by s yields

$$y(x_{n+1}) = y(x_n) + \int_{-1}^{0}\sum_{i=0}^{k}(-1)^i\binom{-s}{i}\nabla^i f_{n+1}ds. \tag{6.4}$$

Simplify

$$y(x_{n+1}) = y(x_n) + h\sum_{i=0}^{k} \gamma_{1,i}^{*} \nabla^i f_n, \tag{6.5}$$

where

$$\gamma_{1,i}^{*} = (-1)^i \int_{-1}^{0} \binom{-s}{i} ds. \tag{6.6}$$

The generating function $G_1^*(t)$ of the coefficients $\gamma_{1,i}^*$ is defined as follows:

$$G_1^*(t) = \sum_{i=0}^{\infty} \gamma_{1,i}^{*} t^i \tag{6.7}$$

or

$$\begin{aligned} G_1^*(t) &= \sum_{i=0}^{\infty} (-t)^i \int_{-1}^{0} \binom{-s}{i} ds, \\ G_1^*(t) &= \int_{-1}^{0} (1-t)^{(-s)} ds, \\ G_1^*(t) &= \int_{-1}^{0} e^{-s \ \log(1-t)} ds \end{aligned} \tag{6.8}$$

which leads to

$$G_1^*(t) = -\left[\frac{1}{\log(1-t)} - \frac{(1-t)}{\log(1-t)}\right]. \tag{6.9}$$

For the case $d = 2$, the approximate solution of y has the form

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \int_{x_n}^{x_{n+1}} \frac{(x_{n+1} - x)}{(1)!}^{(1)} f(x, y, y', y'') dx. \tag{6.10}$$

The coefficients are given by

$$\gamma_{2,i}^{*} = (-1)^i \int_{-1}^{0} \frac{(-s)}{1!} \binom{-s}{i} ds, \tag{6.11}$$

where $\gamma_{2,i}^*$ are the coefficients of the backward difference formulation of (6.11) which can be represented by

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + h^2 \sum_{i=0}^{k} \gamma_{2,i}^* \nabla^i f_n. \tag{6.12}$$

The generating function $G_2^*(t)$ for the coefficients $\gamma_{2,i}^*$ is defined as follows:

$$G_2^*(t) = \sum_{i=0}^{\infty} \gamma_{2,i}^* t^i. \tag{6.13}$$

Substituting (6.11) into $G_2^*(t)$ above,

$$G_2^*(t) = \int_{-1}^{0} \frac{(-s)}{1!} e^{-s \log(1-t)} ds. \tag{6.14}$$

Solving (6.14) with the substitution of (3.8) produces the relationship

$$G_2^*(t) = \frac{1}{1!} \left[\frac{(1-t)}{\log(1-t)} - \frac{1! G_1^*(t)}{\log(1-t)} \right]. \tag{6.15}$$

Integrating (1.1) thrice yields

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(x_n) + \int_{x_n}^{x_{n+1}} \frac{(x_{n+1} - x)^{(2)}}{(2)!} f(x, y, y', y'') dx. \tag{6.16}$$

The coefficients are given by

$$\gamma_{3,i}^* = (-1)^i \int_{-1}^{0} \frac{(-s)^2}{2!} \binom{-s}{i} ds, \tag{6.17}$$

where $\gamma_{3,i}^*$ are the coefficients of the backward difference formulation of (6.17) which can be represented by

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(x_n) + h^3 \sum_{i=0}^{k} \gamma_{3,i}^* \nabla^i f_n. \tag{6.18}$$

The generating function $G_3^*(t)$ of the coefficients $\gamma_{3,i}^*$ is defined as follows:

$$G_3^*(t) = \sum_{i=0}^{\infty} \gamma_{3,i}^* t^i. \tag{6.19}$$

Substituting (6.17) into $G_3^*(t)$ above yields

$$G_3^*(t) = \int_{-1}^{0} \frac{(-s)^2}{2!} e^{-s\log(1-t)} ds. \tag{6.20}$$

Solving (6.11) with the substitution of (6.20) produces the relationship

$$G_3^*(t) = \frac{1}{2!}\left[\frac{(1-t)}{\log(1-t)} - \frac{2!G_2^*(t)}{\log(1-t)}\right]. \tag{6.21}$$

7. The Relationship between the Explicit and Implicit Integration Coefficients

Calculating the integration coefficients directly is time consuming when large numbers of integration are involved. A more efficient way of computing the coefficients is by obtaining a recursive relationship between the coefficients. With this recursive relationship, we are able to obtain the implicit integration coefficient with minimal time consumption. The relationship between the explicit and implicit coefficients is expressed below.

For first-order coefficients,

$$G_1^*(t) = -\left[\frac{1}{\log(1-t)} - \frac{1-t}{\log(1-t)}\right]. \tag{7.1}$$

It can be written as

$$G_1^*(t) = -(1-t)\left[\frac{1}{(1-t)\log(1-t)} - \frac{1}{\log(1-t)}\right]. \tag{7.2}$$

By substituting

$$G_1(t) = \frac{1}{(1-t)\log(1-t)} - \frac{1}{\log(1-t)} \tag{7.3}$$

into (7.2), we have

$$\begin{gathered} G_1^*(t) = (1-t)G_1(t), \\ \left(\sum_{i=0}^{\infty}\gamma_{1,i}^* t^i\right) = (1-t)\left(\sum_{i=0}^{\infty}\gamma_{1,i} t^i\right). \end{gathered} \tag{7.4}$$

Expanding the equation yields

$$\begin{gathered} \left(\gamma_{1,0}^* + \gamma_{1,1}^* t + \gamma_{1,2}^* t^2 + \cdots\right) = \frac{1}{(1+t+t^2+\cdots)}\left(\gamma_{1,0} + \gamma_{1,1}t + \gamma_{1,2}t^2 + \cdots\right), \\ \left(\gamma_{1,0}^* + \gamma_{1,1}^* t + \gamma_{1,2}^* t^2 + \cdots\right)\left(1 + t + t^2 + \cdots\right) = \left(\gamma_{1,0} + \gamma_{1,1}t + \gamma_{1,2}t^2 + \cdots\right). \end{gathered} \tag{7.5}$$

This gives the recursive relationship

$$\sum_{i=0}^{k} \gamma_{1,i}^{*} = \gamma_{1,k}. \tag{7.6}$$

For second-order coefficient,

$$G_2^*(t) = -\frac{1}{1!}\left[\frac{1}{\log(1-t)} - \frac{1!G_1^*(t)}{\log(1-t)}\right]. \tag{7.7}$$

It can be written as

$$G_2^*(t) = \frac{(1-t)}{1!}\left[\frac{1}{\log(1-t)} - \frac{1!G_1^*(t)}{(1-t)\log(1-t)}\right]. \tag{7.8}$$

Substituting (7.4) into the equation above gives

$$G_2^*(t) = \frac{(1-t)}{1!}\left[\frac{1}{\log(1-t)} - \frac{1!(1-t)G_1(t)}{(1-t)\log(1-t)}\right] \tag{7.9}$$

or

$$G_2^*(t) = \frac{(1-t)}{1!}\left[\frac{1}{\log(1-t)} - \frac{1!\ G_1(t)}{\log(1-t)}\right]. \tag{7.10}$$

Substituting (4.5) into (7.10) gives

$$\begin{gathered} G_2^*(t) = (1-t)G_2(t), \\ \left(\sum_{i=0}^{\infty} \gamma_{2,i}^{*} t^i\right) = (1-t)\left(\sum_{i=0}^{\infty} \gamma_{2,i} t^i\right). \end{gathered} \tag{7.11}$$

Expanding the equation, we have

$$\begin{gathered} \left(\gamma_{2,0}^* + \gamma_{2,1}^* t + \gamma_{2,2}^* t^2 + \cdots\right) = \frac{1}{(1+t+t^2+\cdots)}\left(\gamma_{2,0} + \gamma_{2,1} t + \gamma_{2,2} t^2 + \cdots\right), \\ \left(\gamma_{2,0}^* + \gamma_{2,1}^* t + \gamma_{2,2}^* t^2 + \cdots\right)\left(1+t+t^2+\cdots\right) = \left(\gamma_{2,0} + \gamma_{2,1} t + \gamma_{2,2} t^2 + \cdots\right). \end{gathered} \tag{7.12}$$

The above gives the relationship

$$\sum_{i=0}^{k} \gamma_{2,i}^{*} = \gamma_{2,k}. \tag{7.13}$$

For third-order coefficient, we have

$$G_3^*(t) = -\frac{1}{2!}\left[\frac{1}{\log(1-t)} - \frac{2!G_2^*(t)}{\log(1-t)}\right]. \tag{7.14}$$

It can be written as

$$G_3^*(t) = \frac{(1-t)}{2!}\left[\frac{1}{\log(1-t)} - \frac{2!G_2^*(t)}{(1-t)\log(1-t)}\right]. \tag{7.15}$$

Substituting (7.10) into (7.15) gives

$$G_3^*(t) = \frac{(1-t)}{2!}\left[\frac{1}{\log(1-t)} - \frac{2!(1-t)G_2(t)}{(1-t)\log(1-t)}\right] \tag{7.16}$$

or

$$G_3^*(t) = \frac{(1-t)}{2!}\left[\frac{1}{\log(1-t)} - \frac{2!G_2(t)}{\log(1-t)}\right]. \tag{7.17}$$

Substituting

$$G_3(t) = \frac{1}{2!}\left[\frac{1}{\log(1-t)} - \frac{2!G_2(t)}{\log(1-t)}\right] \tag{7.18}$$

into (6.15) leads to

$$\begin{gathered} G_3^*(t) = (1-t)G_3(t), \\ \left(\sum_{i=0}^{\infty k}\gamma_{3,i}^* t^i\right) = (1-t)\left(\sum_{i=0}^{\infty}\gamma_{3,i} t^i\right). \end{gathered} \tag{7.19}$$

Expanding the equation into,

$$\begin{gathered} \left(\gamma_{3,0}^* + \gamma_{3,1}^* t + \gamma_{3,2}^* t^2 + \cdots\right) = \frac{1}{(1+t+t^2+\ldots)}\left(\gamma_{3,0} + \gamma_{3,1} t + \gamma_{3,2} t^2 + \cdots\right), \\ \left(\gamma_{3,0}^* + \gamma_{3,1}^* t + \gamma_{3,2}^* t^2 + \cdots\right)\left(1+t+t^2+\ldots\right) = \left(\gamma_{3,0} + \gamma_{3,1} t + \gamma_{3,2} t^2 + \cdots\right) \end{gathered} \tag{7.20}$$

which leads to a recursive relationship

$$\sum_{i=0}^{k}\gamma_{3,i}^* = \gamma_{3,k}. \tag{7.21}$$

Tables 1 and 2 are a few examples of the explicit and implicit integration coefficients.

Table 1: The explicit integration coefficients for k from 0 to 6.

k	0	1	2	3	4	5	6
$\gamma_{1,k}$	1	1/2	5/12	3/8	251/720	95/288	19087/60480
$\gamma_{2,k}$	1/2	1/6	1/8	19/180	3/32	863/10080	275/3456
$\gamma_{3,k}$	1/6	1/24	7/240	17/720	41/2016	731/40320	8563/518400

Table 2: The implicit integration coefficients for k from 0 to 6.

k	0	1	2	3	4	5	6
$\gamma^*_{1,k}$	1	−1/2	−1/12	−1/24	−19/720	−3/160	−813/60480
$\gamma^*_{2,k}$	1/2	−1/3	−1/24	−7/360	−17/1440	−41/5040	−731/120960
$\gamma^*_{3,k}$	1/6	−1/8	−1/80	−1/180	−11/3360	−89/40320	−5849/3628800

8. Numerical Results

For error calculations, we will be using the three error tests, namely, absolute error, relative error, and mixed error tests. The error formula is given by,

$$\text{Error} = \operatorname*{Max}_{x_n} \left\| \frac{y(x_n) - y_n}{A + By(x_n)} \right\|, \tag{8.1}$$

where $A = 1$, $B = 0$ gives the absolute error test, $A = 0$, $B = 1$ gives the relative error test, and $A = 1$, $B = 1$ gives the mixed error test.

In (8.1), $y(x_n)$ is the exact solution for the problem considered and y_n the computed solution. In a general code when the exact solution is not available for the relative error, $y(x_n)$ is replaced by y_n the computed value.

When $\|y(x_n) - y_n\|$ is small, the error in (8.1) will approximate the absolute error. However, when it is large, the mixed error test will approximate the relative error. The numerical results give the three errors.

The following notations hold MAX ABS: maximum error when using absolute error test, MAX MIX: maximum error when using mixed error test, MAX REL: maximum error when using relative error test, h: step size selected.

For the choice of problems to be tested, we choose four linear problems consisting of a second- and a third-order problem. The third problem is a mix system of second- and first-order equations and the fourth problem is a system of three second-order equations. Our reason for choosing the linear problems is that if the formulae are correct, then they should solve linear problems. The choice of system of equations is to raise the degree of difficulty of solving the problems. The rest of the problems are nonlinear, which occur in physical situations. The choices of the physical problems are those with exact solutions known. We give our comments on the numerical results right after the numerical Tables 3, 4, 5, 6, 7, 8, and 9.

Table 3

h	MAX ABS	MAX MIX	MAX REL
10^{-1}	4.73734 + 026	1.18857 − 003	1.18857 − 003
10^{-2}	4.70433 + 023	1.16697 − 006	1.16697 − 006
10^{-3}	4.42360 + 020	1.18335 − 009	1.18335 − 009
10^{-4}	4.05706 + 020	1.01668 − 009	1.01668 − 009
10^{-5}	3.69529 + 021	9.26023 − 009	9.26024 − 009

Table 4

h	MAX ABS	MAX MIX	MAX REL
10^{-1}	1.60072 + 023	1.40364 − 003	1.40364 − 003
10^{-2}	1.52595 + 020	1.33620 − 006	1.33620 − 006
10^{-3}	1.55425 + 017	1.36098 − 009	1.36098 − 009
10^{-4}	1.27710 + 016	1.11807 − 010	1.11807 − 010
10^{-5}	2.20767 + 017	1.93311 − 009	1.93311 − 009

Problem 1.

$$y'' = 2y' - y, \quad y(0) = 0, \quad y'(0) = 1, \qquad 0 \le x \le 64. \tag{8.2}$$

Solution 1.

$$y = xe^x. \tag{8.3}$$

Source: Krogh [7].

This is a linear equation used by Krogh [7] to test his code. The solution increases exponentially to a maximum value of $y(64) = 64e^{64}$ which is considered large and therefore not suitable for absolute error test and hence the large values of the error.

Problem 2.

$$y''' = 2y'' - 4, \quad 0 \le x \le 30, \qquad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 6. \tag{8.4}$$

Solution 2.

$$y(x) = x^2 + e^{2x}. \tag{8.5}$$

Source: Omar and Suleiman [4].

This is a third-order problem with an exponential solution. The difference between Problems 1 and 2 is that one is third order and the other is second order. Again, absolute error test does not work for the same reason given above.

Table 5

h	MAX ABS	MAX MIX	MAX REL
10^{-1}	1.74071 − 002	1.68285 − 003	6.12179 − 003
10^{-2}	2.32581 − 005	4.04838 − 006	5.34483 − 005
10^{-3}	2.38274 − 008	4.39132 − 009	5.74568 − 007
10^{-4}	5.33390 − 009	5.47762 − 010	9.42528 − 009
10^{-5}	4.54132 − 008	4.54132 − 008	5.25218 − 007

Table 6

h	MAX ABS	MAX MIX	MAX REL
10^{-1}	1.40347 + 001	1.55464 + 000	6.27279 + 002
10^{-2}	1.78122 − 002	1.75004 − 002	1.60254 + 001
10^{-3}	2.32649 − 005	2.32643 − 005	9.97054 − 001
10^{-4}	2.44607 − 008	2.44600 − 008	9.82644 − 001
10^{-5}	9.90539 − 010	9.90531 − 010	9.99724 − 001

Problem 3.

$$\begin{gathered} y_1'' = -2y_1' - 5y_2 + 3, \quad y_2' = y_1' + 2y_2, \\ 0 \leq x \leq 16\pi, \\ y_1(0) = 0, \quad y_1'(0) = 0, \quad y_2(0) = 1. \end{gathered} \tag{8.6}$$

Solution 3.

$$\begin{gathered} y_1(x) = 2\cos x + 6\sin x - 2 - 6x, \\ y_2(x) = -2\cos x + 2\sin x + 3. \end{gathered} \tag{8.7}$$

Source: Bronson [9].

For this problem, all error tests worked well.

Problem 4.

$$\begin{gathered} y_1'' = 1 - y_2 - y_3, \quad y_2'' = y_3 - y_1, \quad y_3'' = y_1' + y_2', \\ 0 \leq x \leq 4\pi, \\ y_1(0) = 0, \quad y_1'(0) = 1, \quad y_2(0) = 0, \quad y_2'(0) = 0, \quad y_3(0) = -1, \quad y_3'(0) = 1. \end{gathered} \tag{8.8}$$

Solution 4.

$$\begin{gathered} y_1(x) = \sin x, \\ y_2(x) = \cos x - 1, \\ y_2(x) = \sin x - \cos x. \end{gathered} \tag{8.9}$$

Source: Bronson [9].

Table 7

h	MAX ABS	MAX MIX	MAX REL
10^{-1}	2.34537 − 001	4.96059 − 004	1.00000 + 000
10^{-2}	2.85286 − 004	2.86491 − 007	1.00000 + 000
10^{-3}	2.91084 − 007	6.19617 − 010	1.00083 + 000
10^{-4}	3.00856 − 008	7.74497 − 013	1.80152 + 000
10^{-5}	2.43305 − 007	1.09024 − 011	7.08729 + 000

This problem does not work for relative error test because of the small value of the solution for certain values of x.

Problem 5.

$$y''' = -\frac{1}{x}y'' + \frac{1}{x^2}y' + \frac{1}{x}, \quad 1 \le x \le 50,$$
$$y(1) = \frac{26}{21}\ln^2(2) + \frac{99}{104}, \quad y'(1) = -\frac{40}{21}\ln(2) - \frac{5}{13}, \quad y''(1) = \frac{3}{26} + \frac{4}{7}\ln(2). \tag{8.10}$$

Solution 5.

$$y(x) = \frac{x^2}{8}\left(2\ln\left(\frac{x}{2}\right) - \frac{33}{13} - \frac{2}{3}\ln(2)\right) + \left(\frac{1}{3} - \frac{26}{21}\ln\left(\frac{x}{2}\right)\right)\ln(2) + \frac{33}{26}. \tag{8.11}$$

Source: Russel and Shampine [10].

This problem is the symmetrical bending of a laterally loaded circular plate.

The numerical results of this problem show the failure to control the error using relative error test. This is because the solution is zero when $x = 2$.

Problem 6.

$$y_1'' = -\frac{y_1}{r^3}, \quad y_1(0) = 1, \quad y_1'(0) = 0,$$
$$y_2'' = -\frac{y_2}{r^3}, \quad y_2(0) = 0, \quad y_2'(0) = 1,$$
$$r = \left(y_1^2 + y_2^2\right)^{1/2},$$
$$0 \le x \le 16\pi. \tag{8.12}$$

Solution 6.

$$y_1(x) = \cos x, \quad y_2(x) = \sin x. \tag{8.13}$$

Source: Shampine and Gordon [11].

Table 8

h	MAX ABS	MAX MIX	MAX REL
10^{-1}	1.01336 – 004	8.29238 – 005	5.15346 – 002
10^{-2}	8.33352 – 008	8.33254 – 008	9.08294 – 004
10^{-3}	9.71863 – 011	9.71317 – 011	4.58972 – 006
10^{-4}	5.45085 – 010	5.45075 – 010	3.45420 – 004
10^{-5}	4.75544 – 009	4.75543 – 009	1.91598 – 002

Table 9

h	MAX ABS	MAX MIX	MAX REL
10^{-1}	6.95439 – 002	5.63803 – 002	2.97860 – 001
10^{-2}	8.31669 – 005	7.12977 – 005	4.99583 – 004
10^{-3}	8.60027 – 008	7.37166 – 008	5.16016 – 007
10^{-4}	8.66052 – 011	7.42330 – 011	5.19631 – 010
10^{-5}	1.86645 – 012	1.55538 – 012	9.33222 – 012

This problem is Newton's equations of motion for the two-body problem.

Again, relative error test does not work too well for this problem because y_n is very small at certain points x_n.

Problem 7.

$$y'' = 2y^3, \quad 0 \le x \le 5, \qquad y(0) = 1, \quad y'(0) = -1. \tag{8.14}$$

Solution 7.

$$y(x) = \frac{1}{x+1}. \tag{8.15}$$

Source: Robert Jr. [12].

For this problem, all error tests worked well.

All the numerical results show that the errors in the mixed error mode give a reliable error estimate for all the problems given. The absolute error mode failed to give meaningful error results for Problems 1 and 2. This is because the value of $\|y_n\|$ increases as x increases and this becomes large. Similarly, for Problems 4, 5, and 6, the relative error failed to give an acceptable result because $\|y_n\|$ is small.

The research work done shows that the method developed for solving higher-order ODEs directly using the backward difference is successful. We recommend that, for multistep method, the error control procedure should use the mixed error test. This research suggests the potential of this work developing a robust code for solving higher-order ODEs directly.

Acknowledgments

This paper has been supported by the UPM Graduate Research Fellowship (GRF) and Science Fund.

References

[1] F. T. Krogh, "Algorithms for changing the step size," *SIAM Journal on Numerical Analysis*, vol. 10, pp. 949–965, 1973.

[2] M. B. Suleiman, "Solving nonstiff higher order ODEs directly by the direct integration method," *Applied Mathematics and Computation*, vol. 33, no. 3, pp. 197–219, 1989.

[3] Z. A. Majid and M. B. Suleiman, "Direct integration implicit variable steps method for solving higher order systems of ordinary differential equations directly," *Sains Malaysiana*, vol. 35, no. 2, pp. 63–68, 2006.

[4] Z. B. Omar and M. B. Suleiman, "Parallel 2-point explicit block method for solving higher order ordinary differential equations directly," *International Journal of Simulation and Process Modelling*, vol. 2, no. 3, pp. 227–231, 2006.

[5] L. Collatz, *The Numerical Treatment of Differential Equations*, Springler, New York, NY, USA, 1966.

[6] C. W. Gear, "The numerical integration of ordinary differential equations," *Mathematics of Computation*, vol. 21, pp. 146–156, 1967.

[7] F. T. Krogh, "A variable step, variable order multistep method for the numerical solution of ordinary differential equations," in *Proceedings of the IFIP Congress in Information Processing*, vol. 68, pp. 194–199, 1968.

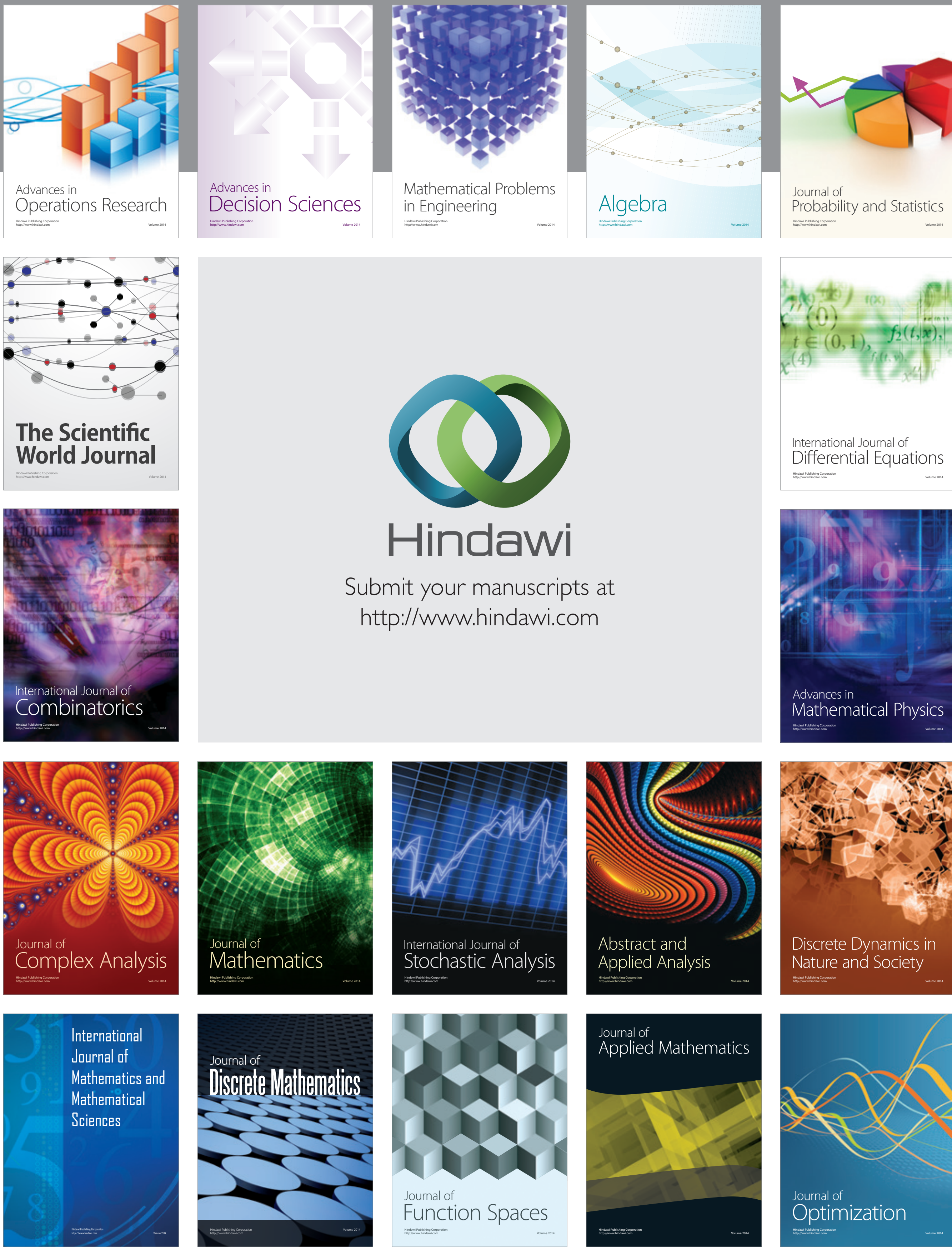
[8] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, Wiley, New York, NY, USA, 1962.

[9] R. Bronson, *Modern Introductory Differential Equation: Schaum's Outline Series*, McGraw-Hill, USA, 1973.

[10] R. D. Russell and L. F. Shampine, "A collocation method for boundary value problems," *Numerische Mathematik*, vol. 19, pp. 1–28, 1972.

[11] L. F. Shampine and M. K. Gordon, *Computer Solution of Ordinary Differential Equations*, W. H. Freeman and Co., San Francisco, Calif, USA, 1975.

[12] C. E. Roberts Jr., *Ordinary Differential Equations. A Computational Approach*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1979.


Advances in
Operations Research
Advances in
Decision Sciences
Mathematical Problems
in Engineering
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
International Journal of
Stochastic Analysis
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
Journal of
Applied Mathematics
Journal of
Optimization