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Research Article

Transfer Matrix Method for Natural Vibration Analysis of Tree System

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The application of Transfer matrix method (TMM) ranges from linear/nonlinear vibration, composite structure, and multibody system to calculating static deformation, natural vibration, dynamical response, and damage identification. Generally TMM has two characteristics: (1) the TMM formulae share similarity to the chain mechanics model in terms of topology structure; then TMM often is selected as a powerful tool to analyze the chain system. (2) TMM is adopted to deal with the problems of the discrete system, continuous system, and especial discrete/continuous coupling system with the uniform matrix form. In this investigation, a novel TMM is proposed to analyze the natural vibration of the tree system. In order to make the TMM of the tree system have the two above advantages of the TMM of the chain system, the suitable state vectors and transfer matrices of the typical components of the tree system are constructed. Then the topology comparability between the mechanics model and its corresponding formulae of TMM can be adopted to assembling the transfer matrices and transfer equations of the global tree system. Two examples of natural vibration problems validating the method are given. The formulation of the proposed TMM is mathematically intuitive and can be held and applied by the engineers easily.

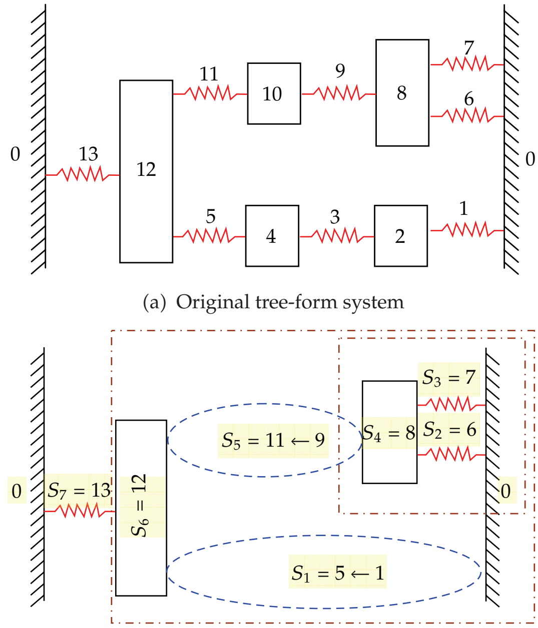
1. Introduction

Transfer matrix method (TMM) has been developed for a long time and has been used widely in engineering mechanics of the linear and nonlinear system. To linear system, Holzer (1921) initially applied TMM to solve the problems of torsion vibrations of rods [1], Myklestad (1945) selected TMM to determine the bending-torsion modes of beams [2], Thomson (1950) used TMM to more general vibration problems [3], Pestel and Leckie (1963) listed transfer matrices for elastomechanical elements up to twelfth order [4], and Rubin (1964, 1967) provided a general treatment for transfer matrices and their relation to other forms of frequency response matrices [5, 6]. Transfer matrices have been applied to a wide variety of engineering programs by a number of researchers, including Targoff [7], Lin [8], Mercer and Seavey [9], Lin and McDaniel [10], Mead [11, 12], Henderson and McDaniel [13], McDaniel [14, 15], and Murthy [16–18], dealing with beams, beam-type periodic structures, skin-stringer panels, rib-skin structures, curved multispan structures, cylindrical shells, stiffened rings, and so forth.

To nonlinear system, Zu and Ji (2002) proposed an improved TMM for steady-state analysis of nonlinear rotor-bearing systems [19]. Gu et al. (2003) analyzed the Transient response analysis of large-scale rotor-bearing system with strong nonlinear elements by a transfer matrix-newmark formulation integration method [20]. Liew et al. (2004) used the TMM for transient analysis of nonlinear rotor-bearing systems [21]. To composite structure, Ellakany et al. (2004) dealt with free vibration analysis of composite beams by a combined transfer matrix and analogue beam method [22]. Yeh and Chen (2006) analyzed the wave propagations problems of a periodic sandwich beam by FEM and TMM [23]. In order to increase the applied field of the TMM, Dokanish (1972) developed finite element-transfer matrix method (FE-TMM) to solve the problems of plate structure vibration analysis, by combining finite element method and transfer matrix method [24]. Many researchers, such as Ohga and Shigematsu (1987), Xue (1994), and Loewv (1985, 1999), studied and improved FE-TMM for structure dynamics [25–28]. Lee (2000) analyzed the one-dimensional structures problems using the spectral TMM [29]. Choi and Man (2001) dealt with the dynamic analysis of the geared rotor-bearing system by TMM [30]. Hsieha et al. (2006) proposed a modified TMM to analyze the coupling lateral and torsional vibrations of symmetric rotor-bearing systems [31]. Horner and Pilkey (1978) proposed Riccati transfer matrix method in order to circumvent the numerical stability of the boundary value problem [32]. By combining the TMM and the numerical procedure, Kumar and Sankar (1986) constructed the discrete time transfer matrix method to analyze the dynamical response of the vibration system [33].

Recently Liu (1999) adopted TMM to analyze the plane frame with variable section and branch [34], and Yu et al. (2002) dealt with furcated structural system by TMM [35]. Huang and Horng (2001) applied the extended TMM with complex numbers analyzing the branched torsional systems [36]. Zou et al. (2003) analyzed the torsional vibration of complicated multibranched shafting systems by the modal synthesis method [37]. Rui et al. (1998, 2005, 2010) developed the discrete time transfer matrix method for multibody system dynamics [38–40].

In this investigation the tree structure system is modeled by TMM. A special attention is focused on how the transfer equations and transfer matrices of the global system can be developed conveniently. By defining the state vectors and deducing the transfer matrices of the typical components of the tree system suitably, some interesting phenomena, which are that the topology structure of the mechanics model is almost similar to that of the interrelated formula, are discovered. Then a systemic TMM is proposed that can be used conveniently to deal with the vibration problem of tree structure. This formulation is mathematically and practically convenient. The text is organized as follows. In Section 2, the general theorem of the method is shown. In Section 3, the transfer matrices of typical elements are developed, including chain lumped mass, branched lumped mass, and spring. In Section 4, some results calculated by TMM and the other method are given that can validate the proposed method. The conclusion and future works are presented in Section 5.



(a) Original tree-form system

(b) Discrete tree-form system

Figure 1: Tree-form system and its linearization.

2. General Theorem of TMM

2.1. Tree System

The tree system is one kind of important nonchain system. It contains many nodes, branches, and hierarchical organizations. Its concrete shape is very similar to the natural biologic tree. In a tree system there is one and only one path leading from one location to any other location in the system.

The tree system includes two kinds of the subsystems: chain subsystem and branched subsystem. Chain subsystem is comprised of some elements with one input end and one output end that are connected with chain form. The branched subsystem at least contains one element with many input ends and one output end, usually named as the branched components. It does not lose generality in this work, the tree system is constituted by the spring, chain lumped mass, and branched lumped mass whose mechanics performance will be introduced in detail in Section 3.

For example, the tree system shown in Figure 1 is made up of thirteen components, where the component 8 and 12 are the branched components that can be regarded as two branched subsystems S_4 and S_6, others are connected with the chain organization. The elements 1, 2, 3, 4, 5 and 9, 10, 11 are connected with chain form and can be regarded as two chain subsystems S_1 and S_5. Apart from these two chain subsystems, the components 6, 7, and 13 may well be denoted as three chain subsystems S_2, S_3, and S_7. Then the original tree structure system depicted in Figure 1(a) has the same function and topology structure with the model shown in Figure 1(b) that includes seven subsystems S_i, $i = 1, 2, \ldots, 7$. It should be noted that all the boundaries here are expressed by the zero.

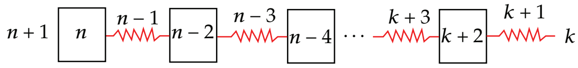


Figure 2: Model of chain subsystem.

2.2. State Vector

Now that the tree structure system includes the spring and the lumped mass undergoing one dimension motion along the ox axis, the state vector of the connected point between the spring and the lumped mass is the function with respect to time, which can be defined as

$$z = \begin{bmatrix} x & q \end{bmatrix}^T, \tag{2.1}$$

where x and q, respectively, represent the displacement coordinate and internal force of the connected point with respect to the direction ox. As for the system with period motion, the displacement coordinate and the internal force can be assumed

$$x(t) = X \exp(i\omega t), \qquad q(t) = Q \exp(i\omega t), \tag{2.2}$$

where ω is the circular frequency, X, Q are the shape functions of the displacement coordinate and internal force, and i is the unit of imaginary number. Then the velocity and acceleration can be expressed as

$$\begin{aligned} \dot{x}(t) &= i\omega X \exp(i\omega t), \\ \ddot{x}(t) &= -\omega^2 X \exp(i\omega t). \end{aligned} \tag{2.3}$$

To substitute (2.3) into (2.2), the corresponding modal state vector of physics state vector z can be written as [4]

$$\mathbf{Z} = \begin{bmatrix} X & Q \end{bmatrix}^T. \tag{2.4}$$

Supposing that the system undergoes period motion, the modal state vector $\mathbf{Z}$ does not vary with the change of the time but remains the function of the displacement coordinates of different connected nodes.

2.3. Transfer Matrix of Chain Subsystem

The transfer equation of the chain subsystem S_i can be obtained by using the transfer equations of the components in turn. For the chain subsystem from the component $k + 1$ to $n \geq k + 1$ that is shown in Figure 2, its transfer equation can be written as

$$\mathbf{Z}_{S_j,S_i} = \mathbf{U}_{S_i}\mathbf{Z}_{S_i,S_l}, \tag{2.5}$$

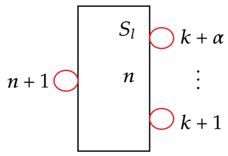


Figure 3: Branched component.

where the global transfer matrix of the chain subsystem can be expressed as

$$\mathbf{U}_{S_i} = \mathbf{U}_n \mathbf{U}_{n-1} \mathbf{U}_{n-2} \cdots \mathbf{U}_{k+2} \mathbf{U}_{k+1}. \tag{2.6}$$

$\mathbf{Z}_{S_j,S_i}$ and $\mathbf{Z}_{S_i,S_l}$ are the state vectors of input and output ends, their definitions can be seen in (2.1), and the subscripts S_i, S_j, S_l of the state vectors denote this subsystem (or component itself) S_i, the output subsystem (or component) of S_i, and the input subsystem (or component) of S_i. A very useful convention adopted in this work is that the first (second) subscript denotes the input (output) component or subsystem.

The topology structure of the chain system in Figure 2 shares the similarity with the transfer matrix (equation) of (2.5) and (2.6) in terms of chain form. This reminds us of whether the relationship between the system and its equation of transfer matrix can be used to assemble the formulation of the global system. In the following section, this topology comparability will be employed to analyze the natural vibration of the tree-like system by TMM. And the component of the tree system, discrete or continuous, does not affect the matrix form of (2.5) and (2.6), as long as the transfer matrices of the components have been obtained ahead of schedule [4].

2.4. Transfer Matrix of Branched Subsystem

The branched subsystem S_l or branched component n is shown in Figure 3, including α input components and one output component, denoted as $k+1, k+2, \ldots, k+\alpha$, and $n+1$, respectively. The transfer equation of the branched subsystem S_l contains two equations: one is the access transfer equation that describes the state vector of the output end as the linear function of the state vectors of input ends; the other is the nodal transfer equation that conveys the relationship between the state vectors of input ends. Generally the access transfer equation can be written as

$$\mathbf{Z}_{n+1,n} = \sum_{j=k+1}^{k+\alpha} \mathbf{U}'_n(j) \mathbf{Z}_{n,j}, \tag{2.7}$$

where $\mathbf{Z}_{n+1,n}$, $\mathbf{Z}_{n,j}$, $(j = k+1, \ldots, k+\alpha)$ are the state vectors of the output and input ends of the branched subsystem, respectively, $\mathbf{U}'_n(j)$, $(j = k+1, \ldots, k+\alpha)$ are the block transfer matrices. Equation (2.7) can be used conveniently to analyze the problems of the single or few branched system [34–36]. To the tree system with many branched subsystems, the formula will be quite complex and not intuitive enough. In order to circumvent this lack of TMM, we try to develop a formulation that resembles the tree system in topology structure. So (2.7) can be rewritten as

$$\mathbf{Z}_{n+1,n} = \mathbf{U}'_{S_l}\begin{bmatrix}\mathbf{Z}_{n,k+1}\\ \mathbf{Z}_{n,k+2}\\ \vdots \\ \mathbf{Z}_{n,k+\alpha}\end{bmatrix}, \tag{2.8}$$

where the access transfer matrix is

$$\mathbf{U}'_{S_l} = \left[\mathbf{U}'_n(k+1)\ \ \mathbf{U}'_n(k+2)\ \ \cdots\ \ \mathbf{U}'_n(k+\alpha)\right]. \tag{2.9}$$

Mathematically, (2.7) is completely equivalent to (2.8). In contrast with (2.7), the topology structure of (2.8) and Figure 3 are similar. This comparability seems more useful when the analysis system is more complicated. One thing worth noting is that (2.8) does not explain the internal relationship among the state vectors of the input ends. A further equation, the nodal transfer equation, is still called for

$$\mathbf{O}_{(i+1)\times 1} = \mathbf{U}''_{S_l}\begin{bmatrix}\mathbf{Z}_{n,k+1}\\ \mathbf{Z}_{n,k+2}\\ \vdots \\ \mathbf{Z}_{n,k+\alpha}\end{bmatrix}, \tag{2.10}$$

where the subscript i is in terms of the state vector's definition and the number α. The detailed derivation of the nodal transfer equation will be given in Section 3. The topology structure of (2.10) and (2.8) is similar to that of the tree system shown in Figure 3. So this comparability seems very useful for assembling of the global system transfer matrix that will be introduced in the following part.

2.5. Transfer Equation and Transfer Matrix of Global System

According to the definition of the state vector and transfer matrix in the above section, the transfer equations of all subsystems of the tree system shown in Figure 1 can be written as

$$\begin{aligned}\mathbf{Z}_{S_6,S_1} &= \mathbf{U}_{S_1}\mathbf{Z}_{S_1,0},\\ \mathbf{Z}_{S_4,S_2} &= \mathbf{U}_{S_2}\mathbf{Z}_{S_2,0},\\ \mathbf{Z}_{S_4,S_3} &= \mathbf{U}_{S_3}\mathbf{Z}_{S_3,0},\\ \mathbf{Z}_{S_6,S_5} &= \mathbf{U}_{S_5}\mathbf{Z}_{S_5,S_4},\end{aligned}$$

$$\begin{aligned}
\mathbf{Z}_{0,S_7} &= \mathbf{U}_{S_7}\mathbf{Z}_{S_7,S_6}, \\
\mathbf{Z}_{S_7,S_6} &= \mathbf{U}'_{S_6}\begin{bmatrix}\mathbf{Z}_{S_6,S_5}\\ \mathbf{Z}_{S_6,S_1}\end{bmatrix}, \\
\mathbf{Z}_{S_5,S_4} &= \mathbf{U}'_{S_4}\begin{bmatrix}\mathbf{Z}_{S_4,S_3}\\ \mathbf{Z}_{S_4,S_2}\end{bmatrix}, \\
0 &= \mathbf{U}''_{S_6}\begin{bmatrix}\mathbf{Z}_{S_6,S_5}\\ \mathbf{Z}_{S_6,S_1}\end{bmatrix}, \\
0 &= \mathbf{U}''_{S_4}\begin{bmatrix}\mathbf{Z}_{S_4,S_3}\\ \mathbf{Z}_{S_4,S_2}\end{bmatrix},
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
\mathbf{U}_{S_1} &= \mathbf{U}_5\mathbf{U}_4\mathbf{U}_3\mathbf{U}_2\mathbf{U}_1, \\
\mathbf{U}_{S_2} &= \mathbf{U}_6, \\
\mathbf{U}_{S_3} &= \mathbf{U}_7, \\
\mathbf{U}'_{S_4} &= \mathbf{U}'_8, \\
\mathbf{U}''_{S_4} &= \mathbf{U}''_8, \\
\mathbf{U}_{S_5} &= \mathbf{U}_{11}\mathbf{U}_{10}\mathbf{U}_9, \\
\mathbf{U}'_{S_6} &= \mathbf{U}'_{12}, \\
\mathbf{U}''_{S_6} &= \mathbf{U}''_{12}, \\
\mathbf{U}_{S_7} &= \mathbf{U}_{13}.
\end{aligned} \tag{2.12}$$

The transfer matrices of the right parts of the equation can be referred in Section 3. The global system transfer equation that connects the boundary state vectors can be obtained by the iterative computation of the transfer equations of all components,

$$\begin{aligned}
\mathbf{Z}_{0,S_7} &= \mathbf{U}_{S_7}\mathbf{U}'_{S_6}\begin{bmatrix}\mathbf{U}_{S_5}\mathbf{U}'_{S_4}\begin{bmatrix}\mathbf{U}_{S_3}\mathbf{Z}_{S_3,0}\\ \mathbf{U}_{S_2}\mathbf{Z}_{S_2,0}\end{bmatrix}\\ \mathbf{U}_{S_1}\mathbf{Z}_{S_1,0}\end{bmatrix}, \\
0 &= \mathbf{U}''_{S_6}\begin{bmatrix}\mathbf{U}_{S_5}\mathbf{U}'_{S_4}\begin{bmatrix}\mathbf{U}_{S_3}\mathbf{Z}_{S_3,0}\\ \mathbf{U}_{S_2}\mathbf{Z}_{S_2,0}\end{bmatrix}\\ \mathbf{U}_{S_1}\mathbf{Z}_{S_1,0}\end{bmatrix}, \\
0 &= \mathbf{U}''_{S_4}\begin{bmatrix}\mathbf{U}_{S_3}\mathbf{Z}_{S_3,0}\\ \mathbf{U}_{S_2}\mathbf{Z}_{S_2,0}\end{bmatrix}.
\end{aligned} \tag{2.13}$$

Up to now, we have found a very interesting phenomenon. The topology structure of the first formulation in (2.13) nearly equals to that of Figure 1. The mechanics model given in this figure is branched at the subsystems S_4 and S_6; formulation (2.13) seems also "branched" at the transfer matrices of these two subsystems. The subsystems S_1 and S_5 in Figure 1 are chain, and (2.13) seems also chain at these stations. On the other hand, assuming one part of Figure 1(a) that starts from the subsystem S_6 and can be separated by the dash dot line box plotted in Figure 1(b), the topology structure of the figure in this box is similar to that of the second formulation of (2.13). And the third formulation in (2.13) is like the figure inside the little dash dot line box of Figure 1(b). So this comparability can be used to put together the transfer matrix of the global system. It is noted that the nodal transfer equation of each branched component will cause an extra transfer equation of the global system. So as for the system shown in Figure 1 which includes two branched component, three transfer equations can be developed constituting one access transfer equation of the global system and two nodal transfer equations yielded via the branched subsystems S_4 and S_6, respectively. Furthermore, (2.13) can be rewritten as one formulation that can be named as the transfer equation of the global system

$$\begin{bmatrix} \mathbf{Z}_{0,13} \\ 0 \\ 0 \end{bmatrix} = \mathbf{U}_{\text{all}} \begin{bmatrix} \mathbf{Z}_{7,0} \\ \mathbf{Z}_{6,0} \\ \mathbf{Z}_{1,0} \end{bmatrix}, \tag{2.14}$$

where the transfer matrix of the global system is

$$\begin{aligned}
\mathbf{U}_{\text{all}} &= \begin{bmatrix} \mathbf{U}_{\text{all},1}^T & \mathbf{U}_{\text{all},2}^T & \mathbf{U}_{\text{all},3}^T \end{bmatrix}^T, \\
\mathbf{U}_{\text{all},1} &= \mathbf{U}_{S_7}\mathbf{U}'_{S_6} \begin{bmatrix} \mathbf{U}_{S_5}\mathbf{U}'_{S_4} \begin{bmatrix} \mathbf{U}_{S_3} & \mathbf{O}_{2\times2} \\ \mathbf{O}_{2\times2} & \mathbf{U}_{S_2} \end{bmatrix} & \mathbf{O}_{2\times2} \\ \mathbf{O}_{2\times4} & \mathbf{U}_{S_1} \end{bmatrix}, \\
\mathbf{U}_{\text{all},2} &= \mathbf{U}''_{S_6} \begin{bmatrix} \mathbf{U}_{S_5}\mathbf{U}'_{S_4} \begin{bmatrix} \mathbf{U}_{S_3} & \mathbf{O}_{2\times2} \\ \mathbf{O}_{2\times2} & \mathbf{U}_{S_2} \end{bmatrix} & \mathbf{O}_{2\times2} \\ \mathbf{O}_{2\times4} & \mathbf{U}_{S_1} \end{bmatrix}, \\
\mathbf{U}_{\text{all},3} &= \begin{bmatrix} \mathbf{U}''_{S_4} \begin{bmatrix} \mathbf{U}_{S_3} & \mathbf{O}_{2\times2} \\ \mathbf{O}_{2\times2} & \mathbf{U}_{S_2} \end{bmatrix} & \mathbf{O}_{1\times2} \end{bmatrix}.
\end{aligned} \tag{2.15}$$

Then natural frequency of the tree system can be resolved like the common transfer matrix method. Since the unknown quantities in the boundary state vector have nonzero solution, the shape function and the natural frequencies can be calculated.

3. Transfer Matrix of Typical Component

3.1. Transfer Matrix of Spring Undergoing Longitudinal Vibration

The spring neglecting the mass vibrates by natural circular frequency ω in one-dimension direction. The stiffness of the spring i is K_i, whose input and output components are denoted as $i-1$ and $i+1$. State vectors of the input or output ends of the spring i are denoted as $\mathbf{Z}_{i,i-1}$ or $\mathbf{Z}_{i+1,i}$. The equilibrium equation of the spring can be obtained

$$\begin{aligned} x_{i+1,i} &= x_{i,i-1} - \frac{1}{K_i} q_{i,i-1}, \\ q_{i+1,i} &= q_{i,i-1}. \end{aligned} \tag{3.1}$$

Since the system is harmonic vibration, (2.3) can be substituted into (3.1), and the transfer equation of the spring can be written as

$$\mathbf{Z}_{i+1,i} = \mathbf{U}_i \mathbf{Z}_{i,i-1}, \tag{3.2}$$

where transfer matrix is [4]

$$\mathbf{U}_i = \begin{bmatrix} 1 & -\dfrac{1}{K_i} \\ 0 & 1 \end{bmatrix}. \tag{3.3}$$

3.2. Transfer Matrix of Lumped Mass Undergoing Longitudinal Vibration

The dynamics equation of the lumped mass m_i undergoing longitudinal vibration can be written as

$$\begin{aligned} x_{i+1,i} &= x_{i,i-1}, \\ q_{i,i+1} &= -m_i \ddot{x}_{i,i-1} + q_{i,i-1}. \end{aligned} \tag{3.4}$$

It is the same that (2.3) also can be used, the transfer equation can be obtained

$$\mathbf{Z}_{i+1,i} = \mathbf{U}_i \mathbf{Z}_{i,i-1}, \tag{3.5}$$

where

$$\mathbf{U}_i = \begin{bmatrix} 1 & 0 \\ m_i\omega^2 & 1 \end{bmatrix} \tag{3.6}$$

is the transfer matrix of the lumped mass m_i [4].

3.3. Transfer Matrix of Lumped Mass with Branched Connection Form

Considering that the lumped mass n connected by the branched form is shown in Figure 3, its mass can be denoted as m_n and one output component and α input components can be denoted as $n+1$ and $k+j$, $j = 1, 2, \ldots, \alpha$, respectively. The dynamics equation of this mass can be written as

$$\begin{aligned} x_{n+1,n} &= x_{n,k+j}, \quad j = 1, 2, \ldots, \alpha, \\ q_{n+1,n} &= -m_n \ddot{x}_{n,k+1} + \sum_{j=1}^{\alpha} q_{n,k+j}. \end{aligned} \tag{3.7}$$

It is also similar to the above section where (2.3) can be used, (3.7) can be expressed as

$$\begin{aligned} X_{n+1,n} &= X_{n,k+j}, \quad j = 1, 2, \ldots, \alpha, \\ Q_{n+1,n} &= \omega^2 m_n \ddot{x}_{n,k+1} + \sum_{j=1}^{\alpha} Q_{n,k+j}. \end{aligned} \tag{3.8}$$

So we can select the first one of (3.8) and set $j = 1$, then we can obtain

$$X_{n+1,n} = X_{n,k+1}. \tag{3.9}$$

The above equation and the second one of (3.8) can be expressed as the matrix form. The state vector of the output ends can be expressed by the state vectors of the input ends by this matrix equation that is similar to the access transfer equation (2.8) of the branched component,

$$\mathbf{Z}_{n+1,n} = \mathbf{U}'_n \left[\mathbf{Z}^T_{n,k} \;\; \mathbf{Z}^T_{n,k+1} \;\; \cdots \;\; \mathbf{Z}^T_{n,k+\alpha} \right]^T, \tag{3.10}$$

where the access transfer matrix of the branched lumped mass n can be written as

$$\begin{aligned} \mathbf{U}'_n &= \left[\mathbf{U}_n \;\; \mathbf{B} \;\; \cdots \;\; \mathbf{B} \right], \\ \mathbf{U}_n &= \begin{bmatrix} 1 & 0 \\ m_n \omega^2 & 1 \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.11}$$

However to the formulations in (3.8) when $j \neq 1$,

$$X_{n+1,n} = X_{n,k+j}, \quad j = 2, 3, \ldots, \alpha. \tag{3.12}$$

If we rewrite the above equations as the matrix form directly, that is,

$$\mathbf{A}\mathbf{Z}_{n+1,n} = \mathbf{D}\left[\mathbf{Z}_{n,k+1}^{T} \quad \mathbf{Z}_{n,k+2}^{T} \quad \cdots \quad \mathbf{Z}_{n,k+\alpha}^{T}\right]^{T}, \tag{3.13}$$

where

$$\mathbf{A} = \left[\mathbf{C}^{T} \quad \mathbf{C}^{T} \quad \mathbf{C}^{T}\right]^{T},$$
$$\mathbf{D} = \begin{bmatrix} \mathbf{C} & \mathbf{O}_{1\times 2} & \cdots & \mathbf{O}_{1\times 2} \\ \mathbf{O}_{1\times 2} & \mathbf{C} & \cdots & \mathbf{O}_{1\times 2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_{1\times 2} & \mathbf{O}_{1\times 2} & \cdots & \mathbf{C} \end{bmatrix}, \tag{3.14}$$
$$\mathbf{C} = \left[1 \quad 0\right].$$

If (3.13) is directly used to assemble the global system transfer matrix, the whole task will be huge since variables are unknown for both sides of the equal sign in (3.13). A new equation is required in order to simplify the process. In the rewritten form all the variables are available except in one side of the equal sign. So the method used in Section 2 can also be used to determine a new form of (3.13) whose topology structure is similar to that of the branched component. Then we can substitute the first one of (3.8)

$$X_{n+1,n} = X_{n,k+1} \tag{3.15}$$

into (3.12), the transfer equation of the branched lumped mass, that is, the same with (2.10) can be obtained, where the transfer matrix is

$$\mathbf{U}_n'' = \begin{bmatrix} \mathbf{C} & -\mathbf{C} & \mathbf{O}_{1\times 2} & \cdots & \mathbf{O}_{1\times 2} \\ \mathbf{C} & \mathbf{O}_{1\times 2} & -\mathbf{C} & \cdots & \mathbf{O}_{1\times 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{C} & \mathbf{O}_{1\times 2} & \mathbf{O}_{1\times 2} & \cdots & -\mathbf{C} \end{bmatrix}. \tag{3.16}$$

The aim of this investigation is limited in the multiple-branched system by using TMM. So in order to narrate conveniently, the research object is selected as the relative simple component, such as spring and lumped mass undergoing one-dimensional motion. Furthermore, the transfer matrices of complex components can be found in [6] for chain form system. And the assembling of global transfer matrix and the deduction of transfer matrix of complex component for the branched component are almost the same with the approach given here.

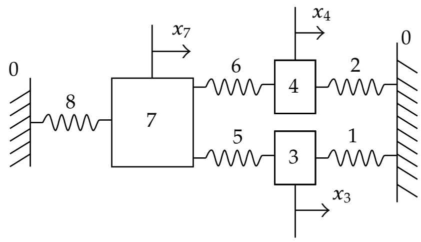


Figure 4: Simple tree system model.

4. Applied Example

Two applied examples are given to validate the method of this paper.

4.1. Simple Example

As for the simple tree system, according to the structure characteristic of the system, the component from one end (root node of the tree system) to the others (leaf node of the tree system) can be denoted as $i = 8, 7, \ldots, 1$ shown in Figure 4. All the boundaries are fixed and denoted as 0. Of course it is not difficult to consider other type boundary conditions for the natural vibration problems. So the model shown in Figure 4 contains three lumped mass $3, 4, 7$ and five springs $1, 2, 5, 6, 8$ with one-dimensional motion. The serial number of the entire component in the system can be determined by one uniform technique, which is different with the method often used in multibody system dynamics [38, 39]. That is because the transfer matrices of the spring and lumped mass have same importance and function in the solution process. According to the formula given in Section 3, the transfer equations of the components shown in Figure 3 can be written as

$$
\begin{gathered}
\mathbf{Z}_{3,1} = \mathbf{U}_1 \mathbf{Z}_{1,0}, \\
\mathbf{Z}_{4,2} = \mathbf{U}_2 \mathbf{Z}_{2,0}, \\
\mathbf{Z}_{5,3} = \mathbf{U}_3 \mathbf{Z}_{3,1}, \\
\mathbf{Z}_{6,4} = \mathbf{U}_4 \mathbf{Z}_{4,2}, \\
\mathbf{Z}_{7,5} = \mathbf{U}_5 \mathbf{Z}_{5,3}, \\
\mathbf{Z}_{7,6} = \mathbf{U}_6 \mathbf{Z}_{6,4}, \\
\mathbf{Z}_{8,7} = \mathbf{U}'_7 \left[\mathbf{Z}^T_{7,6} \quad \mathbf{Z}^T_{7,5} \right]^T, \\
0 = \mathbf{U}''_7 \left[\mathbf{Z}^T_{7,6} \quad \mathbf{Z}^T_{7,5} \right]^T, \\
\mathbf{Z}_{0,8} = \mathbf{U}_8 \mathbf{Z}_{8,7},
\end{gathered}
\tag{4.1}
$$

where $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_5, \mathbf{U}_6, \mathbf{U}_8$ can be referred as (3.3), $\mathbf{U}_3, \mathbf{U}_4$ are defined in (3.6), the access transfer matrix $\mathbf{U}'_7$ and the nodal transfer matrix $\mathbf{U}''$ of branched component can be seen in (3.11) and (3.16), and $\mathbf{Z}_{i,j}$ is the state vector at respective node i, j. By using the iterative method, the global transfer equations that reflect the relations between all the state vectors in the system boundary nodes can be obtained

$$\begin{aligned}
\mathbf{Z}_{0,8} &= \mathbf{U}_8\mathbf{U}'_7\begin{bmatrix}\mathbf{U}_6\mathbf{U}_4\mathbf{U}_2\mathbf{Z}_{2,0}\\ \mathbf{U}_5\mathbf{U}_3\mathbf{U}_1\mathbf{Z}_{1,0}\end{bmatrix}, \\
0 &= \mathbf{U}''_7\begin{bmatrix}\mathbf{U}_6\mathbf{U}_4\mathbf{U}_2\mathbf{Z}_{2,0}\\ \mathbf{U}_5\mathbf{U}_3\mathbf{U}_1\mathbf{Z}_{1,0}\end{bmatrix}.
\end{aligned} \tag{4.2}$$

Equation (4.2) can be rewritten as

$$\begin{bmatrix}\mathbf{Z}_{0,8}\\ 0\end{bmatrix} = \mathbf{U}_{\text{all}}\begin{bmatrix}\mathbf{Z}_{2,0}\\ \mathbf{Z}_{1,0}\end{bmatrix}, \tag{4.3}$$

where the transfer matrix of global system is

$$\begin{aligned}
\mathbf{U}_{\text{all}} &= \begin{bmatrix}\mathbf{U}^T_{\text{all},1} & \mathbf{U}^T_{\text{all},2}\end{bmatrix}^T, \\
\mathbf{U}_{\text{all},1} &= \mathbf{U}_8\mathbf{U}'_7\begin{bmatrix}\mathbf{U}_6\mathbf{U}_4\mathbf{U}_2 & \mathbf{O}_{2\times 2}\\ \mathbf{O}_{2\times 2} & \mathbf{U}_5\mathbf{U}_3\mathbf{U}_1\end{bmatrix}, \\
\mathbf{U}_{\text{all},2} &= \mathbf{U}''_7\begin{bmatrix}\mathbf{U}_6\mathbf{U}_4\mathbf{U}_2 & \mathbf{O}_{2\times 2}\\ \mathbf{O}_{2\times 2} & \mathbf{U}_5\mathbf{U}_3\mathbf{U}_1\end{bmatrix}.
\end{aligned} \tag{4.4}$$

As to the fixed boundary condition, state vectors in the boundary nodes can be expressed as

$$\begin{aligned}
\mathbf{Z}_{1,0} &= \begin{bmatrix}0 & q_1\end{bmatrix}, \\
\mathbf{Z}_{2,0} &= \begin{bmatrix}0 & q_2\end{bmatrix}, \\
\mathbf{Z}_{0,8} &= \begin{bmatrix}0 & q_8\end{bmatrix}.
\end{aligned} \tag{4.5}$$

In order to calculate conveniently, we can set

$$m_3 = m_4 = m_7 = m, \qquad k_1 = k_2 = k_5 = k_6 = k_8 = k. \tag{4.6}$$

Since the system boundary state vectors have nonzero solution, the eigenfrequency equation of the system can be obtained from (4.3)

$$\left(k - m\omega^2\right)\left(2k - m\omega^2\right)\left(4k - m\omega^2\right) = 0. \tag{4.7}$$

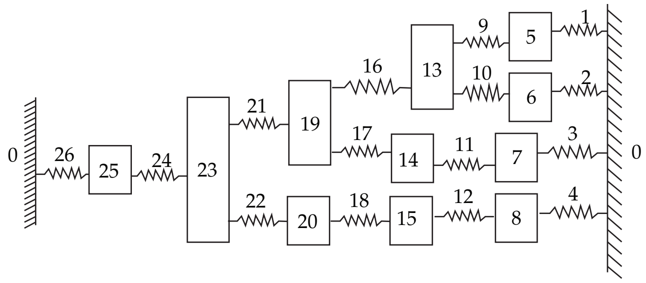


Figure 5: Another branched system model.

The positive solution of the above equation can be obtained easily

$$\omega_1 = 2\sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{2k}{m}}, \qquad \omega_3 = \sqrt{\frac{k}{m}}. \tag{4.8}$$

Of course, global mass matrix M and stiffness matrix K can also be written

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 3k & -k & -k \\ -k & 2k & 0 \\ -k & 0 & 2k \end{bmatrix} \tag{4.9}$$

to obtain the eigenfrequency equation and natural frequencies of the system. Then (4.7) and (4.8) can be gotten again. This proves that the proposed transfer matrix method is effective for the simple tree vibration system.

4.2. Another Example

The model of Figure 5 includes 26 components undergoing one dimensional motion. Fifteen springs can be denoted as $1, 2, 3, 4, 9, 10, 11, 12, 16, 17, 18, 21, 22, 24, 26$ and eleven lumped mass can be expressed as the number $5, 6, 7, 8, 13, 14, 15, 19, 20, 23, 25$, where $13, 19, 23$ particularly express the branched components especially. The mass of all lumped mass and the stiffness of all springs are m and k, respectively. The Lagrange method can be used to obtain the global mass and stiffness matrices that is given in the appendix. So the eigenfrequency equation of the system can be obtained by the dynamics equation of the global system

$$\begin{aligned} &\left(k - \omega^2 m\right)\left(2k - \omega^2 m\right)\left(2k^2 - 4k\omega^2 m + m^2\omega^4\right) \\ &\quad \times \left(53k^6 - 285k^5 m\omega^2 + 440k^4 m^2\omega^4 - 297k^3 m^3\omega^6 + 99k^2 m^4\omega^6 - 16km^5\omega^{10} + m^6\omega^{12}\right) = 0. \end{aligned} \tag{4.10}$$

If setting

$$m = 1\,\text{kg}, \qquad k = 10\,\text{N/m}, \tag{4.11}$$

the eigenfrequency column matrix can be written numerically

$$\begin{aligned}\omega = [&1.72715, 2.4203, 3.16228, 3.47478, 4.47214, 4.47214,\\ &4.81352, 5.60707, 5.84313, 6.3794, 7.04534]^T\ \text{rad/s}.\end{aligned} \tag{4.12}$$

On the other hand, the proposed transfer matrix method can be used to solve the above problem. According to the topology structure of this model, the transfer matrix of the chain subsystem can be written as

$$\begin{aligned}
\mathbf{U}_{S_1} &= \mathbf{U}_9\mathbf{U}_5\mathbf{U}_1,\\
\mathbf{U}_{S_2} &= \mathbf{U}_{10}\mathbf{U}_6\mathbf{U}_2,\\
\mathbf{U}_{S_3} &= \mathbf{U}_{17}\mathbf{U}_{14}\mathbf{U}_{11}\mathbf{U}_7\mathbf{U}_3,\\
\mathbf{U}_{S_4} &= \mathbf{U}_{22}\mathbf{U}_{20}\mathbf{U}_{18}\mathbf{U}_{15}\mathbf{U}_{12}\mathbf{U}_8\mathbf{U}_4,\\
\mathbf{U}_{S_5} &= \mathbf{U}_{26}\mathbf{U}_{25}\mathbf{U}_{24}.
\end{aligned} \tag{4.13}$$

The transfer equation of the global system can be expressed as by the method of Section 2.5,

$$\begin{aligned}
\mathbf{Z}_{0,26} &= \mathbf{U}_{26}\mathbf{U}_{25}\mathbf{U}_{24}\mathbf{U}'_{23}\begin{bmatrix}\mathbf{U}_{21}\mathbf{U}'_{19}\begin{bmatrix}\mathbf{U}_{16}\mathbf{U}'_{13}\begin{bmatrix}\mathbf{U}_{S_1}\mathbf{Z}_{1,0}\\ \mathbf{U}_{S_2}\mathbf{Z}_{2,0}\end{bmatrix}\\ \mathbf{U}_{S_3}\mathbf{Z}_{3,0}\end{bmatrix}\\ \mathbf{U}_{S_4}\mathbf{Z}_{4,0}\end{bmatrix},\\
0 &= \mathbf{U}''_{23}\begin{bmatrix}\mathbf{U}_{21}\mathbf{U}'_{19}\begin{bmatrix}\mathbf{U}_{16}\mathbf{U}'_{13}\begin{bmatrix}\mathbf{U}_{S_1}\mathbf{Z}_{1,0}\\ \mathbf{U}_{S_2}\mathbf{Z}_{2,0}\end{bmatrix}\\ \mathbf{U}_{S_3}\mathbf{Z}_{3,0}\end{bmatrix}\\ \mathbf{U}_{S_4}\mathbf{Z}_{4,0}\end{bmatrix},\\
0 &= \mathbf{U}''_{19}\begin{bmatrix}\mathbf{U}_{16}\mathbf{U}'_{13}\begin{bmatrix}\mathbf{U}_{S_1}\mathbf{Z}_{1,0}\\ \mathbf{U}_{S_2}\mathbf{Z}_{2,0}\end{bmatrix}\\ \mathbf{U}_{S_3}\mathbf{Z}_{3,0}\end{bmatrix},\\
0 &= \mathbf{U}''_{13}\begin{bmatrix}\mathbf{U}_{S_1}\mathbf{Z}_{1,0}\\ \mathbf{U}_{S_2}\mathbf{Z}_{2,0}\end{bmatrix},
\end{aligned} \tag{4.14}$$

where the transfer matrices of the components in (4.13) and (4.14) can refer to the corresponding equations in Section 3. Since the system boundary state vectors

$$\mathbf{Z}_{0,26} = \begin{bmatrix} 0 & q_{26} \end{bmatrix}, \qquad \mathbf{Z}_{1,0} = \begin{bmatrix} 0 & q_1 \end{bmatrix}, \qquad \mathbf{Z}_{2,0} = \begin{bmatrix} 0 & q_2 \end{bmatrix}, \qquad \mathbf{Z}_{3,0} = \begin{bmatrix} 0 & q_3 \end{bmatrix}, \qquad \mathbf{Z}_{4,0} = \begin{bmatrix} 0 & q_4 \end{bmatrix} \tag{4.15}$$

have nonzero solution, the eigenfrequency equation of the system can be obtained that is the same with (4.10). Considering (4.11) again, the numerical solution is the same with (4.12). It can be found from (4.14) that, as for the tree system including three branched components, four independent transfer equations of the global system can be obtained. The first one is the access transfer equation of the global system, and three others are caused by the relevant branched components, especially by the nodal transfer equations of the branched components.

From the two examples we find that the proposed transfer matrix method can be used to solve the natural frequencies of the vibration system without developing the global system dynamics equation. The only things which must be done are to know the transfer matrix of typical components that have been deduced aforehand, the topology structure, and the boundary condition of the global system. So we can obtain the eigenfrequency equation of the system to calculate the natural frequencies by the proposed transfer matrix method. This method is intuitive, simple, and can be held by the engineers easily.

5. Discussion and Conclusion

On the basis of classic transfer matrix method for the chain system, this investigation constructs the transfer matrix method for tree structure natural vibration from the totally novel angle of view. When this method is used, the transfer matrix of the global system can be obtained like assembling the toy block, and the correlative formulations can be verified by comparing the topology structure of the vibration system and corresponding transfer equations. So the error probability can be reduced enormously, and computer programming or manual derivation is very convenient.

For the discrete/continuous coupling system that includes discrete and continuous components, as long as the transfer matrices of these components have been deduced ahead of schedule, the natural frequency of the tree structure system can be calculated by the method. If other methods are used to analyze the natural frequency of this kind of the system, the hybrid ordinary and partial differential equations have to be resolved generally. This requires good mathematical technique and cannot be applied easily by the engineers. An important work in TMM, deducing the transfer matrix of the general elastic body, is complicated and does not obtain easily. The fortunate thing is that FE-TMM can be used to develop the transfer matrix of these components conveniently [19–23]. To combine the proposed method and FE-TMM, the complex tree system can be analyzed. And if the method is combined with the numerical method, the proposed method can be used to deal with the problems of nonlinear tree structure system [25, 29] and even multibody system dynamics [39, 40]. If complex mode theory is used, the damped system also can be dealt with. And the transfer matrix method for the network system including the multi-in and multi-out components will be discussed in another paper in detail. Actually as long as the derived tree system of the network system is obtained, the proposed method can be adopted to resolve the network system problems. So the application field of the proposed method can be expanded.

It is noted that the formulation and computation of this paper is finished by symbolic software Mathematica and the computational convergence and stability are not essential problems. And by the method of symbolic computation, if the eigenfrequency equation is obtained, such as (4.7) or (4.10), the multiplicational calculations among the transfer matrices must not be implemented as to the same topology structure system.

Appendix

There are

$$M = m\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$K = k\begin{bmatrix} 2&-1&0&0&0&0&0&0&0&0&0\\ -1&3&-1&-1&0&0&0&0&0&0&0\\ 0&-1&2&0&0&0&-1&0&0&0&0\\ 0&-1&0&3&-1&-1&0&0&0&0&0\\ 0&0&0&-1&2&0&0&0&0&-1&0\\ 0&0&0&-1&0&3&0&-1&-1&0&0\\ 0&0&-1&0&0&0&2&0&0&0&-1\\ 0&0&0&0&0&-1&0&2&0&0&0\\ 0&0&0&0&0&-1&0&0&2&0&0\\ 0&0&0&0&-1&0&0&0&0&2&0\\ 0&0&0&0&0&0&-1&0&0&0&2 \end{bmatrix}, \tag{A.1}$$

where m and k are the mass of the lumped mass and the stiffness of the spring.

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