Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 602094, 26 pages
doi:10.1155/2012/602094

Research Article

Performance Analysis of a Manufacturing Line Operated under Optimal Surplus-Based Production Control

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Received 24 September 2011; Accepted 14 January 2012

Academic Editor: Wei-Chiang Hong

We examine optimality and performance of a tandem manufacturing line driven by a surplus-based decentralized production control strategy. The main objective of this type of production strategies is to guarantee that the cumulative number of produced products follows the cumulative production demand on the output of the given network. The basic idea of surplus-based control strategy is presented for the case of one manufacturing machine. We prove that this strategy is optimal. Then, a flow model of a line composed of arbitrarily many machines with bounded buffers is analyzed. We prove that the surplus-based production control enables this network to efficiently follow the product demand and establish the relation between the efficiency in the production tracking error and the intermediate inventory levels of a line. Performance and robustness of the flow model of the closed-loop manufacturing line are illustrated by computer simulations.

1. Introduction

Nowadays, the highly dynamic market environment requires that production control policies implemented in manufacturing industries should be capable of providing quick and accurate responses to constant and rapid changes in the production demand. This strongly shifts the interests of manufacturers to the need of theoretical analysis of the currently existing policies, that is, the study of conformity between the production output and the demand of their product, as well as the relation between stock level (buffer content) and the production surplus of the manufacturing network.

Currently, there is a substantial literature on manufacturing control policies and their performance. A number of classifications of these policies were introduced by various authors. In this paper, we follow the classification introduced in [1], which puts the control policies into three categories: token-based, time-based, and surplus-based, respectively. In token-based approaches, the so-called tokens are generated and utilized to trigger certain events in the manufacturing system. The famous examples of such a policy are Kanban [2, 3], Conwip [4], and Basestock [5]. The time-based approaches offer to make the control decisions on the basis of the time when a certain operation should take place, they are exemplified by Material Resource Planning, Least Stack, and Earliest Due Date strategies (see, e.g., [6]). In the surplus approach, control decisions are based on the production tracking error, (here the term "production tracking" refers to the action of following or keeping track of the production demand trajectory on the output of a given network) which is the difference between the cumulative demand and output of the system. For extensive surveys and further details concerning, in particular, the production line control mechanisms, we refer the reader to [1, 7–9].

In the afore-mentioned literature, considerable research effort has been invested into the issue of optimality. In [10], the authors showed that for unreliable manufacturing systems with parameters from a certain domain, the zero-inventory policies are optimal even if there is an uncertainty in the manufacturing capacity. This was demonstrated under the assumption that the demand rate is constant and the production rate can be adjusted proceeding from the deviation from the optimal inventory level. In [11], the optimality of pull controlled flow shop was established in the case where the performance index encompasses the buffer holding costs and system shortfall/inventory costs. This research treated the production demand and processing rates as deterministic.

In [12], a broad class of dynamic scheduling problems associated with single-server, multiclass, continuous-flow, flexible manufacturing systems was considered. The objective was to minimize the integral of an instantaneous cost function defined on the inventory/backlog state of the system. The authors provide sufficient conditions under which the optimal solution comes to implementation of the myopic scheduling policy. The paper also presents examples and counterexamples that explicitly illustrate the behavior and limitations of the myopic scheduling policies.

In [13], a model based on stochastic discrete-time controlled dynamical systems was developed in order to derive optimal policies for controlling the material flow in supply networks. Contrary to most studies in the area, which typically assume a given (parameterized) control strategy and analyze how the dynamics depends on the parameters, the authors do not assume a certain family of strategies a priori, thus allowing any control law in the form of a function of the current state of the system. The individual nodes are controlling their inflows in a decentralized fashion by placing orders to their immediate suppliers. An explicit optimal state-feedback control is derived with respect to the cost functional that typically takes into account both inventory holding costs and ordering costs.

Extended reviews on optimal control in production networks can be found in, for example, [8, 9, 14].

This paper offers the extension of our previous results (see [15]) on an analytical analysis for pull-type policies that concern a single machine and a line of machines with both being driven by the cumulative production demand. In addition to our previous work (see [15]), where we also examined the performance of one machine and the line, here we present some novel results. In this paper, in case of one machine, we find the optimal control policy that minimizes the mismatch between the cumulative production output and cumulative demand. Then, we examine a tandem line, where each machine is driven by the obtained optimal policy. For this network, we obtain the bounds of the production surplus for all machines, as well as in [15], but now assuming that the manufacturing line has a limited inventory (buffer content) for storing its intermediate production. Also in this paper we provide the analytical results obtained on the tradeoff relationship between the production tracking error bounds and the inventory level (buffer capacity) of a manufacturing line. In our simulation results, we show that this relation have an important meaning and can be used as a tool in practice.

Thus, we first tackle the problem of optimal control for one manufacturing machine that nominally produces products in lots of a given size and is controlled by carrying out only two commands ("on" and "off"), unlike [13], where the size of the lot was controllable and the set of feasible controls was a polygon. Another difference is that we deal with a deterministic model of the system. Similarly to [13] and contrary to the bulk of the literature in the area, we do not limit the class of the strategies *a priori* and take into account all control laws that are fed by the currently available data, which concerns not only the current system state but also the past states. The system is influenced by uncertain disturbances that are due to both market fluctuations in demand rate and random fluctuations in the production rate of the machine. By assuming known bounds on these disturbances and by applying the min-max dynamical programming, we prove that surplus-based pull control policy is optimal.

Further, we apply classical tools from the control theory in order to evaluate the performance of this technique for a unidirectional production line of N machines with limited capacity intermediate buffers.

The production flow process is described by means of difference equations and in order to analyze performance, Lyapunov theory approach is employed (see, e.g., [16, 17] and references therein). Each machine in the line coordinates its individual production with those of the rest of the system. Its primary objective may be viewed as manufacturing of sufficient quantity of parts to satisfy the demand of its immediate downstream machine and some desired amount as backup material storage in its downstream buffer. The production strategy itself is intuitive and we will show that it can be associated with existing techniques.

To the best of our knowledge, concerning the previous results on performance analysis of surplus-based approaches (see, e.g., [9, 10, 18–23], as well as the references from our review), the novelty of our result can be summarized as follows. The studied tandem production model is considered in discrete time, where the production speed of each machine is defined as deterministic with bounded perturbations. The future production demand is assumed to be unknown and with bounded fluctuations. As the result, for a line of N machines with limited capacity intermediate buffers, strict "worst case" bounds on production tracking errors were obtained as well as their relation with the system inventory levels. In our simulation results, it is shown that the obtained relation can be a very important tool to be used in the decision-making process by a production line manager.

The paper is organized as follows. In Section 2, a discrete-time flow model of a single manufacturing machine is presented and the optimal control strategy is obtained. Section 3 is devoted to study of a flow model of a unidirectional manufacturing line. Performance of the closed-loop production line is illustrated by computer simulations in Section 4. Finally, Section 5 presents conclusions and discussions of future work.

2. Analysis of One Manufacturing Machine

2.1. Flow Model

At discrete time k and for one manufacturing machine, the cumulative number of produced products can be described as the sum of its production rates at each time step till time k. So, the flow model of one manufacturing machine in discrete time is defined as

$$y(k+1) = y(k) + u(k) + f(k), \quad \forall k \in N, \tag{2.1}$$

where $y(k) \in \mathbb{R}$ is the cumulative output of the machine in time k, $u(k) \in \mathbb{R}$ is the control signal, $f(k) \in \mathbb{R}$ is an unknown external disturbance that affects on the production rate of the machine and N is a set of naturals.

Under the assumption that there is always sufficient quantity of the raw material to feed the machine, the control aim is to track the nondecreasing cumulative production demand. We define the cumulative production demand by using $y_d(k) \in \mathbb{R}$ given by

$$y_d(k) = y_{d0} + v_d k + \varphi(k), \tag{2.2}$$

where y_{d0} is a positive constant that represents the initial production demand, v_d is a positive constant that defines the average desired demand rate, and $\varphi(k) \in \mathbb{R}$ is the bounded fluctuation that is imposed on the linear demand $v_d k$, for example, market fluctuation.

At each time step k, the control input u can take only two values: 1 or 0. With regard to (2.1), this means that the nominal size of the lot produced by the machine for the unit time is assumed to be 1. (The unity is taken for the definiteness. The case of arbitrary production rate can be reduced to that at hand by proper scaling of all variables involved.)

To efficiently fulfill the control aim, we are going to minimize the production error $\varepsilon(k) = y_d(k) - y(k)$ over the class of all control strategies fed by the available data:

$$u(k) = U_k\left[y(0), \dots, y(k), y_d(0), \dots, y_d(k)\right] \in \{0; 1\}. \tag{2.3}$$

Due to (2.1) and (2.2), the evolution of the production error $\varepsilon(k)$ obeys the following equation:

$$\varepsilon(k+1) = \varepsilon(k) - u(k) + \xi(k), \quad \text{where } \xi(k) := v_d + \Delta\varphi(k) - f(k), \tag{2.4}$$

and $\Delta\varphi(k) = \varphi(k+1) - \varphi(k)$ is the deviation of the demand rate $\Delta\varphi(k) + v_d$ from its average desired value v_d. We assume that despite of the disturbances, the demand rate is realistic, that is, it lies within the nominal bounds $u(k)+f(k)|_{u(k)=0} = f(k)$ and $u(k)+f(k)|_{u(k)=1} = f(k)+1$ of the actual production rate $u(k) + f(k)$. As a result, the signed deviation $\varepsilon(k) = y_d(k) - y(k)$ of the cumulative output from the cumulative demand is controllable: it can be both decreased $\varepsilon(k+1) \le \varepsilon(k)$ and increased $\varepsilon(k+1) \ge \varepsilon(k)$ by applying the proper feasible control $u(k) = 0, 1$.

This implies the following condition (also known as capacity condition):

$$0 < \xi(k) < 1, \quad \forall k \in \mathbb{N}. \tag{2.5}$$

2.2. Optimal Control Strategy

In this section, we examine the following two optimization problems:

$$J_T = \sup_{\xi(0),\dots,\xi(T-1)} \sum_{k=0}^{T} |\varepsilon(k)|^p \longrightarrow \min_U, \tag{2.6}$$

$$J_\infty = \limsup_{k\to\infty} \sup_{\xi(\cdot)} |\varepsilon(k)|^p \longrightarrow \min_U. \tag{2.7}$$

Here, $p \ge 1$ is a given constant, sup is taken over all $\xi(\cdot)$ satisfying (2.5), and $U = \{U_k(\cdot)\}_{k=0}^{\infty}$ is the set of all possible control strategies given by (2.3).

The problem (2.6) is to optimize the worst-case summary error, where the sum is over the given and finite time horizon T and the worst case is considered with respect to feasible perturbations $\varphi(k)$, $f(k)$, and demand rate v_d. This optimization provides the best performance guarantees in the face of uncertainties in $v_d, \varphi(k)$, and $f(k)$. The performance criterion (2.7) is aimed at minimization of the worst-case production error at large enough times via, maybe, sacrificing the exactness of tracking the production demand during an initial time interval. Though the control objectives in (2.6) and (2.7) are different, the main result of this section (Theorem 2.1) shows that they are achieved by a common control strategy.

In (2.6) and (2.7), the larger values of the parameter p are used to penalize large production errors more severely. In the literature, the most popular values are $p = 1, 2$. We consider arbitrary $p \in [1, +\infty)$ partly in order to show that the solution of the optimization problem is not influenced by the value of p. This is presumably due to the use of discrete controls $u = 0, 1$; if the set of feasible controls is not finite, the solution typically depends on p.

The following theorem is the main result of the section.

Theorem 2.1. *The following control strategy*

$$u(k) = \operatorname{sign}_+(\varepsilon(k)) \tag{2.8}$$

is optimal with respect to the performance index (2.6) *for any given T, as well as with respect to the performance criterion* (2.7)*. This is true irrespective of the choice of $p \in [1, +\infty)$.*

Here, the control action is given by

$$u = \operatorname{sign}_+(\varepsilon) := \begin{cases} 1 & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon < 0 \\ 0, 1 & \text{if } \varepsilon = 0. \end{cases} \tag{2.9}$$

Basically, this controller works as a pull control that authorizes the machine to produce if the product surplus is negative, stops the machine if the surplus is positive, and randomly selects between the two previous decisions if the surplus is on the boundary.

The proof of Theorem 2.1 is given in Appendix A.

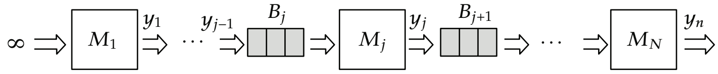

Figure 1: Schematics of a line of N manufacturing machines.

Now that for one machine the optimal tracking controller is derived, we can extend our analysis of this strategy by applying it to a line of N manufacturing machines with bounded intermediate buffers.

3. A Line of Machines with Bounded Buffers

3.1. Flow Model

The flow model of a manufacturing line is presented in this section. Figure 1 shows a schematics of a line of N manufacturing machines with machines M_j, buffers B_j, and infinite product supply. The optimal control strategy from the previous section is modified with respect to the buffers and the machines present in the line. New limitations such as desired buffer content and buffer capacity restriction are considered in the model.

The flow model of the manufacturing line is defined as

$$y_1(k+1) = y_1(k) + \beta_1(k)\mathrm{sign}_{-}\left(w_2(k) - \gamma_2\right), \tag{3.1}$$

$$y_j(k+1) = y_j(k) + \beta_j(k)\mathrm{sign}_{\mathrm{Buff}}\left(w_j(k) - \beta_j(k)\right)\mathrm{sign}_{-}\left(w_{j+1}(k) - \gamma_{j+1}\right), \quad j = 2,\ldots,N-1, \tag{3.2}$$

$$y_N(k+1) = y_N(k) + \beta_N(k)\mathrm{sign}_{\mathrm{Buff}}\left(w_N(k) - \beta_N(k)\right), \tag{3.3}$$

where $y_j(k)$ is the cumulative output of machine M_j in time k, $w_j(k) = y_{j-1}(k) - y_j(k)$ is the content of buffer B_j, $\beta_j(k) = u_j(k) + f_j(k)$, for all $j = 1,\ldots,N$, f_j is the external disturbance affecting machine M_j (e.g., production speed variations, undesired delay or setup time), u_j is the control input of machine M_j, $\mathrm{sign}_{\mathrm{Buff}}(x) = (1,$ if $x \geq 0 \mid 0$, otherwise), $\mathrm{sign}_{-}(x) = (1,$ if $x \leq 0 \mid 0$, otherwise), and γ_{j+1} is the threshold value of the buffer content w_{j+1}.

In order to give a solution to the demand tracking problem, we propose the following control inputs:

$$\begin{gathered} u_j(k) = \mu_j\ \mathrm{sign}_{+}\left(\varepsilon_{j+1}(k) + w_{d_{j+1}} - w_{j+1}(k)\right), \quad \forall j = 1,\ldots,N-1, \\ u_N(k) = \mu_N\ \mathrm{sign}_{+}\left(y_d(k) - y_N(k)\right), \end{gathered} \tag{3.4}$$

where μ_j is the constant processing speed of machine j, $w_{d_{j+1}}$ is the constant that represents the desired inventory level of buffer B_{j+1}, and ε_{j+1} is the production error of machine M_{j+1}. Here, for simplicity, we restrict the value of sign_{+} function, which was defined in the previous

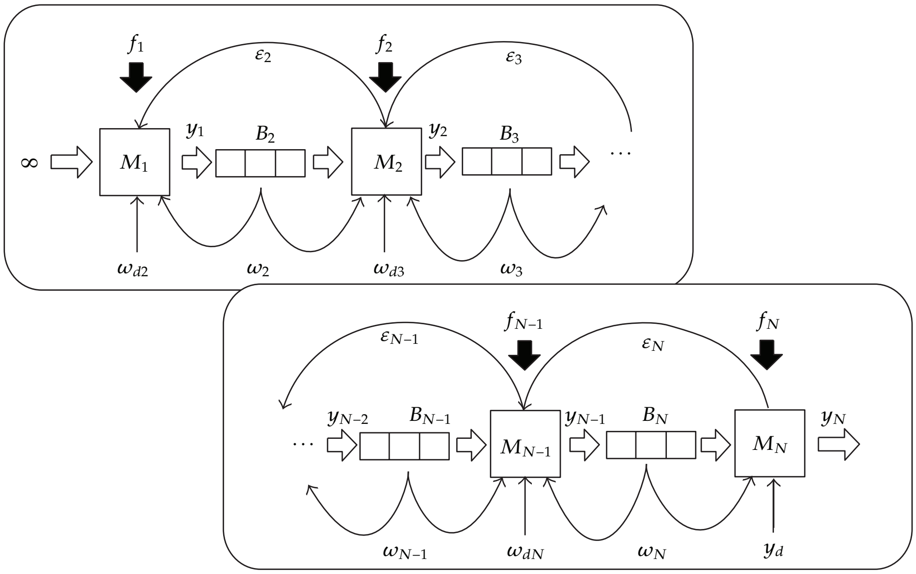

Figure 2: Flow model diagram for a line of N manufacturing machines.

section, to $\operatorname{sign}_+(x) = (1,\ \text{if}\ x > 0 \mid 0, \text{otherwise})$. The tracking error of each machine is given by

$$\varepsilon_j(k) = \varepsilon_{j+1}(k) + \left(w_{d_{j+1}} - w_{j+1}(k)\right), \quad \forall j = 1, \ldots, N-2, \tag{3.5}$$

$$\varepsilon_{N-1}(k) = \varepsilon_N(k) + (w_{d_N} - w_N(k)), \tag{3.6}$$

$$\varepsilon_N(k) = y_d(k) - y_N(k). \tag{3.7}$$

It follows from (3.7) that the error of machine M_N is defined exactly as for the single machine case. The buffer restriction, as seen from (3.3), is the only difference in the flow model of machine M_N with the flow model of (2.1). For (3.5), (3.6) new considerations are applied for the production error of each machine M_j, where $j = 1, \ldots, N-1$. Here, $\varepsilon_j(k)$ depends on number of produced products $y_j(k)$ with respect to current demand $y_d(k)$ and desired buffer content (buffer inventory level) $w_{d_{j+1}}$ of each downstream buffer. This means that every upstream machine needs to produce $w_{d_{j+1}}$ lots more than the downstream one. Constant parameter w_d is introduced in order to prevent downstream machines from starvation, for example, in case of a sudden growth of the product demand.

Figure 2 shows a schematic of the information flow throughout the manufacturing line of N machines. The squares represent the manufacturing machines each with inside label M_j and outside short thick black arrows denoting the external perturbations (f_i), which are affecting its production rates. Each machine (exept for M_1) has a buffer connected to it, each one denoted by 3 joined squares. The product flow directions are denoted by a thick white arrows with a thin black frame. The transferring of the production tracking error (ε_j) information is shown by arched thin black arrows, going from one machine to another in upstream manner. For each machine, the upstream and downstream inventory level (w_j) information transfer is depicted by a curved thin black arrow coming from each buffer and the desired downstream buffer inventory level is shown by a short thin black arrow pointing to each machine.

The substantial difference in the model for N machines compared to the model of one machine can be appreciated through the flow model (3.2) as well as graphically through the Figure 2. For each machine, we introduce an extra restriction on production that is based on the buffer content of its upstream and downstream buffer. Functions $\text{sign}_{\text{Buff}}(w_j(k) - \beta_j(k))$ and $\text{sign}_{-}(w_{j+1}(k) - \gamma_{j+1})$ together with the control input $u_j(k)$ are acting as the restrictions that are imposed on production of M_j. Thus, any machine M_j, with $j = 2, \dots, N-1$, is activated only if three authorizations are given. The first authorization comes from control input of M_j, which is based on the current production error value of this machine $(\varepsilon_j(k))$. The second authorization comes from the restriction on the upstream buffer content that is granted if the buffer contains at least the minimal number of products required $(\beta_j(k))$ in order for the machine M_j to start its work. Finally, the third authorization comes from the downstream buffer of given machine. This authorization is possible only if the downstream buffer have sufficient storage in order to accept incoming production.

Note that we could easily associate this control algorithm with Basestock policy (see [7] and references there in) as well as with Hedging Point policy (see [1] and references there in). From (3.5) the production error of each intermediate machine can be interpreted as $\varepsilon_j(k) = y_d(k) - y_j(k) + w_{d_{j+1}} + \cdots + w_{d_N}$, $j = 1, \dots, N-2$.

This means that each machine is keeping track of the current demand as well as of its Hedging point or its Basestock level, which in this case is the sum of the desired buffer contents of all the downstream buffers of M_j. Also, due to our intermediate buffer content limitation (γ_j), this policy could be associated to a Kanban or a combination of local Conwip controllers (one for each intermediate machine) and a surplus-based pull control (for the output machine M_N).

It is also important to take into account that the control actions are decentralized throughout the network. In other words, the control action of each machine in the line depends only on the production error of its neighboring downstream machine (except for machine M_N, which control action depends directly on cumulative demand input) and the current buffer content of its upstream and downstream buffer (see Figure 2). This gives our flow model an extra robustness with respect to the undesired events such as temporal machine setup or breakdown.

For further analysis, let us rewrite flow model (3.1), (3.2), and (3.3) in feedback with (3.4) as

$$\Delta\varepsilon_1(k) = v_d + \Delta\varphi(k) - f_1(k) - \mu_1 \, \text{sign}_{+}(\varepsilon_1(k))\text{sign}_{-}\left(w_2(k) - \gamma_2\right), \tag{3.8}$$

$$\Delta\varepsilon_j(k) = v_d + \Delta\varphi(k) - f_j(k) - \mu_j \, \text{sign}_{+}\left(\varepsilon_j(k)\right)\text{sign}_{\text{Buff}}\left(w_j(k) - \beta_j(k)\right)\text{sign}_{-}\left(w_{j+1}(k) - \gamma_{j+1}\right), \tag{3.9}$$

$$\Delta\varepsilon_N(k) = v_d + \Delta\varphi(k) - f_N(k) - \mu_N \, \text{sign}_{+}(\varepsilon_N(k))\text{sign}_{\text{Buff}}\left(w_N(k) - \beta_N(k)\right), \tag{3.10}$$

where $\Delta\varepsilon_j(k) = \varepsilon_j(k+1) - \varepsilon_j(k)$.

For system (3.8), (3.9), and (3.10), let us introduce the following assumptions.

Assumption 3.1 (Boundedness of perturbations). There are constants α_1, α_2, and α_3 such that $W_j(k) = \Delta\varphi(k) - f_j(k)$, for all $j = 1, \dots, N$, satisfies

$$\alpha_1 < W_j(k) < \alpha_2, \quad \forall k \in \mathbb{N}, \tag{3.11}$$

and $f_j(k)$ satisfies

$$f_j(k) \leq \alpha_3, \quad \forall k \in \mathbb{N}. \tag{3.12}$$

Assumption 3.2 (Capacity condition). Constants α_1, α_2 satisfy the following inequalities

$$\alpha_2 < \mu_j - v_d, \tag{3.13}$$

$$\alpha_1 > -v_d. \tag{3.14}$$

Thus, from (3.11), (3.13), and (3.14), the following condition holds:

$$0 < v_d + W_j(k) < \mu_j, \quad \forall j = 1, \dots, N. \tag{3.15}$$

It is important to notice that each machine M_j in the line has a processing speed of μ_j lots per time unit, which can differ from the rest of the machines.

The buffer content condition is considered as

$$\beta_j(k) \leq w_j(k) < \gamma_j, \quad \forall j = 2, \dots, N. \tag{3.16}$$

Note that the physical restriction on buffer content is given as

$$0 \leq w_j(k) \leq \gamma_j + \mu_{j-1} + \alpha_3, \quad \forall j = 2, \dots, N. \tag{3.17}$$

Here, $\gamma_j = \mu_j + \alpha_2 - \alpha_1 + w_{d_j}$. Thus, from (3.5), (3.6) and (3.16), the following tracking error condition holds

$$\varepsilon_j(k) \geq \beta_j(k) - w_{d_j} + \varepsilon_{j-1}(k), \quad \forall j = 2, \dots, N, \tag{3.18}$$

where w_{d_j} satisfies the following assumption.

Assumption 3.3 (Desired buffer content condition). The constants w_{d_j} comply with the following inequality:

$$w_{d_j} \geq \mu_j + \mu_{j-1} + \alpha_3 + \alpha_2 - \alpha_1, \tag{3.19}$$

from where it follows that

$$w_{d_j} \geq \beta_j(k) + \mu_{j-1} + \alpha_2 - \alpha_1, \quad \forall k \in \mathbb{N},\ j = 2, \dots, N. \tag{3.20}$$

One can note the following relation between the buffer content and the tracking errors of its neighboring machines. That is, if the first part of inequality (3.16) is not satisfied, that is, the buffer content is lower than the minimum, then

$$\varepsilon_{j-1}(k) \overset{(3.6),(3.17)}{>} \mu_{j-1} + \alpha_2 - \alpha_1 + \varepsilon_j(k). \tag{3.21}$$

In case the second part of (3.16) is unsatisfied, that is, the buffer is full, then

$$\varepsilon_j(k) \overset{(3.6)}{\geq} \mu_j + \alpha_2 - \alpha_1 + \varepsilon_{j-1}(k). \tag{3.22}$$

3.2. Results on Performance

In this section, we present obtained results on the production tracking error trajectories behavior of flow model (3.8).

Theorem 3.4. *Assume that the discrete time system defined by* (3.8), (3.9), *and* (3.10) *satisfies Assumptions 3.1, 3.2, and 3.3. Then, all solutions of* (3.8), (3.9), *and* (3.10) *are uniformly ultimately bounded by*

$$\begin{gathered}\limsup_{k\to\infty} \varepsilon_j(k) \leq v_d + \alpha_2, \\ \liminf_{k\to\infty} \varepsilon_j(k) \geq v_d + \alpha_1 - \mu_j.\end{gathered} \tag{3.23}$$

Proof. The proof of Theorem 3.4 is given in Appendix B. □

Theorem 3.5. *Assume that the discrete time system defined by* (3.8), (3.9), *and* (3.10) *satisfies Assumptions 3.1 and 3.2. Then, all solutions of* (3.8), (3.9), *and* (3.10) *are uniformly ultimately bounded by*

$$\limsup_{k\to\infty} \varepsilon_j(k) \leq v_d + \alpha_2 + x_j, \tag{3.24}$$

$$\liminf_{k\to\infty} \varepsilon_j(k) \geq v_d + \alpha_1 - \mu_j, \tag{3.25}$$

where $x_1 = 0$ *and* $x_j = \sum_{i=2}^{j} \max((\mu_{i-1} - \alpha_1 + \alpha_2 - w_{di} + \underbrace{(\mu_i + \alpha_3)}_{\beta_i}), 0)$ *for all* $j = 2, \dots, N$.

Proof. The proof of Theorem 3.5 is given in Appendix C. □

Now, in order to support the proposed development, let us supplement our analysis by a simulation example.

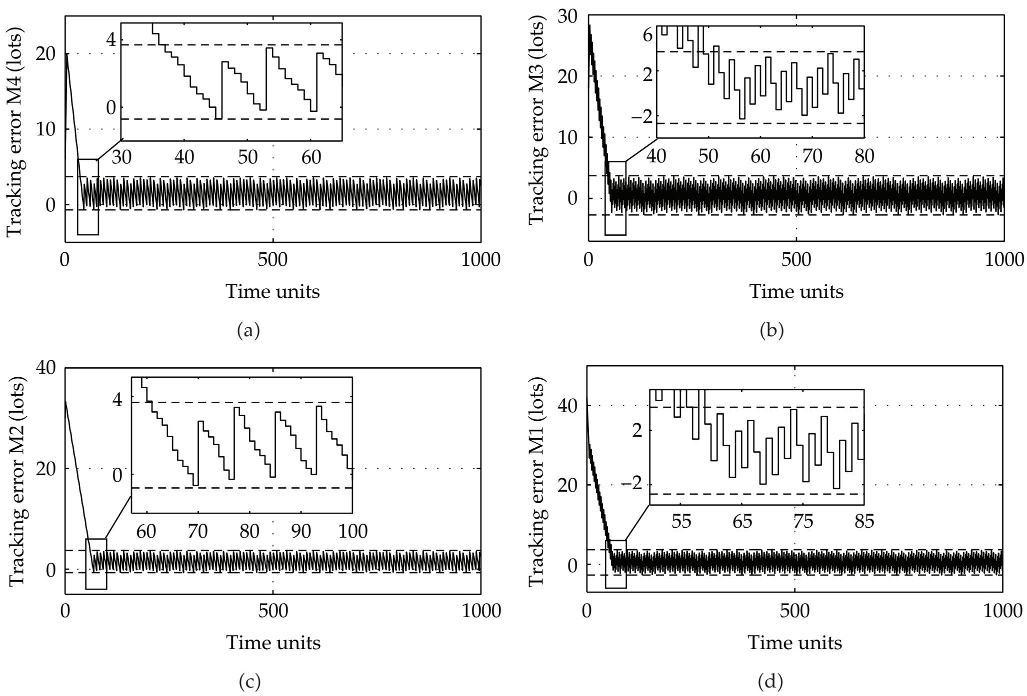

Figure 3: Tracking error $\varepsilon_j(k)$.

4. Simulation Examples

Consider the following example of a production line of 4 manufacturing machines (see Figure 1) operating under surplus-based regulators (3.4). The processing speed for each machine is set to $\mu_j = 6$ lots per time unit for all $j = 1, 3$ and $\mu_j = 4$ lots per time unit for all $j = 2, 4$, the desired inventory level of each buffer is selected considering (3.20) as $w_{d_j} = 12$ lots, with $j = 2, \ldots, 4$, and the mean demand rate $v_d = 3.5$ lots per time unit with fluctuation rate of $\Delta\varphi(k) = 0.2\sin(5k)$. The tracking error of each machine in the line is depicted in Figure 3. Here, the initial conditions $(y_{d0}, y_1(0), y_2(0), y_3(0), y_4(0))$ were set to the zero value. After the first 40 time steps, as it is shown in Figures 4 and 3, the system reaches its steady state. Tracking errors (see the dashed lines of Figure 3) are maintained inside $[-2.7, 3.7]$ lots for machine M_1, $[-0.7, 3.7]$ lots for machine M_2, $[-2.7, 3.7]$ lots for machine M_3, and $[-0.7, 3.7]$ lots for machine M_4, which satisfy the bounds (3.23). From Figure 5 it can be observed that the inventory level of each buffer satisfies the buffer limit given by the second part of inequality (3.17) and the capacity condition (3.16) is sometimes violated due to the discrete nature of the model. Here, $\gamma_2 = 14.8$ (lots), $\gamma_3 = 16.8$ (lots), $\gamma_4 = 14.8$ (lots).

Now, let us show the effectiveness of Theorem 3.5 by means of a following example. Consider a similar production line of 4 manufacturing machines (see Figure 1) operating under surplus-based regulators (3.4). The nominal speed for each machine is $\mu_1 + f_1(k) = 10.56 + 0.5\sin(180k)$, $\mu_2 + f_2(k) = 10.7 + 0.5\sin(20k)$, $\mu_3 + f_3(k) = 15.5 + 0.5\sin(45k)$ and $\mu_4 + f_4(k) = 20.5 + 0.5\sin(90k)$ lots per time unit. The desired inventory level of each buffer is selected as $w_{d_j} = \mu_j\rho$ lots, with $j = 2, \ldots, 4$ and ρ as a constant. The experiment was executed 37 times. Thus, the value of constant ρ was modified 37 times as well, starting from $\rho = 1$

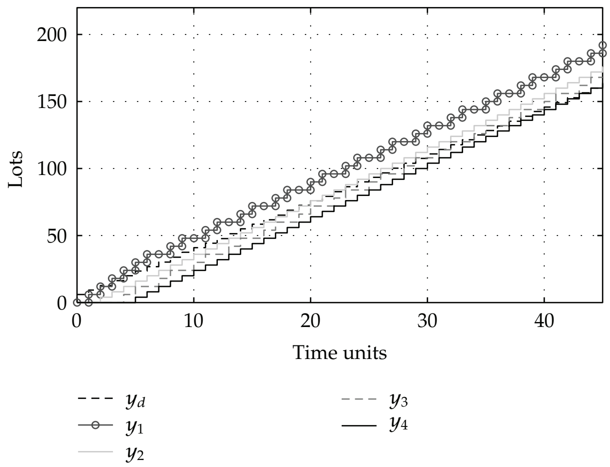

Figure 4: Outputs $y_j(k)$ versus demand $yd(k)$.

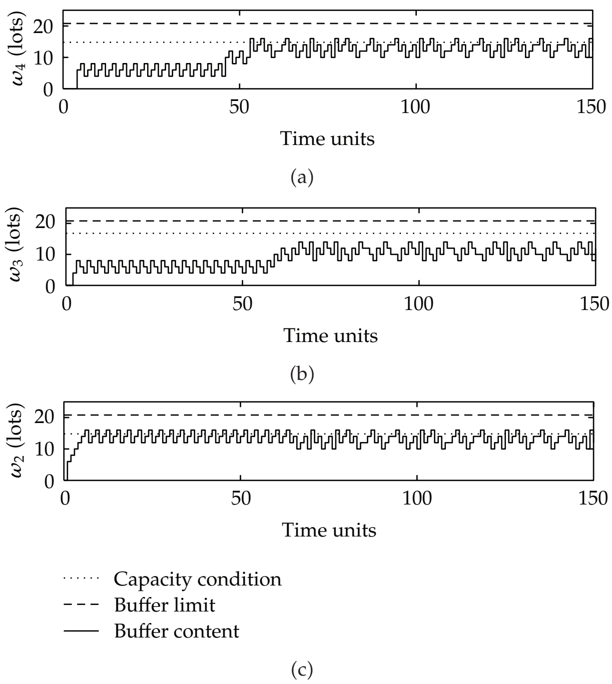

Figure 5: Buffer content $w_j(k)$.

and with increments of 0.25 units till $\rho = 10$. The demand rate was selected as $v_d + \Delta\varphi(k) = 10 + 0.05\cos(45k)$ lots per time unit. Here, each γ_j was selected according to (3.16). For this experiment, the initial conditions $(y_{d0}, y_1(0), y_2(0), y_3(0), y_4(0))$ were set to the zero value.

The relation between the maximal value of steady-state (SS) production tracking errors of machines M_2, M_3, M_4 (see Figure 1) and the desired inventory levels of there upstream buffers is depicted in Figure 6. Each graphic of Figure 6 shows the maximal value of each ε_j in SS with its values from simulation and from analytical result given by (3.24). It could

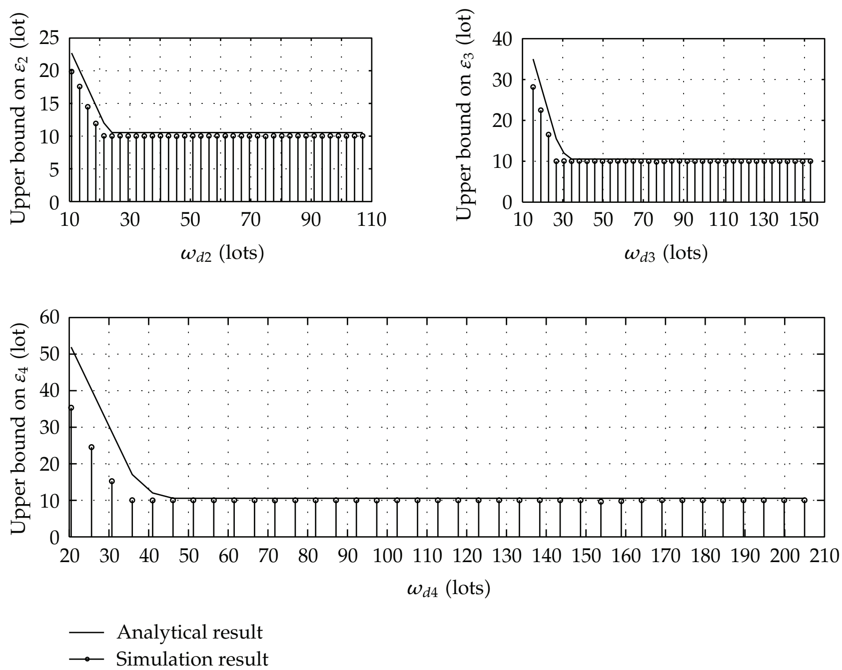

Figure 6: Production tracking errors versus desired inventory levels.

be appreciated that our analytical result describes in rather accurate manner the tradeoff between the desired inventory level and the accuracy of the production demand satisfaction. It is important to notice that the production tracking error precision is limited. Thus, for each of 3 machines, the production surplus can be decreased by incrementing the desired inventory level till some threshold value, which is given by (3.20), after which the bound on the surplus value remains constant. Doing so improves the service level of the network but increases the inventory costs. At the same time, lowering the inventory level further than the threshold decreases the service level of the network. This trade-off relationship plays an important role in the decision-making process of a production line manager. By using the relation provided beforehand in Theorem 3.5, a number of scheduling related decisions can be significantly reduced, which in consequence can decrease a computational time spent on planning for an efficient distribution of resources in a production line. In conclusion, presented simulation results reflect the desired flow model behavior, that is, all technical conditions proposed in this section correspond to analytical results described in Section 3. Also, the result shown on Figure 6 underlines the practical importance of the obtained theoretical results.

5. Conclusion

In the case of a single manufacturing machine in isolation, a surplus-based pull controller was proven to be optimal for the cumulative production demand tracking problem. This control strategy was extended to a tandem production line with variable processing speed of each machine and restrictions on the sizes of intermediate buffers. The performance of the closed-loop system was addressed in the form of bounds on the production tracking errors that occur for each machine in the line, respectively. The analytical results describing the tradeoff relationship between the production tracking errors and the inventory level in the manufacturing line were obtained. All theoretical results were illustrated and confirmed by computer simulation. In simulation results, it was shown that our analytical results on performance can be used as an important tool in the decision-making process of the production line manager.

Our future work includes study of performance of re-entrant networks, and multiple part type production systems. A special attention will be given to detailed production analysis of these networks in the presence of unexpected delays and setup times.

Appendices

A. Proof of Theorem 2.1

Based on (2.4), it is easy to see that without any loss of generality, the class of admissible control strategies (2.3) can be reduced to those processing only the tracking errors:

$$u(k) = U_k[\varepsilon(0), \dots, \varepsilon(k)] \in \{0; 1\}. \tag{A.1}$$

We start with the problem (2.6). The proof is based on the min-max dynamic programming. So, we first introduce the cost-to-go:

$$V_\tau(a) = \min_{U_\tau(\cdot),\dots,U_{T-1}(\cdot)} \sup_{\xi(\cdot)} \sum_{k=\tau}^{T} |\varepsilon(k)|^p, \qquad V_T[a] := |a|^p, \tag{A.2}$$

where the minimum is over all functions $U_k(\varepsilon_k, \dots, \varepsilon_{T-1}) \in \{0; 1\}$, and $\varepsilon(k)$ is obtained from (2.4), where $k = \tau, \dots, T-1$ and $\varepsilon(\tau) = a$. This function satisfies the Bellman equation ([24]):

$$V_{\tau-1}(a) = \min_{u=0;1} \sup_{\xi\in(0;1)} \left\{ |a|^p + V_\tau[a - u + \xi] \right\}, \tag{A.3}$$

and the optimal strategy is given by $u(\tau-1) = U^0_{\tau-1}[\varepsilon(\tau-1)]$, where $U^0_{\tau-1}[a]$ is the point u furnishing the minimum in (A.3).

Lemma A.1. *The cost-to-go* (A.2) *is the piece-wise smooth even function depicted in Figures 7 and 8, and*

$$U^0_\tau(a) = \operatorname{sign}_+(a) \quad \text{for } \tau = 0, \dots, T-1. \tag{A.4}$$

Proof. We first note that (A.3) can be shaped into

$$V_{\tau-1}(a) = \min \left\{ \overbrace{\sup_{\xi\in(0;1)} V_\tau[a+\xi]}^{S_0};\ \overbrace{\sup_{\xi\in(0;1)} V_\tau[a-\xi]}^{S_1} \right\} + |a|^p. \tag{A.5}$$

Here, S_0 and S_1 correspond to $u = 0$ and $u = 1$, respectively. So, $U^0_{\tau-1}(a) = \sigma_{\min}$, where $\sigma_{\min} = 0, 1$ is the index of the term S_σ furnishing the minimum in (A.5). We also note that since

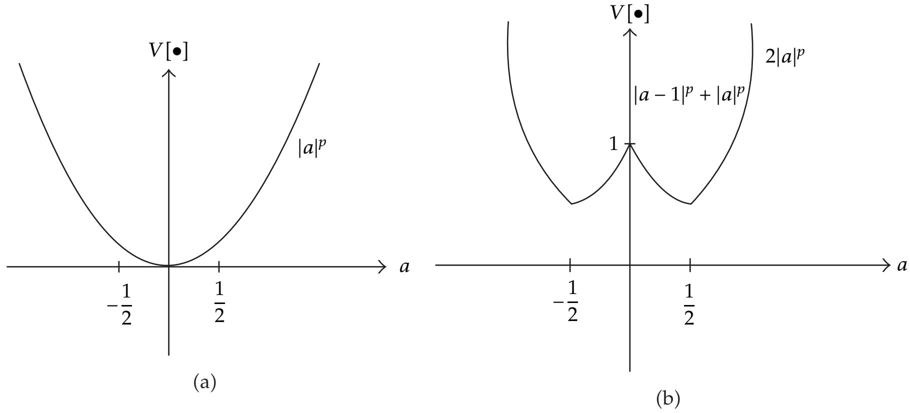

Figure 7: (a) The graph of V_T; (b) the graph of V_{T-1}.

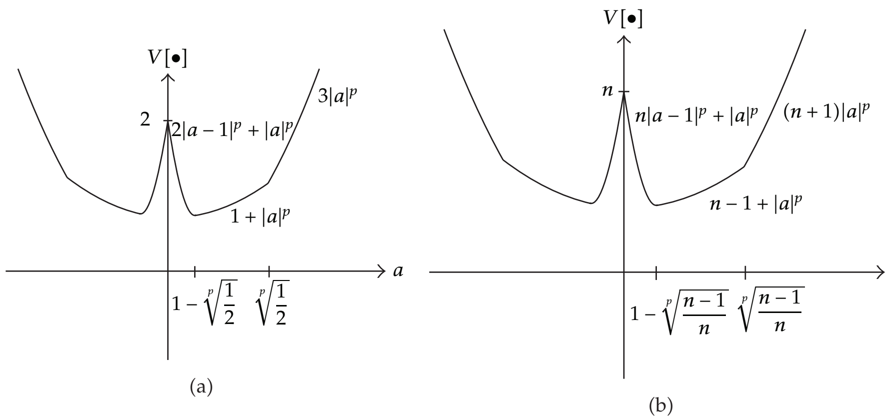

Figure 8: (a) The graph of V_{T-2}; (b) the graph of V_{T-n} with $n \geq 3$.

the function $a \mapsto |a|^p$ is even, simple induction on $\tau = T, \ldots, 0$ and the last equation from (A.2) show that $V_\tau[\cdot]$ is even for any τ. With this in mind, it becomes clear that firstly, $\sigma_{\max} = 0, 1$ for $a = 0$ and secondly, substitution $a := -a$ in (A.5) switches $\sigma_{\min}$ to the alternative value. This permits us to focus on $a > 0$ in the subsequent proof. For $a > 0$, formula (A.4) (to be justified) takes the form $U_\tau^0(a) = 1$.

We proceed with immediate proof of the lemma, arguing by induction on $\tau = T - n, n = 0, 1, \ldots$.

(i) $n = 0$. The claim is immediate from the last equation in (A.2).

(ii) $n = 1$.

(a) $a \geq 1/2$. Then evidently, $S_1 = |a|^p$, and $S_0 = |a + 1|^p > S_1$. So, due to (A.5), $V_{T-1}(a) = 2|a|^p$, as is depicted in Figure 7(b), and $U_\tau^0(a) = 1$.

(b) $0 < a < 1/2$. Since $V_T(\cdot)$ is even, $S_1 = |a - 1|^p < |a + 1|^p = S_0$. So $V_{T-1} = |a - 1|^p + |a|^p$, as is depicted in Figure 7(b), and $U_\tau^0(a) = 1$.

(iii) $n = 2$.

(a) $a \geq 1$. Similarly, in (A.5), the supremum S_0 is equal to $2|a+1|^p$, whereas $S_1 = 2|a|^p < S_0$.

(b) $1/2 \leq a < 1$:

$$S_1 = \begin{Bmatrix} 2|a|^p & a > \sqrt[p]{1/2} \\ 1 & a < \sqrt[p]{1/2} \end{Bmatrix} < 2|a+1|^p = S_0, \tag{A.6}$$

(c) $0 \leq a < 1/2$:

$$S_1 = \begin{Bmatrix} 2|a-1|^p & a < 1 - \sqrt[p]{1/2} \\ 1 & a > 1 - \sqrt[p]{1/2} \end{Bmatrix} < 2|a+1|^p = S_0. \tag{A.7}$$

Thus

$$V_{T-2}(a) = \begin{cases} 3|a|^p & a \geq \sqrt[p]{1/2} \\ 1 + |a|^p & 1 - \sqrt[p]{1/2} \leq a < \sqrt[p]{1/2} \\ 2|a-1|^p + |a|^p & a < 1 - \sqrt[p]{1/2}, \end{cases} \tag{A.8}$$

as is depicted in Figure 8(a), so $U_\tau^0(a) = 1$.

Figure 8(a) is a particular case of Figure 8(b). So to complete the proof, it suffices to show that:

(D) Figure 8(b) is correct and $U_{T-n}^0(a) = 1$ for $n = 2, 3, \ldots$, arguing by induction on n.

Suppose that (D) is true for some $n \geq 2$. To compute $V_{T-n-1}(a)$, we consider separately several cases.

(i) $a \geq \sqrt[p]{n/(n+1)}$. Here $\sqrt[p]{n/(n+1)} > \sqrt[p]{(n-1)/n}$. It follows that in (A.5) the supremum S_1 is attained at $\xi = 0$ and thus equals $(n+1)|a|^p$, whereas $S_0 = (n+1)|a+1|^p > S_1$. Thus, (D) does hold for $n := n+1$.

(ii) $\sqrt[p]{(n-1)/n} \leq a \leq \sqrt[p]{n/(n+1)}$. Then evidently $S_1 = n$, whereas $S_0 = (n+1)|a+1|^p > S_1$. Thus (D) does hold for $n := n+1$.

(iii) $1 - \sqrt[p]{(n-1)/n} \leq a \leq \sqrt[p]{(n-1)/n}$. Since the left end $a-1$ of the interval $[a-1, a]$ is still to the right of the first fracture point of the graph from Figure 3(b), the situation replicates the previous one.

(iv) $1 - \sqrt[p]{n/(n+1)} \leq a \leq 1 - \sqrt[p]{(n-1)/n}$. That end is to the left of the first fracture point. So either $S_1 = n$ (and is attained at the third fracture point) or $S_1 = (n+1)|a-1|^p$ (and is attained at $\xi = 1$). Elementary comparison shows that in fact $S_1 = n$, and so the situation still replicates the previous two ones.

(v) $0 \leq a \leq 1 - \sqrt[p]{n/(n+1)}$. Then, conversely, $S_1 = (n+1)|a-1|^p$, whereas $S_0 = (n+1)|a+1|^p > S_1$. Thus, (D) does hold for $n := n+1$, which completes the proof. □

For the performance index (2.6), Theorem 2.1 is straightforward from Lemma A.1 and the dynamic programming principle ([24]).

To deal with (2.7), we introduce the following intermediate performance criterion:

$$J_{\text{av}} = \limsup_{T\to\infty} \sup_{\xi(0),\ldots,\xi(T-1)} \frac{1}{T}\sum_{k=0}^{T} |\varepsilon(k)|^p. \tag{A.9}$$

It is clear that

$$\inf_U J_{\text{av}} \geq \limsup_{T\to\infty} \frac{1}{T} \min_U J_T \overset{\text{(A.2)}}{=\!=} \limsup_{T\to\infty} \frac{V_0^T(a)}{T}, \tag{A.10}$$

where the upper index ^{T} in V_τ^T underscores that the cost-to-go is computed for the time horizon $[0:T]$. As a result, Lemma A.1 and the evident inequality $J_\infty \geq J_{\text{av}}$ imply the following lower estimates:

$$\inf_U J_\infty \geq \inf_U J_{\text{av}} \geq \begin{cases} |a|^p & \text{if } |a| \geq 1 \\ 1 & \text{otherwise.} \end{cases} \tag{A.11}$$

Now, we are going to show that this lower estimate of J_∞ is attained at the control strategy (2.8), which will complete the proof.

Let the system (2.4) be driven by the control law (2.8). By invoking (2.5), we conclude that

$$\varepsilon[k+1] \in \begin{cases} (\varepsilon[k]-1, \varepsilon[k]) & \text{if } \varepsilon(k) > 0 \\ (\varepsilon[k], \varepsilon[k]+1) & \text{if } \varepsilon(k) < 0 \\ (\varepsilon[k]-1, \varepsilon[k]+1) & \text{if } \varepsilon(k) = 0. \end{cases} \tag{A.12}$$

Hence,

$$f_-[\varepsilon(k)] \leq \varepsilon(k+1) \leq f_+[\varepsilon(k)], \quad \text{where } f_-(\varepsilon) := \min\{\varepsilon; -1\},\ f_+(\varepsilon) := \max\{\varepsilon; 1\}. \tag{A.13}$$

It follows that

$$\varepsilon_-(k) \leq \varepsilon(k) \leq \varepsilon_+(k) \quad \forall k, \tag{A.14}$$

where $\varepsilon_-(k)$ and $\varepsilon_+(k)$ are the solutions of the following recursions:

$$\varepsilon_\pm(k+1) = f_\pm[\varepsilon_\pm(k)], \qquad \varepsilon_\pm(0) = a. \tag{A.15}$$

It is evident that

$$\varepsilon_\pm(k) \in [\min\{-|a|, -1\};\ \max\{|a|; 1\}], \tag{A.16}$$

which completes the proof.

B. Proof of Theorem 3.4

Proof. Let us prove that Theorem 3.4 holds for a line of 2 manufacturing machines ($j = 1, 2$) defined by (3.8) and (3.10). With this goal, let us introduce the following Lyapunov function:

$$V^{2M}(\varepsilon_1; \varepsilon_2) = \max\{V_1(\varepsilon_1), V_2(\varepsilon_2)\}, \tag{B.1}$$

where

$$V_j(\varepsilon_j) = \max\{-\varepsilon_j - \mu_j + v_d + \alpha_1, \varepsilon_j - v_d - \alpha_2, 0\} > 0, \quad \forall \varepsilon_j \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2], \ j = 1, 2. \tag{B.2}$$

Here, for the sake of brevity $V^{2M}(\varepsilon_1(k), \varepsilon_2(k)) = V_k^{2M}$, $V_j(\varepsilon_j(k)) = V_{j,k}$, with $V^{2M} = 0$ for all $\varepsilon_j \in [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$.

Thus, ΔV_k^{2M} along the solutions of $\varepsilon_1(k)$ and $\varepsilon_2(k)$ is given by

$$\Delta V_k^{2M} = V_{k+1}^{2M} - V_k^{2M} = \max\{V_{1,k+1}, V_{2,k+1}\} + \min\{-V_{1,k}, -V_{2,k}\}, \tag{B.3}$$

where

$$V_{j,k+1} = \max\left\{\begin{matrix} -\varepsilon_j(k) - W_j(k) + \alpha_1 - \mu_j + \mu_j\eta_{j,k}, \\ \varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j\eta_{j,k}, \\ 0 \end{matrix}\right\}, \quad j = 1, 2. \tag{B.4}$$

Here, for the sake of brevity we introduce $\eta_{j,k}$ as

$$\begin{aligned} \eta_{1,k} &= \operatorname{sign}_+(\varepsilon_1(k))\operatorname{sign}_-(w_2(k) - \gamma_2), \\ \eta_{2,k} &= \operatorname{sign}_+(\varepsilon_2(k))\operatorname{sign}_{\text{Buff}}(w_2(k) - \beta_2(k)). \end{aligned} \tag{B.5}$$

In order to perform a more detailed analysis on ΔV_k^{2M}, let us divide this proof into 3 cases.

Case 1 (Sufficient buffer content). Suppose that $w_2(k)$ satisfies the following inequality:

$$\beta_2(k) \le w_2(k) < \gamma_2, \tag{B.6}$$

which means that machine M_2 has sufficient material in its buffer B_2 in order to start working and machine M_1 always has an access to the infinite raw material supply. Thus, these machines have an independent behavior and it will be sufficient to analyse the increment of only one of the functions $V_{j,k}$ in order to determine the behavior of ΔV_k^{2M}.

Let us assume that $\varepsilon_j(k)$ satisfies the following condition:

$$\varepsilon_j(k) > 0, \tag{B.7}$$

and in consequence from (B.5) it follows that $\eta_{j,k} = 1$.

Then, $\Delta V_{j,k}$ along the solutions of $\varepsilon_j(k)$ is given by

$$\Delta V_{j,k} = \underbrace{\max\left\{\begin{matrix} -\varepsilon_j(k) - W_j(k) + \alpha_1, \\ \varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j, \\ 0 \end{matrix}\right\}}_{V_{j,k+1}} + \underbrace{\min\left\{\begin{matrix} \varepsilon_j(k) + \mu_j - v_d - \alpha_1, \\ -\varepsilon_j(k) + v_d + \alpha_2, \\ 0 \end{matrix}\right\}}_{-V_{j,k}}, \quad j = 1, 2. \tag{B.8}$$

From where with help of Assumptions 3.1 and 3.2, it can be easily deduced that

$$\Delta V_{j,k} = \begin{cases} 0 & \text{if } \varepsilon_j(k) \le v_d + \alpha_2, \\ -\varepsilon_j(k) + v_d + \alpha_2 < 0 & \text{if } v_d + \alpha_2 < \varepsilon_j(k) \le \mu_j + \alpha_2 - W_j(k), \\ -\mu_j + v_d + W_j(k) < 0 & \text{if } \varepsilon_j(k) > \mu_j + \alpha_2 - W_j(k). \end{cases} \tag{B.9}$$

Now, suppose that for $\varepsilon_j(k)$ the following condition holds:

$$\varepsilon_j(k) \le 0, \tag{B.10}$$

and in consequence from (B.5) it yields that $\eta_{j,k} = 0$. Then, $\Delta V_{j,k}$ along the solutions of $\varepsilon_j(k)$ is given by

$$\Delta V_{j,k} = \underbrace{\max\left\{\begin{matrix} -\varepsilon_j(k) - W_j(k) + \alpha_1 - \mu_j, \\ \varepsilon_j(k) + W_j(k) - \alpha_2, \\ 0 \end{matrix}\right\}}_{V_{j,k+1}} + \underbrace{\min\left\{\begin{matrix} \varepsilon_j(k) + \mu_j - v_d - \alpha_1, \\ -\varepsilon_j(k) + v_d + \alpha_2, \\ 0 \end{matrix}\right\}}_{-V_{j,k}}, \quad j = 1, 2. \tag{B.11}$$

Here, with help of Assumptions 3.1 and 3.2 it can be easily deduced that

$$\Delta V_{j,k} = \begin{cases} 0 & \text{if } v_d + \alpha_1 - \mu_j \le \varepsilon_j(k) \le 0, \\ -v_d - W_j(k) < 0 & \text{if } \varepsilon_j(k) < -\mu_j + \alpha_1 - W_j(k), \\ \varepsilon_j(k) + \mu_j - v_d - \alpha_1 < 0 & \text{if } -\mu_j + \alpha_1 + W_j(k) \le \varepsilon_j(k) < v_d + \alpha_1 - \mu_j. \end{cases} \tag{B.12}$$

Summarizing, for conditions (B.6), (B.7), and (B.10) from (B.9) and (B.12) it holds that if $V_{j,k} > 0$, its increment $\Delta V_{j,k} < 0$. From the definition of min, it yields that for ΔV_k^{2M} given by (B.3) the following inequality is satisfied:

$$\Delta V_k^{2M} \le V_{i,k+1} - V_{i,k} \le 0, \tag{B.13}$$

where $i = \arg\max_{j=1,2}\{V_{j,k+1}\}$. Note that $\Delta V_{j,k} = V_{j,k+1} - V_{j,k} = 0$ only if either first condition of (B.9) or first condition of (B.12) is satisfied and for all $\varepsilon_j(k) \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$ it follows that $\Delta V_k^{2M} < 0$. Thus, in this case it holds that for $V_k^{2M} > 0$ given by (B.1) its increment $\Delta V_k^{2M} < 0$.

Case 2 (Insufficient buffer content). Let us assume that $w_2(k)$ satisfies the following inequality:

$$w_2(k) < \beta_2(k), \quad \forall k \in \mathbb{N}, \tag{B.14}$$

and $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) \le 0, \quad \forall k \in \mathbb{N}. \tag{B.15}$$

Then, from (3.21) it holds that

$$\varepsilon_1(k) > \mu_1 + \alpha_2 - \alpha_1 + \varepsilon_2(k). \tag{B.16}$$

Here, similarly to Case 1, the behavior of these two machines can be considered independently. Thus, for $\varepsilon_1(k)$ satisfying (B.16), it holds that $\Delta V_{1,k}$ is given by (B.9) or (B.12) if $\varepsilon_2(k) < -\mu_1 - \alpha_2 + \alpha_1$ or $\Delta V_{1,k}$ is given by (B.9) if $-\mu_1 - \alpha_2 + \alpha_1 \le \varepsilon_2(k) \le 0$. For $\varepsilon_2(k)$ satisfying (B.15) the increment $\Delta V_{2,k}$ is given by (B.12). In consequence for ΔV_k^{2M} given by (B.3), the inequality (B.13) in this case is also satisfied.

Now, let us assume that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) > 0, \quad \forall k \in \mathbb{N}. \tag{B.17}$$

Then, from (3.21), it holds that $\varepsilon_1(k)$ is given by (B.16). In this case, M_2 has a positive tracking error, but its buffer B_2 has insufficient raw material content (B.14) in order to start working ($\eta_{2,k} = 0$). Machine M_1 has a positive error as well, but due to its infinite raw material supply access, it can immediately initiate its production process ($\eta_{1,k} = 1$). Thus, for (B.17) and (B.16), let us rewrite ΔV_k^{2M} from (B.3) as

$$\Delta V_k^{2M} = \underbrace{\max\begin{Bmatrix} -\varepsilon_1(k) - W_1(k) + \alpha_1, \\ \varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1, \\ -\varepsilon_2(k) - W_2(k) + \alpha_1 - \mu_2, \\ \varepsilon_2(k) + W_2(k) - \alpha_2, \\ 0 \end{Bmatrix}}_{V_{k+1}^{2M}} + \underbrace{\min\begin{Bmatrix} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ -\varepsilon_1(k) + v_d + \alpha_2, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ -\varepsilon_2(k) + v_d + \alpha_2, \\ 0 \end{Bmatrix}}_{-V_k^{2M}}. \tag{B.18}$$

It follows from (3.11), (B.17), and (B.16) that ΔV_k^{2M} from (B.18) can be reduced to

$$\Delta V_k^{2M} = \underbrace{\max\begin{Bmatrix} \varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1, \\ \varepsilon_2(k) + W_2(k) - \alpha_2, \\ 0 \end{Bmatrix}}_{V_{k+1}^{2M}} + \underbrace{\min\begin{Bmatrix} -\varepsilon_1(k) + v_d + \alpha_2, \\ -\varepsilon_2(k) + v_d + \alpha_2, \\ 0 \end{Bmatrix}}_{-V_k^{2M}}. \tag{B.19}$$

Now, let us prove that, for $\varepsilon_1(k)$ given by (B.16), inequality

$$\varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1 > \varepsilon_2(k) + W_2(k) - \alpha_2 \tag{B.20}$$

is satisfied.

Indeed, from condition (B.16), it yields that

$$\varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1 > \varepsilon_2(k) + W_1(k) - \alpha_1 \overset{(3.11)}{>} \varepsilon_2(k) + W_2(k) - \alpha_2. \tag{B.21}$$

Thus, inequality (B.20) is satisfied. Also, from (B.21), it holds that

$$\varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1 \overset{(3.11),(B.17)}{>} 0. \tag{B.22}$$

Now, considering (B.20) and (B.22), we can rewrite V_{k+1}^{2M} given by the first term of (B.19) as

$$V_{k+1}^{2M} = \varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1. \tag{B.23}$$

Let us prove that, for $\varepsilon_1(k)$ given by (B.16), inequality

$$-\varepsilon_2(k) + v_d + \alpha_2 > -\varepsilon_1(k) + v_d + \alpha_2 \tag{B.24}$$

is satisfied. Here, from condition (B.16), it yields that

$$-\varepsilon_2(k) + v_d + \alpha_2 > -\varepsilon_1(k) + \mu_1 + \alpha_2 - \alpha_1 + v_d + \alpha_2 \overset{(3.11),(3.14)}{>} -\varepsilon_1(k) + v_d + \alpha_2. \tag{B.25}$$

Thus, inequality (B.24) is satisfied. From inequalities (B.16), (3.13), it follows that

$$-\varepsilon_1(k) + v_d + \alpha_2 < 0. \tag{B.26}$$

From (B.24), (B.26), we can rewrite V_k^{2M} given by the second term of (B.19) as

$$V_k^{2M} = \varepsilon_1(k) - v_d - \alpha_2 \overset{(B.26)}{>} 0. \tag{B.27}$$

Having V_{k+1}^{2M} given by (B.23) and V_k^{2M} given by (B.27), we can finally reduce ΔV_k^{2M} from (B.19) to

$$V_k^{2M} = -\mu_1 + v_d + W_1(k) \overset{(3.15)}{<} 0. \tag{B.28}$$

Thus, for this case, it holds that for $V_k^{2M} > 0$ given by (B.1) its increment $\Delta V_k^{2M} < 0$.

Case 3 (Limited buffer content). Suppose that $w_2(k)$ satisfies the following inequality:

$$w_2(k) \geq \gamma_2, \quad \forall k \in \mathbb{N}, \tag{B.29}$$

and let us first assume that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) \leq 0, \quad \forall k \in \mathbb{N}. \tag{B.30}$$

Then, from (3.22), it holds that

$$\varepsilon_1(k) \leq \varepsilon_2(k) - \mu_2 - \alpha_2 + \alpha_1, \tag{B.31}$$

where $-\mu_2 - \alpha_2 + \alpha_1 \overset{(3.11)}{<} 0$.

In this case, machines M_1 and M_2 are not working ($\eta_{j,k} = 0$) and their behavior can be considered similar to the first part of Case 2. It follows that for (B.30) and (B.31) the increments $\Delta V_{1,k}$ and $\Delta V_{2,k}$ are given by (B.12), respectively. Thus, for ΔV_k^{2M} given by (B.3) the inequality (B.13) in this case is also satisfied.

Now, let us assume that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) > 0, \quad \forall k \in \mathbb{N}. \tag{B.32}$$

In this case, M_2 has sufficient material to start working ($\eta_{2,k} = 1$) and M_1 is stopped ($\eta_{1,k} = 0$) due to the limited capacity of its downstream buffer B_2. Thus, two situations may occur. Firs, consider that M_1 is stopped, but its tracking error $\varepsilon_1(k) \leq 0$. This may occur if $\varepsilon_2(k)$ satisfies $0 < \varepsilon_2(k) \overset{(B.31)}{\leq} \mu_2 + \alpha_2 - \alpha_1$. The behavior of these 2 machines can be considered independently and by following the procedure from Case 1, we arrive to the conclusion that for $V_k^{2M} > 0$ given by (B.1) its increment $\Delta V_k^{2M} < 0$. In the second situation, consider that $\varepsilon_1(k)$ satisfies

$$\varepsilon_1(k) > 0, \quad \forall k \in \mathbb{N}, \tag{B.33}$$

which by (B.31) implies that

$$\varepsilon_2(k) > \mu_2 + \alpha_2 - \alpha_1. \tag{B.34}$$

Then, for (B.33) and (B.34), let us rewrite ΔV_k^{2M} from (B.3) as

$$\Delta V_k^{2M} = \underbrace{\max\left\{\begin{array}{c} -\varepsilon_1(k) - \mu_1 - W_1(k) + \alpha_1, \\ \varepsilon_1(k) + W_1(k) - \alpha_2, \\ -\varepsilon_2(k) - W_2(k) + \alpha_1, \\ \varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2, \\ 0 \end{array}\right\}}_{V_{k+1}^{2M}} + \underbrace{\min\left\{\begin{array}{c} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ -\varepsilon_1(k) + v_d + \alpha_2, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ -\varepsilon_2(k) + v_d + \alpha_2, \\ 0 \end{array}\right\}}_{-V_k^{2M}}. \tag{B.35}$$

It follows from (3.11) and (3.15) that ΔV_k^{2M} from (B.35) can be reduced to

$$\Delta V_k^{2M} = \underbrace{\max\left\{\begin{array}{c} \varepsilon_1(k) + W_1(k) - \alpha_2, \\ \varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2, \\ 0 \end{array}\right\}}_{V_{k+1}^{2M}} + \underbrace{\min\left\{\begin{array}{c} -\varepsilon_1(k) + v_d + \alpha_2, \\ -\varepsilon_2(k) + v_d + \alpha_2, \\ 0 \end{array}\right\}}_{-V_k^{2M}}. \tag{B.36}$$

Now, let us derive from (B.31) that the following inequality

$$\varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2 > \varepsilon_1(k) + W_1(k) - \alpha_2 \tag{B.37}$$

is satisfied. Indeed, from (B.31), it holds that

$$\begin{aligned} \varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2 &\geq \varepsilon_1(k) + \mu_2 + \alpha_2 - \alpha_1 + W_2(k) - \alpha_2 - \mu_2 \\ &\overset{(3.11)}{>} \varepsilon_1(k) + W_1(k) - \alpha_2. \end{aligned} \tag{B.38}$$

Thus, inequality (B.37) is satisfied. From (B.34) and (3.11), it also holds that

$$\varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2 > 0. \tag{B.39}$$

Considering (B.37) and (B.39), we can rewrite V_{k+1}^{2M} from the first part of (B.36) as

$$V_{k+1}^{2M} = \varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2. \tag{B.40}$$

Let us show that from (B.31) the following inequality:

$$-\varepsilon_1(k) + v_d + \alpha_2 > -\varepsilon_2(k) + v_d + \alpha_2 \tag{B.41}$$

is satisfied. Here, from condition (B.31), it yields that

$$-\varepsilon_1(k) + v_d + \alpha_2 > -\varepsilon_2(k) + v_d + \alpha_2 + \mu_2 + \alpha_2 - \alpha_1 \overset{(3.11),(3.14)}{>} -\varepsilon_2(k) + v_d + \alpha_2. \tag{B.42}$$

Thus, inequality (B.41) is satisfied. From inequalities (B.31), (3.13), it follows that

$$-\varepsilon_2(k) + v_d + \alpha_2 < 0. \tag{B.43}$$

From (B.41), (B.43), we can rewrite V_k^{2M} given by the second part of (B.36) as

$$V_k^{2M} = \varepsilon_2(k) - v_d - \alpha_2 \overset{(B.43)}{>} 0. \tag{B.44}$$

Having V_{k+1}^{2M} given by (B.40) and V_k^{2M} given by (B.44), we can finally reduce ΔV_k^{2M} from (B.36) to

$$\Delta V_k^{2M} = -\mu_2 + v_d + W_2(k) \overset{(3.15)}{<} 0. \tag{B.45}$$

Thus, for this case it holds that for $V_k^{2M} > 0$ given by (B.1) its increment $\Delta V_k^{2M} < 0$. Summarizing for 3 cases, we have shown that for $V_k^{2M} > 0$ given by (B.1) its increment $\Delta V_k^{2M} < 0$ for all $\varepsilon_j(k) \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$ and $\Delta V_k^{2M} = 0$ for all $\varepsilon_j(k) \in [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$. Thus, $\limsup_{k\to\infty} V_k^{2M} = 0$ which completes our proof.

In this proof, we have analyzed the increment of the proposed Lyapunov function by means of 3 cases. Now, for a line of N manufacturing machines ($j = 1, \ldots, N$) defined by (3.8), (3.9), and (3.10), the Lyapunov function (B.1) is extended to

$$V_k^{NM} = \max\{V_1(\varepsilon_1), \ldots, V_N(\varepsilon_N)\}. \tag{B.46}$$

Here, the same reasoning is followed as for the proof for 2 machines. □

C. Proof of Theorem 3.5

For a different value of the desired buffer inventory level constant, the Lyapunov function is modified as follows. The function for $j = 1$ remains as in (B.2), and for $j = 2, \ldots, N$, it is now given by

$$V_j(\varepsilon_j) = \max\left\{\begin{matrix} -\varepsilon_j - \mu_j + v_d + \alpha_1, \\ \varepsilon_j - v_d - \alpha_2 - \underbrace{\sum_{i=2}^{j} \max\left(\left(\left(\mu_{i-1} - \alpha_1 + \alpha_2 - w_{di} + \left(\underbrace{\mu_i + \alpha_3}_{\beta_i}\right)\right)\right), 0\right)}_{x_j}, \\ 0 \end{matrix}\right\} > 0,$$

$$\forall \varepsilon_j \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2 + x_j], \quad j = 2, \ldots, N. \tag{C.1}$$

Thus, ΔV_k^{NM} along the solutions of $\varepsilon_j(k)$ is given by

$$\Delta V_k^{NM} = V_{k+1}^{NM} - V_k^{NM} = \max\{V_{1,k+1}, \ldots, V_{N,k+1}\} + \min\{-V_{1,k}, \ldots, -V_{N,k}\}, \tag{C.2}$$

where for $j = 2, \ldots, N$,

$$V_{j,k+1} = \max\left\{\begin{matrix} -\varepsilon_j(k) - W_j(k) + \alpha_1 - \mu_j + \mu_j \eta_{j,k}, \\ \varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j \eta_{j,k} - x_j, \\ 0 \end{matrix}\right\}. \tag{C.3}$$

Here, $\eta_{j,k}$ is given by

$$\begin{aligned} \eta_{1,k} &= \operatorname{sign}_+(\varepsilon_1(k))\operatorname{sign}_-(w_2(k) - \gamma_2), \\ \eta_{j,k} &= \operatorname{sign}_+(\varepsilon_j(k))\operatorname{sign}_{\text{Buff}}(w_j(k) - \beta_j(k))\ \operatorname{sign}_-(w_{j+1}(k) - \gamma_{j+1}), \\ \eta_{N,k} &= \operatorname{sign}_+(\varepsilon_N(k))\operatorname{sign}_{\text{Buff}}(w_N(k) - \beta_N(k)). \end{aligned} \tag{C.4}$$

The analysis of (C.2) is subdivided into the same 3 cases. Case 1 (Sufficient buffer content) and the first part of Case 2 ($w_j(k) < \beta_j(k)$ and $\varepsilon_j(k) \leq 0$, for all $j = 2, \ldots, N$) are solved identically to the proof for the line of 2 machines. For the second part of Case 2 and for the Case 3 (just as in proof of Theorem 3.4), the proof relies on the condition (3.16) in combination with (3.5) and (3.6). Due to the extensive technical details and the similarity of the procedure with the proof of Theorem 3.5, we omit the complete analysis for a line of N machines and restrict ourselves by only giving this general idea of the procedure.

Acknowledgments

The research leading to these results has received partial funding from the European Community's Seventh Framework Programme (FP7/2007–2013) under grant agreement no. INFSO-ICT-223844. This work was supported by the Russian Foundation for Basic Research (Grant 09-08-00803) and the Russian Federal Programm (Grants N 2010-1.1-111-128-033 and 16.740.11.0042).

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