CUTTHROAT, AN ALL-SMALL GAME ON GRAPHS.

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Abstract

Cutthroat is an all-small game played on a graph with vertices that are colored white, black or green. For colorings using only black and white, we give a strategy, based on atomic weights, for playing disjunctive sums of stars; we also give the game theoretic values for all colorings of complete graphs, some colorings of complete bipartite graphs and the values for paths with vertices colored alternately white and black.

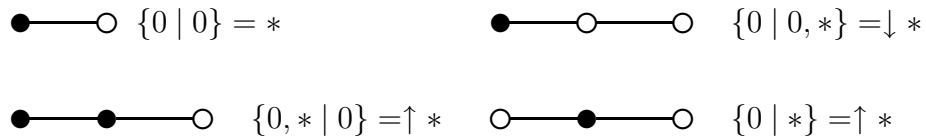**Key words:** all-small game, vertex deletion, Cutthroat, atomic weight, graph.

1. Introduction.

Cutthroat is a combinatorial game played on a graph each vertex of which has been colored black, white or green. There are two players, Left and Right, who move alternately, Left removes a black or a green vertex and Right removes a white or a green vertex. After a move, or at the beginning of play, any monochromatic, connected component is also removed from the graph. For example, if the graph were a single edge with one white and one black vertex then Left moving first would win since he would leave just the white vertex, a monochromatic component which would then be removed. Similarly, Right would win moving first. The game was first introduced in [5].

Cutthroat originated as a variant of Clobber [1]. In Clobber, the vertices have black or white tokens and, on their turn, a player slides one of their tokens along an edge to a vertex occupied by an opposing token. No moves can be played in a connected component occupied by tokens of just one color. Effectively, this sub-graph is removed from the game. Clobber is difficult. In [1], it is shown that calculating the atomic weight of a Clobber position is NP-hard, even when the game has just one black token. In contrast, for Cutthroat, atomic weights are helpful. Atomic weights are introduced later in this section and the necessary mechanisms for determining atomic weights are given in Section 2 (see also [2] and [4]).

Cutthroat is an *all-small* game since either both players have a move, or neither player does. As a consequence, the value of any position is an infinitesimal, [2,p 229]. For example, consider the following small (in size) positions. We use ● for black vertices and ○ for white.

$$\bullet \!\!-\!\!\!-\!\! \circ \quad \{0 \mid 0\} = * \qquad\qquad \bullet \!\!-\!\!\!-\!\! \circ \!\!-\!\!\!-\!\! \circ \quad \{0 \mid 0, *\} = \downarrow *$$

$$\bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \circ \quad \{0, * \mid 0\} = \uparrow * \qquad \circ \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \circ \quad \{0 \mid *\} = \uparrow *$$

In this paper, we consider *White-Black-Cutthroat,* i.e. the vertices will only be colored white or black. A **star**, $K_{1,p}$ is a graph with one vertex adjacent to p vertices, these p vertices have no edges between them. Let $\bullet_{\circ m}^{\bullet n}$ ($\circ_{\circ m}^{\bullet n}$) denote a star with a black (white) center, n black leaves and m white leaves. Determining the outcome classes of $\bullet_{\circ m}^{\bullet n}$ is simple. The game $\bullet_{\circ 0}^{\bullet n}$ is clearly a second player win; $\bullet_{\circ 1}^{\bullet n}$ is a first player win since either player can leave a monochromatic sub-graph; and $\bullet_{\circ m}^{\bullet n}$, $m > 1$ is a Left win. These results are shown in Table 1.

$n\backslash m$	0	1	2	3	4	$\ldots$	k
0	P	N	L	L	L		L
1	P	N	L	L	L		L
2	P	N	L	L	L		L
$\vdots$						$\ddots$	
j	P	N	L	L	L		L

Table 1: Outcome classes of stars with black centers: $\bullet_{\circ m}^{\bullet n}$

Although the outcomes for stars are simple, the same cannot be said for the values. Table 2 has the canonical forms of some stars. The values are defined recursively, $\bullet_{\circ k}^{\bullet j} = \{0, \bullet_{\circ k}^{\bullet j-1} \mid \bullet_{\circ k-1}^{\bullet j}\}$, $j, k > 0$, provided the options exist. In general, the values of $\bullet_{\circ m}^{\bullet n}$ do not simplify, are rather lengthy in presentation and, because of their form, are difficult to compare. It is difficult to evaluate their sums. For example,

$$\bullet_{\circ 4}^{\bullet 3} + \circ_{\circ 5}^{\bullet 4} = \{0^3 \parallel 0, A \mid 0\} + \{0^4 \parallel 0, \{0, A \mid 0\} \mid 0)\}$$
$$= \{\{0^3 || 0, A|0|||\{0||0, A|0\}, \{0^2||0, A|0\}||||\{0, A|0\}, \{0||0, A|0\}\}|0, \{0, A|0\}||0^2\}$$

where $A = \{0, \uparrow* \mid 0\}$ and $\{0^n \mid B\} = \{0 \mid \{0 \mid \{0 \mid \ldots \{0 \mid B\} \ldots\}\}\}$ where there are n 0's. The

$n\backslash m$	1	2	3
0	$*$	$\uparrow$	$\Uparrow + *$
1	$\uparrow*$	$\Uparrow$	$3.\uparrow + *$
2	$A = \{0, \uparrow* \mid 0\}$	$\{0 \mid\mid A\}$	$\{0^2 \mid A\}$
3	$\{0, A \mid\mid 0\}$	$\{0 \mid\mid 0, A \mid 0\}$	$\{0^2 \mid\mid 0, A \mid 0\}$
4	$\{0, \{0, A \mid 0\} \mid 0\}$	$\{0 \mid\mid 0, \{0, A \mid 0\} \mid 0\}$	$\{0^2 \mid\mid 0, \{0, A \mid 0\} \mid 0\}$

Table 2: The stars $\bullet^{\bullet n}_{\circ m}$, $A = \{0, \uparrow* \mid 0\}$

values do not lead to an easy-to-understand strategy. Atomic weights were introduced in [4] (see also [2], [3]), to help with exactly this type of situation. Atomic weights are a homomorphism of the set of games to itself where $\mathrm{aw}(\uparrow) = 1$. This means that atomic weights are additive:

$$\mathrm{aw}(G + H) = \mathrm{aw}(G) + \mathrm{aw}(H);$$

and atomic weights are order preserving:

$$\text{If } G \geq H \text{ then } \mathrm{aw}(G) \geq \mathrm{aw}(H).$$

Essentially, $\mathrm{aw}(G) = n$ means that the position is $n.\uparrow$ plus infinitesimals of a higher order. A more intuitive, but not completely accurate, interpretation of the atomic weight is that if $\mathrm{aw}(G) = n$, $n \geq 2$ then Left can let Right have j moves, for any $j < n$, and Left can still end the game on his move. Sometimes, knowledge of the atomic weight suffices to determine the outcome the game. Specifically, *if* $\mathrm{aw}(G) \geq 2$ *then* $G > 0$; *if* $\mathrm{aw}(G) \leq -2$ *then* $G < 0$; if $2 > \mathrm{aw}(G) > -2$ but there is no general relationship between the outcome and atomic weight if $-2 < \mathrm{aw}(G) < 2$.

Cutthroat stars are a good example of the use of atomic weights. The actual values of positions are hard to manipulate but all the atomic weights are integers and are easy to calculate. This leads to a winning strategy that is easy for a player to remember. In general, an atomic weight is itself a game and can be arbitrarily complicated (see [1]). The stars in Cutthroat all have integer atomic weights and it is not known if there are Cutthroat positions with hot or even non-integral atomic weights.

In the next section, we develop a winning strategy for stars based on the atomic weights. In contrast, only a few moments of reflection are needed for the reader to come up with winning strategies for Cutthroat played on a single complete graph. The values for a path in which the colors alternate is a little harder. These we consider in the last section.

All the necessary game theory background can be found in [2] or [4].

2. Stars.

We let $\bullet^{w}_{\circ b}$ be black centered star with w white leaves and b black leaves. The first lemma finds the values for $\bullet^{n}_{\circ m}$, $n = 0, 1$. These values form the basis for most of the induction proofs in this section. Recall that if m is even then $m.* = 0$ and if m is odd then $m.* = *$.

Lemma 1.

a: For $m \geq 1$, $\bullet^{0}_{\circ m} = (m - 1).\uparrow + m.*$.

b: For $m \geq 1$, $\bullet^{1}_{\circ m} = m.\uparrow + m.*$.

Proof: We use induction on m.

a: As already seen, $\bullet^{0}_{\circ 1} = *$, and $\bullet^{0}_{\circ 2} = \uparrow$. Now suppose $\bullet^{0}_{\circ m-1} = (m - 2).\uparrow + (m - 1).*$ and consider $\bullet^{0}_{\circ m}$, with $m \geq 3$. Left's only move is to 0 by removing the central vertex, and Right's move is to $\bullet^{0}_{\circ m-1}$, thus $\bullet^{0}_{\circ m} = \{0 \mid (m - 2).\uparrow + (m - 1).*\} = (m - 1).\uparrow + m. *$.

b: As already observed, $\bullet^{1}_{\circ 1} = \downarrow *$. Now suppose $\bullet^{1}_{\circ m-1} = (m - 1).\uparrow + (m - 1).*$, and consider $\bullet^{1}_{\circ m}$ where $m \geq 3$. Left has two moves: to 0 by taking the center vertex, and to take the black leaf, which leads to $\bullet^{0}_{\circ m} = (m-1).\uparrow + m.*$. Since $m \geq 3$, the game $(m-1).\uparrow + m.*$ is positive and so Left's move to 0 is dominated. Right's moves are all equivalent and therefore

$$\bullet^{1}_{\circ m} = \{(m - 1).\uparrow + m. * \mid (m - 1).\uparrow + (m - 1).*\} = m.\uparrow + m. * . \quad \blacksquare$$

Lemma 2. $\bullet^{n+1}_{\circ m} > \bullet^{n}_{\circ m}$ when $m \geq 1$.

Proof: The proof is by induction on $n + m$. For $n + m = 1$, we have $m = 1$ and $n = 0$. We see that $\bullet^{1}_{\circ 1} = \uparrow * > \bullet^{0}_{\circ 1} = *$. For the other cases where $n = 0$, we have that $\bullet^{0}_{\circ m} = (m - 1).\uparrow + m.* < \bullet^{1}_{\circ m} = m.\uparrow + m.*$. Now suppose the statement holds for $n + m \leq t - 1$. We wish to show that $\bullet^{n+1}_{\circ m} > \bullet^{n}_{\circ m}$ with $m > 1$, so we play the difference game

$$\bullet^{n+1}_{\circ m} - \bullet^{n}_{\circ m} = \bullet^{n+1}_{\circ m} + \circ^{m}_{\bullet n}.$$

We shall show that Left can win regardless of who starts. If it is Left to play first, then he has a winning move by playing to $\bullet^{n}_{\circ m} - \bullet^{n}_{\circ m} = 0$. If it is Right to play first, she has three options. If she plays to $\bullet^{n+1}_{\circ m-1} - \bullet^{n}_{\circ m}$, then Left can reply by playing to $\bullet^{n+1}_{\circ m-1} - \bullet^{n}_{\circ m-1}$, which he wins by hypothesis. If Right plays to $\bullet^{n+1}_{\circ m} - \bullet^{n-1}_{\circ m}$, then Left can play to $\bullet^{n}_{\circ m} - \bullet^{n-1}_{\circ m}$, which he also wins by induction. If Right plays to $\bullet^{n+1}_{\circ m}$ by taking the center of the $-\bullet^{m}_{\circ n}$, she leaves a Left-win game (Table 1). So Left always wins the difference game and hence $\bullet^{n+1}_{\circ m} > \bullet^{n}_{\circ m}$ when $m \geq 1$. $\quad \blacksquare$

Next, we need a technical lemma to aid in the atomic weight calculations. Let $\star = \{0, *, *2, *3, \ldots \mid 0, *, *2, *3, \ldots\}$. The game $\star$ is called *a remote star*.

Lemma 3. $\bullet^{\bullet n}_{\circ m} > \star$ for all $n \geq 0$, $m > 0$ except $(n, m) = (0, 1)$.

Proof: From Lemma 1, $\bullet^{\bullet 0}_{\circ m} = (m-1).\uparrow + m.* > \star$ for $m \geq 2$. From Lemma 2 then, we have $\bullet^{\bullet n}_{\circ m} > \bullet^{\bullet 0}_{\circ m} > \star$ for $n > 0$.

If $n = 1$ then $\bullet^{\bullet 1}_{\circ 1} = \uparrow * > \star$. Again, from Lemma 2, we obtain $\bullet^{\bullet n}_{\circ 1} > \bullet^{\bullet 1}_{\circ 1} > \star$

Hence $\bullet^{\bullet n}_{\circ 1} > \star$ for $n \geq 1$.   ∎

We now turn our attention to the atomic weights of stars. The *atomic weight* of a game, $\mathrm{aw}(g)$, is defined recursively: *If $g = \{a, b, c, \ldots \mid d, e, f, \ldots\}$ where a, b, c, d, e, f, $\ldots$ have atomic weights A, B, C, D, E, $\ldots$, then the atomic weight of g is*

$$G_0 = \{A - 2, B - 2, C - 2, \ldots \mid D + 2, E + 2, F + 2, \ldots\}$$

UNLESS G_0 is an integer and either $g > \star$ or $g < \star$. In these exceptional cases, if $g > \star$ then $\mathrm{aw}(g)$ is the largest integer G for which $G \mid > D + 2, E + 2, F + 2, \ldots$. Similarly, if if $g < \star$ then $\mathrm{aw}(g)$ is the least integer for which $G < \mid A - 2, B - 2, C - 2, \ldots$.

Table 3 lists the atomic weights of stars and the next lemma gives the proof.

$n \backslash m$	0	1	2	3	4	$\ldots$	k
0	0	0	1	2	3		$k-1$
1	0	1	2	3	4		k
2	0	1	2	3	4		k
$\vdots$						$\ddots$	
j	0	1	2	3	4		k

Table 3: Atomic weights of stars with black centers: $\bullet^{\bullet n}_{\circ m}$

Lemma 4. $\mathrm{aw}(\bullet^{\bullet 0}_{\circ 0}) = 0$, $\mathrm{aw}(\bullet^{\bullet 0}_{\circ m}) = m - 1$ for $m \geq 1$, and $\mathrm{aw}(\bullet^{\bullet n}_{\circ m}) = m$ for $n \geq 1$, as shown in Table 3.

Proof: For the base cases, $\mathrm{aw}(\bullet^{\bullet 0}_{\circ 0}) = \mathrm{aw}(\bullet^{\bullet 0}_{\circ 1}) = 0$, $\mathrm{aw}(\bullet^{\bullet 0}_{\circ m}) = \mathrm{aw}((m-1).\uparrow + m.*) = m - 1$, and $\mathrm{aw}(\bullet^{\bullet 1}_{\circ m}) = \mathrm{aw}(m.\uparrow + m.*) = m$.

Next, we verify that $\mathrm{aw}(\bullet^{\bullet n}_{\circ 1}) = 1$ when $n \geq 1$ by induction on n. For $n = 1$ we have already seen that the claim is true. Now suppose it is true when $n \leq t - 1$ and consider a star with $n = t$. We have that

$$\{\mathrm{aw}(0) - 2, \mathrm{aw}(\bullet^{\bullet n-1}_{\circ 1}) - 2 \mid \mathrm{aw}(0) + 2\} = \{-2, -1 \mid 2\} = 0$$

which is an integer. From Lemma 3, $\bullet^{\bullet n}_{\circ 1} > \star$, thus $\mathrm{aw}(\bullet^{\bullet n}_{\circ 1}) = 1$. Now consider $\mathrm{aw}(\bullet^{\bullet n}_{\circ 2})$ with $n \geq 2$ and proceed by induction on n. Suppose $\mathrm{aw}(\bullet^{\bullet n}_{\circ 2}) = 2$ for $n \leq t - 1$ and consider $n = t$.

We have

$$
\begin{aligned}
\mathrm{aw}(\bullet^{\bullet n}_{\circ 2}) &= \{\mathrm{aw}(0) - 2, \mathrm{aw}(\bullet^{\bullet n-1}_{\circ 2}) - 2 \mid \mathrm{aw}(\bullet^{\bullet n}_{\circ 1}) + 2\} \\
&= \{-2, -1 \mid 3\} \\
&= \{-1 \mid 3\}.
\end{aligned}
$$

By Lemma 3, $\bullet^{\bullet n}_{\circ 2} > \star$ and thus $\mathrm{aw}(\bullet^{\bullet n}_{\circ 2}) = 2$.

Finally, we investigate the stars $\bullet^{\bullet n}_{\circ m}$ with $n \geq 2$ and $m \geq 2$. We proceed by induction on $n + m$. Assume that $\mathrm{aw}(\bullet^{\bullet n}_{\circ m}) = m$ holds for $n + m = t - 1$ and consider when $n + m = t$. We have

$$
\begin{aligned}
\mathrm{aw}(\bullet^{\bullet n}_{\circ m}) &= \{\mathrm{aw}(0) - 2, \mathrm{aw}(\bullet^{\bullet n-1}_{\circ m}) - 2 \mid \mathrm{aw}(\bullet^{\bullet n}_{\circ m-1}) + 2\} \\
&= \{-2, m - 2 \mid m + 1\} \\
&= \{m - 2 \mid m + 1\}
\end{aligned}
$$

and since $\bullet^{\bullet n}_{\circ m} > \star$ (by Lemma 3), we conclude that $\mathrm{aw}(\bullet^{\bullet n}_{\circ m}) = m$. ∎

Corollary 5. $\bullet^{\bullet n}_{\circ m+i} > \bullet^{\bullet n}_{\circ m}$ for $i \geq 2$.

Proof: Since $\mathrm{aw}(\bullet^{\bullet n}_{\circ m+i}) - \mathrm{aw}(\bullet^{\bullet n}_{\circ m}) = i \geq 2$ then from the properties of atomic weights $\bullet^{\bullet n}_{\circ m+i} - \bullet^{\bullet n}_{\circ m} > 0$. ∎

Not all the stars can be compared. As the next lemma shows, adding one white leaf does not improve the situation for either player.

Lemma 6. $\bullet^{\bullet n}_{\circ m}$ is incomparable with $\bullet^{\bullet n}_{\circ m+1}$ for $n \geq 0$ and $m \geq 0$.

Proof: We shall prove this by induction on $n + m$. When $n + m = 0$, we have $\bullet^{\bullet 0}_{\circ 1} = *$ which is incomparable with $0 = \bullet^{\bullet 0}_{\circ 0}$. When $n + m = 1$, we have $\bullet^{\bullet 0}_{\circ 2} = \uparrow$ is incomparable with $* = \bullet^{\bullet 0}_{\circ 1}$, and $\bullet^{\bullet 1}_{\circ 1} = \uparrow\!*$ is incomparable with $0 = \bullet^{\bullet 1}_{\circ 0}$.

Now suppose the statement holds for $n + m \leq t - 1$. Now consider a star $\bullet^{\bullet n}_{\circ m}$, with $n + m = t$. Consider the difference game

$$
\bullet^{\bullet n}_{\circ m+1} - \bullet^{\bullet n}_{\circ m} = \bullet^{\bullet n}_{\circ m+1} + \circ^{\bullet m}_{\circ n}.
$$

If it is Right's turn to start she plays to $\bullet^{\bullet n}_{\circ m} - \bullet^{\bullet n}_{\circ m} = 0$ and wins. If it is Left's turn to start, he may play in $-\bullet^{\bullet n}_{\circ m}$ to $\bullet^{\bullet n}_{\circ m+1} - \bullet^{\bullet n}_{\circ m-1}$, which is greater than zero (Corollary 5) and so he wins. Thus, the game $\bullet^{\bullet n}_{\circ m+1} - \bullet^{\bullet n}_{\circ m}$ is a first player win and the Lemma follows. ∎

We see from all of this that adding a black leaf and adding two white leaves are both operations which increase the value of a black centered star. Now a natural question presents

itself: how many additional black leaves are needed to give the same boost in value to the star as adding two white leaves? The surprising answer is in the next result.

Corollary 7. $\bullet_{\circ n+2}^{\bullet m} > \bullet_{\circ n}^{\bullet m+i}$ for any $i \geq 0$; that is, adding any number of black leaves gains less advantage than adding two white leaves.

Proof: This follows immediately from Corollary 5 and Lemma 2. ∎

As a final comment on the order relations of stars, Lemma 6 shows that adding one white leaf to a black centered star does not increase the value but adding a further black leaf does; that is

$$\bullet_{\circ m+1}^{\bullet n+1} > \bullet_{\circ m}^{\bullet n} \text{ when } m \geq 1.$$

The proof of this we leave to the reader.

3. A Cutthroat Strategy for Stars.

If one uses the exact values, it is difficult to find a strategy for an arbitrary disjunctive sum of stars. However, the atomic weights help find a winning strategy if one exists. Note that because of the suppression of higher order infinitesimals, this strategy might not be a winning strategy if Cutthroat stars is part of a disjunctive sum with other games. We first consider the case that the atomic weight is at least 2, then we outline the situation if the atomic weight is 1. It is not known what the best plays are if the atomic weight is zero.

We will examine a position from Left's viewpoint. From Lemma 2, it seems reasonable that, unless forced to do so, Left should not remove a leaf from a black-centered star since this reduces the value of the game. In contrast, removing a black leaf from a white-centered star increases the atomic weight but does not increase the value, however if white does not decrease the atomic weight, Left can remove another black leaf from a white-centered star, increase the atomic weight by a further 1 and, in two moves, has increased the value of the game. This observation gives us our first heuristic.

Players should move so as to change the atomic weight in their favor.

This is only a heuristic since, as we shall see, it can lead the players astray if the atomic weight is -1, 0 or 1.

Suppose Left is playing a game consisting of (possibly both) black- and white-centered stars and the combined atomic weight is 2 or more. Note that the stars $\bullet_{\circ 1}^{\bullet 0}$ and $\circ_{\circ 0}^{\bullet 1}$ are identical. For clarity during our discussion, they will not be considered as stars but as *isolated edges.*

In general, when the atomic weight is one, the game may be positive or confused with zero. If a game of Cutthroat stars has atomic weight 2 with Right to play or if it has atomic

weight 1 with Left to play, Left wins by modifying Phase 2 which appears below. This modification is left to the reader.

Cutthroat Stars Strategy: Phase 1 *Suppose a Cutthroat game, G, is made up of both black- and white-centered stars and k (possibly zero) isolated edges (i.e. these are isolated edges with a white and a black vertex), and $\mathrm{aw}(G) \geq 2$ and it is Left to play. Left takes leaves from white-centered stars. When there are no white-centered stars left the atomic weight is still at least 2; then Left takes leaves from black-centered stars until the atomic weight reaches exactly two.*

Using this strategy, if Left plays first in a game that has atomic weight at least two at the outset, it will still have atomic weight at least two after Left has destroyed all the white-centered stars by taking their black leaves. So now suppose that the game involves only black-centered stars and paths of length two and its atomic weight is two or greater.

Right will reduce the atomic weight each time she takes a leaf from a black-centered star and Left will be leave it unchanged with each of his moves. Thus it will be Left to play when the atomic weight reaches two. There will still be black-centered stars with leaves since the game is positive. The game must take one of five forms. Let k denote the number of isolated edges. In addition to these the game is composed of one of the following:

1. $\bullet_{\circ 1}^{\bullet c}$ and $\bullet_{\circ 1}^{\bullet d}$ with $c \geq 1$ and $d \geq 1$,

2. $\bullet_{\circ 1}^{\bullet c}$ and $\bullet_{\circ 2}^{\bullet 0}$ with $c \geq 1$,

3. $\bullet_{\circ 2}^{\bullet 0}$ and $\bullet_{\circ 2}^{\bullet 0}$,

4. $\bullet_{\circ 2}^{\bullet f}$ with $f \geq 1$, or

5. $\bullet_{\circ 3}^{\bullet 0}$.

Cutthroat Stars Strategy: Phase 2 *Left to play: In Phase 2 play, If k is even then Left should start in the star or stars that remain, avoiding playing in the isolated edges until Right does (he matches any play in a path with another). He plays each of the following moves depending on the game. We outline the possible Right responses, omitting any intervening pair of isolated edge destruction moves. Since the number of isolated edges is even, Left destroys the last of these, and each of the scenarios below ensures he is last to play overall.*

1. Left takes a leaf to leave the game $\bullet_{\circ 1}^{\bullet c-1} + \bullet_{\circ 1}^{\bullet d}$. From here, Right can destroy one star to play to either $\bullet_{\circ 1}^{\bullet c-1}$ or $\bullet_{\circ 1}^{\bullet d}$. In either case, Left takes the center of the remaining star to play last.

2. Left takes the center of the first star to leave only $\bullet_{\circ 2}^{\bullet 0}$. Now Right must play to $\bullet_{\circ 1}^{\bullet 0}$ and Left takes the center of this to play last.

3. Left takes a center of one star to again leave $\bullet_{\circ 2}^{\bullet 0}$, which he will finish as above.

4. Left takes the center of $\bullet^{\bullet f}_{\circ 2}$, and no stars are left.

5. Left takes the center of $\bullet^{\bullet 0}_{\circ 3}$, and no stars remain.

If k is odd, Left's first move is to take a vertex from an isolated edge and so destroy it leaving an even number of isolated edges. Now Right is the first to move. If she plays in a path of length two, Left responds by playing in another. Thus, Left will be the last to play in these edges. In this way, Right will also be the first to play in the star or stars that remain. For each of her possible first moves in the star or stars, we give a strategy for Left that allows him to play last.

1. If Right plays first then only one of $\bullet^{\bullet c}_{\circ 1}$ and $\bullet^{\bullet d}_{\circ 1}$ survives. From either of these, Left takes the central vertex to play last in the stars.

2. When Right begins she has two moves. One is to $\bullet^{\bullet c}_{\circ 1} + \bullet^{\bullet 0}_{\circ 1}$. From here, Left plays to $\bullet^{\bullet c-1}_{\circ 1} + \bullet^{\bullet 0}_{\circ 1}$. Right now must eliminate one of these and move to either $\bullet^{\bullet c-1}_{\circ 1}$ or $\bullet^{\bullet 0}_{\circ 1}$. Left takes the center vertex of whichever star is left and is the last to play. Right's other starting move is to $\bullet^{\bullet 0}_{\circ 2}$, from which Left takes the center and he is the last to play.

3. When Right begins, she leaves the game $\bullet^{\bullet 0}_{\circ 1} + \bullet^{\bullet 0}_{\circ 2}$. Left eradicates the first star to leave $\bullet^{\bullet 0}_{\circ 2}$. Now Right must play to $\bullet^{\bullet 0}_{\circ 1}$ and Left finishes off by taking the central vertex.

4. Right's only move is to $\bullet^{\bullet f}_{\circ 1}$, and then Left is the last to play by deleting the center.

5. Right must play to $\bullet^{\bullet 0}_{\circ 2}$, which, as before, Left finishes by taking the center and leaving nothing for Right.

If the atomic weight is 1 then best play for both players is to play the heuristic. This continues until there are k isolated edges and one $\bullet^{\bullet n}_{\circ 1}$ or one $\bullet^{\bullet 0}_{\circ 2}$.

Left to move: if k is even then Left takes the central vertex of the star leaving a zero position. If k is odd and there is a $\bullet^{\bullet n}_{\circ 1}$ then Left plays in the star to $\bullet^{\bullet n-1}_{\circ 1}$ and then Left regards this star as an isolated edge making an even number over all. (Note that $n > 0$ since the star is not an isolated edge.) If k is odd and there is a $\bullet^{\bullet 0}_{\circ 2}$ then Left plays in an isolated edge. Eventually, Right must play in the $\bullet^{\bullet 0}_{\circ 2}$ and leave an odd number of isolated edges. In all cases Left wins.

Right to move: if k is even then Right takes the central vertex of the star leaving a zero position — Right wins! If k is odd then we leave it to the reader to show that Right has no good move.

4. Cutthroat Played on Complete and Special Complete Bipartite Graphs.

For complete graphs, size matters or, as is explained in the next section, 'less is more'. The same was not true for stars.

We denote by K_w^b the complete graph with b vertices colored black and w colored red.

Lemma 8. $K_w^b = 0$ if $b = 0$ or $w = 0$ otherwise $K_w^b = (w - b).\!\uparrow +(b + w - 1).\!*$.

Proof: When $b = 0$ or $w = 0$ then the graph is monochromatic and so has value 0. The rest of the proof is by induction on $b + w$. When $b + w = 1$, the complete graph is monochromatic, neither player can move, so the game has value 0. Suppose that $b = 1$. Left's move is to 0 and Right's move is to K_{w-1}^1 which by induction is $(w - 2).\!\uparrow +(w - 1).\!*$. Therefore,
$$K_w^1 = \{0 \mid (w - 2).\!\uparrow +(w - 1).\!*\} = (w - 1).\!\uparrow +w.\!* .$$
Suppose that $b + w = t > 1$, $b, w > 1$ and that the hypothesis holds for complete graphs with $t - 1$ vertices or fewer. Then

$$
\begin{aligned}
K_b^w &= \{K_w^{b-1} \mid K_{w-1}^b\} \\
&= \{(w - b + 1).\!\uparrow +(b + w - 2).\!* \mid (w - b - 1).\!\uparrow +(b + w - 2).\!*\} \\
&= (w - b).\!\uparrow +(b + w - 1).\!* . \quad \blacksquare
\end{aligned}
$$

Suppose the game consists of the disjunctive sum of only complete graphs. The player with fewer initial nodes wins! If the nodes are of equal numbers then the player who moves first wins. This result translates into an easy-to-remember strategy.

Strategy: *Left should never remove a vertex from a K_w^1 unless forced to do so; when forced to, he should always move in the smallest.*

Cutthroat played on a complete bipartite graph, where each of the partitions are monochromatic is just as easy. Let the notation $K'_{b,w}$ denote the complete bipartite graph with the partition of size b colored black and the partition of size w colored red.

Corollary 9. The value of a Cutthroat game played on $K'_{b,w}$ with monochromatic partitions is $(w - b).\!\uparrow +(b + w + 1).\!*$ when $b \geq 1$ and $w \geq 1$.

Because all the black and white vertices are joined, edges between black vertices and between white vertices do not affect the options and therefore the value of the game. This observation gives us our next result. Let b be the number of vertices in graph G, w in graph H and $G \oplus H$ be the graph with edges $\mathrm{E}(G)$ and $\mathrm{E}(H)$ and in which every vertex of G is adjacent to every vertex of H.

Corollary 10. If every vertex of G is colored black and every vertex of H is white then $G \oplus H$ has value $(w - b).\uparrow + (b + w + 1). *$.

This observation can be taken further to get an approximation for the value of a graph. Let A be an induced monochromatic subgraph of G and let G'_A be the graph with all edges removed that have both endpoints in A.

Theorem 11. For any graph G and any monochromatic black subgraph A, $G \geq G'_A$.

Proof. All we have to prove is that Left going second in $G - G'_A$ can win. If Right plays in G or $-G'_A$ then Left plays the corresponding vertex in $-G'_A$ or G respectively. Note that by playing, say x, in $-G'_A$, Right's move may eliminate part of A in $-G'_A$ that is not eliminated when Left plays x in G. But this is only reducing the options for Right and Left will still be able to play his strategy of deleting corresponding vertices.   ∎

5. Cutthroat Played on Alternating Paths.

In black-white Cutthroat, a path is called *alternating* if the colors alternate and one end vertex is black. We denote such a path of n vertices by AP_n. We already know that the alternating path $AP_2 = \bullet - \circ = *$, $AP_3 = \bullet - \circ - \bullet = \downarrow$ and, by definition $AP_1 = 0$. The values of the paths from order 1 through 16 are given below. We use $\pm(\uparrow, *)$ to mean $\{\uparrow, * \mid \downarrow, *\}$ and $\uparrow^2 = \{0 \mid \downarrow *\}$. (See [2] and [3], page 108, for G^n in general.) Note that $\uparrow^2$ is positive since Left wins regardless of who goes first but that $\uparrow^2 .k < \uparrow$ for any positive number k.

$n =$	0	2	4	6	8	10	12	14	16	18	20	22
AP_n	0	$*$	$\pm(\uparrow,*)$	0	$\pm(\uparrow *,0)$	$*$	$\pm(\uparrow,*)$	0	$\pm(\uparrow *,0)$	$*$	$\pm(\uparrow,*)$	0

$n =$	1	3	5	7	9	11	13	15	17	19	21	23
AP_n	0	$\downarrow$	$*$	$\downarrow *$	$\uparrow^2$	$\downarrow$	$\uparrow^2 + *$	$\downarrow *$	$\uparrow^2$	$\downarrow$	$\uparrow^2 + *$	$\downarrow *$

The period for AP_{2k}, starts at $2k = 2$. However, overall, there is a period of length 8, given in Table 4. This we prove in the next theorem.

n (mod 8)	0	1	2	3	4	5	6	7
AP_n	$\pm(\uparrow,*) + *$	$\uparrow^2$	$*$	$\downarrow$	$\pm(\uparrow,*)$	$\uparrow^2 + *$	0	$\downarrow *$

Table 4: Values for alternating paths in Cutthroat, $n \geq 6$.

Theorem 12. The values of AP_n, $n \geq 7$, have a period of 8 and are those given in Table 4.

Proof: By induction on k for paths of order $8k + i$, with $i = 0, 1, \ldots, 7$. The values for $n \leq 23$ were given earlier although the induction only requires $n \leq 16$.

First, note that each of the even order values are their own negatives. There are 8 cases to consider in order to prove the periodicity of the values. We present the first in detail but since the other 7 are similar, we leave these to the interested reader.

For AP_{8k}, with $k \geq 3$, Left's options are pairs of paths which have total order $8k - 1$, with an even and an odd path. The odd path begins and ends with a white vertex and hence is the negative of the value given in Table 4. The table below gives Left's options from AP_{8k}. The top and bottom section of the table contain the options which contain a path which is not part of the period. (Recall that the period for n even starts at $n = 2$.)

Path orders	Value	
$8k - 1$ and 0	$\uparrow * + 0$	$= \uparrow *$
$8k - 3$ and 2	$\downarrow^2 + * + *$	$= \downarrow^2$
$8k - 5$ and 4	$\uparrow + \pm (\uparrow, *)$	$= \uparrow + \pm (\uparrow, *)$
$8k - 7$ and 6	$\downarrow^2 + 0$	$= \downarrow^2$
$8k - 9$ and 8	$\uparrow * + \pm(\uparrow, *) + *$	$= \uparrow + \pm (\uparrow, *)$
$\vdots$	$\vdots$	
7 and $8k - 8$	$\uparrow * + \pm(\uparrow, *) + *$	$= \uparrow + \pm (\uparrow, *)$
5 and $8k - 6$	$* + *$	$= 0$
3 and $8k - 4$	$\uparrow * + \pm(\uparrow, *) + *$	$= \uparrow + \pm (\uparrow, *)$
1 and $8k - 2$	$0 + 0$	$= 0$

Left's options are $\uparrow *, \downarrow^2, \uparrow + \pm (\uparrow, *), 0$, and of these, 0 dominates $\downarrow^2$, since it is negative. The games $\uparrow + \pm (\uparrow, *)$ and 0 are incomparable. The Right options of the order $8k$ path are the same as Left's options but with the colors reversed, i.e. they are the negatives of Left options. Hence

$$AP_{8k} = \{\uparrow *, \uparrow + \pm (\uparrow, *), 0 \mid \downarrow *, \downarrow + \pm (\uparrow, *), 0\}.$$

We can verify that this equals $\pm(\uparrow, *) + *$ by playing the difference game

$$\{\uparrow *, \uparrow + \pm (\uparrow, *), 0 \mid \downarrow *, \downarrow + \pm (\uparrow, *), 0\} - \pm(\uparrow, *) + *$$
$$= \{\uparrow *, \uparrow + \pm (\uparrow, *), 0 \mid \downarrow *, \downarrow + \pm (\uparrow, *), 0\} + \pm(\uparrow, *) + *$$

and show that it is a second player win. Suppose Left is to play first, he has five options:

Option	Left's move	Right's winning response
1:	$\uparrow * + \pm(\uparrow, *) + * = \uparrow + \pm (\uparrow, *)$	$\uparrow + \downarrow = 0$
2:	$\uparrow + \pm (\uparrow, *) + \pm(\uparrow, *) + * = \uparrow *$	0
3:	$0 + \pm(\uparrow, *) + *$	$* + * = 0$
4:	$AP_{8k} + \uparrow + *$	$\downarrow * + \uparrow + * = 0$
5:	$AP_{8k} + \pm(\uparrow, *)$	$\downarrow + \pm (\uparrow, *) \pm (\uparrow, *) = \downarrow$

If Right plays first the games are their own negatives so the argument is the same. Therefore, we conclude that $AP_{8k} = \pm(\uparrow, *) + *$.   ∎.

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