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ON A CONJECTURE OF HAMIDOUNE FOR SUBSEQUENCE SUMS

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Abstract

Let G be an abelian group of order m, let S be a sequence of terms from G with k distinct terms, let $m \wedge S$ denote the set of all elements that are a sum of some m-term subsequence of S, and let $|S|$ be the length of S. We show that if $|S| \geq m+1$, and if the multiplicity of each term of S is at most $m-k+2$, then either $|m \wedge S| \geq \min\{m, |S|-m+k-1\}$, or there exists a proper, nontrivial subgroup H_a of index a, such that $m \wedge S$ is a union of H_a-cosets, $H_a \subseteq m \wedge S$, and all but e terms of S are from the same H_a-coset, where $e \leq \min\{\lfloor \frac{|S|-m+k-2}{|H_a|} \rfloor - 1,\, a-2\}$ and $|m \wedge S| \geq (e+1)|H_a|$. This confirms a conjecture of Y. O. Hamidoune.

Let $(G, +, 0)$ be an abelian group. If $A, B \subseteq G$, then their *sumset*, $A+B$, is the set of all possible pairwise sums, i.e. $\{a+b \mid a \in A, b \in B\}$. A set $A \subseteq G$ is H_a*-periodic*, if it is the union of H_a-cosets for some subgroup H_a of G (note this definition of periodic differs slightly from the usual by allowing H_a to be trivial). A set which is maximally H_a-periodic, with H_a the trivial group, is *aperiodic*, and otherwise we refer to A as *nontrivially periodic*. For notational convenience, we use $\phi_a : G \to G/H_a$ to denote the natural homomorphism. If S is a sequence of terms from G, then an *n-set partition* of S is a collection of n nonempty subsequences of S, pairwise disjoint as sequences, such that every term of S belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct. Thus such subsequences can be considered as sets. Let $A = A_1, \ldots, A_n$ be an n-set partition of a sequence S of terms from G whose sumset (i.e. the sumset of whose terms) is H_a-periodic. Let $y \in G/H_a$. If $y \in \phi_a(A_i)$ for all i, then y is an H_a*-nonexception*, and otherwise y is an H_a*-exception*. The number of $y \in G/H_a$ that are H_a-nonexceptions of A is denoted by $N(A, H_a)$. The number of terms x of S such that $\phi_a(x)$ is an H_a-exception of A is denoted by $E(A, H_a)$. Note $N(A, H_a) = \frac{1}{|H_a|}|\bigcap_{i=1}^{n}(A_i+H_a)|$ and $E(A, H_a) = \sum_{j=1}^{n}(|A_j| - |A_j \cap \bigcap_{i=1}^{n}(A_i+H_a)|)$. A sequence is *zero-sum* if the sum of its terms is zero. Also, $|S|$ denotes the cardinality of S, if S is a set, and the length of S, if S is a sequence. If S' is a subsequence of S, then $S \setminus S'$ denotes the subsequence of S obtained by deleting all terms in S'. Finally, $n \wedge S$ denotes the set of elements that can be represented as a sum of some n-term subsequence of S.

In 1961, Erdős, Ginzburg, and Ziv showed that any sequence of $2m-1$ terms from an abelian group of order m contains an m-term zero-sum subsequence [6]. Their result inspired numerous generalizations in extremal combinatorics. In 1967, Mann gave an easy extension of this theorem, by showing that if m is prime, $|S| = m+n-1$, and every term of S has multiplicity at most n, then $n \wedge S = G$ [15]. In 1977, Olson generalized this result in the case $n = m$ to an arbitrary abelian group of order m, by showing that if $|S| = 2m-1$, and if every term of S has multiplicity at most m, then either $m \wedge S = G$, or there exists a proper, nontrivial subgroup H_a of index a such that $H_a \subseteq m \wedge S$, and all but at most $a-2$ terms of S are from the same H_a-coset [17]. Unfortunately, while the conclusion of Olson's Theorem was quite strong, including a structure restriction on the sequence S, it failed to cover sequences with length smaller than $2m-1$. In an effort to alleviate this restriction, Bolobás and Leader obtained a weaker version of Olson's result valid for sequences of any length; they showed that if $0 \notin m \wedge S$, then $|m \wedge S| \geq |S| - m + 1$ [4]. Hamidoune improved upon this result—extending, as in Mann's result, from m-sums to arbitrary n-sums—by showing that either $|n \wedge S| \geq |S| - n + 1$ or else there exists a term x of S with $nx \in n \wedge S$ [9]. Finally, a recent composite analog of the Cauchy-Davenport Theorem [5] was proved in [7] that fully generalized the previous results of Mann, Olson, Bolobás and Leader, and Hamidoune. It is the case $S = S'$ in Theorem 2 below—which will be the main tool used in this paper, along with its easily derived consequence, Theorem 3.

In [2], Bialostocki and Dierker addressed the question of tightness in the Erdős-Ginzburg-Ziv Theorem, and showed that if there were at least three distinct terms in a sequence S from the cyclic group $\mathbb{Z}_m$, and if $|S| = 2m-2$, then $0 \in m \wedge S$. In the case of m prime, Bialostocki and Lotspeich generalized the previous result by showing that $|S| = 2m-k+1$ guaranteed an m-term zero-sum in a sequence S with at least k distinct terms [3]. Hamidoune, Ordaz, and Ortuño extended this result in the weak Olson sense by showing that if $|S| = 2m-k+1$, and if every term of S has multiplicity at most $m-k+2$, then there exists a nontrivial subgroup H_a such that $H_a \subseteq m \wedge S$ [10]. In an attempt to further generalize the result to sequences of smaller length along lines of the Bollobás-Leader result, Hamidoune made the following conjecture [9].

Conjecture 1. *Let G be a cyclic group of order m, and let S be a sequence of terms from G with $|S| \geq m+1$ and at least k distinct terms. If the multiplicity of every term of S is at most $m-k+2$, then either*

(i) $|m \wedge S| \geq |S| - m + k - 1$,

(ii) there exists a nontrivial subgroup H_a such that $H_a \subseteq m \wedge S$.

Hamidoune was able to prove a weakened from of Conjecture 1, where the inequality in (i) was replaced by $|m \wedge S| \geq |S| - m + k - 2$, and additionally showed that result to be valid for abelian groups with cyclic or trivial 2-torsion subgroup [9].

The main result of this paper is Theorem 1, which confirms Conjecture 1 for an arbitrary abelian group, and which gives a more complete generalization of Olson's result [17] in that it includes the corresponding structural coset condition on S. Theorem 1 also implies that if $|m \wedge S| < |S| - m + k - 1$, then $m \wedge S$ is nontrivially periodic, a conclusion similar to the classical result of Kneser for sumsets [13, 14, 12, 11, 16].

Theorem 1. *Let G be an abelian group of order m, and let S be a sequence of terms from G that has at least k distinct terms. If $|S| \geq m+1$ and the multiplicity of each term of S is at most $m-k+2$, then either:*

(i) $|m \wedge S| \geq \min\{m, |S| - m + k - 1\}$,

(ii) there exists a proper, nontrivial subgroup H_a of index a, such that $m \wedge S$ is H_a-periodic and $H_a \subseteq m \wedge S$, and there exists $\alpha \in G$, such that the coset $\alpha + H_a$ contains all but e terms of S, where $e \leq \min\{\lfloor \frac{|S|-m+k-2}{|H_a|} \rfloor - 1,\ a-2\}$ and $|m \wedge S| \geq (e+1)|H_a|$.

The following are two simple propositions from [1] that we will need for the proof. The first was originally stated only in the case $n_1 = n_0$, but the construction in [1] easily modifies to prove the more general statement given here, while the second was originally stated with G a finite abelian group, but the proof given works even for G an abelian monoid.

Proposition 1. *Let n_1 and n_0 be positive integers with $n_0 \leq n_1$. A sequence S of terms from G has an n_1-set partition $A = A_1, \ldots, A_{n_1}$ with $|A_i| = 1$ for $i > n_0$ (and $||A_i| - |A_j|| \leq 1$ for $i, j \leq n_0$) if and only if $|S| \geq n_1$, and for every nonempty subset $X \subseteq G$ with $|X| \leq \frac{|S|-n_1-1}{n_0} + 1$ there are at most $n_1 + (|X|-1)n_0$ terms of S from X. In particular, S has an n_1-set partition if and only if $|S| \geq n_1$ and the multiplicity of every term of S is at most n_1.*

Proposition 2. *Let S be a finite sequence of terms from an abelian group G, and let $A = A_1, \ldots, A_n$ be an n-set partition of S, where $|\sum_{i=1}^{n} A_i| = r$. Then there exists a subsequence S' of S of length at most $n+r-1$, and an n-set partition $A' = A'_1, \ldots, A'_n$ of S', where $A'_i \subseteq A_i$ for $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} A'_i = \sum_{i=1}^{n} A_i$.*

The following is a refinement of the composite analog to the Cauchy-Davenport Theorem proved in [7], strengthened along lines of a result from [8]. Observe that Theorem 2 implies $|\sum_{i=1}^{n} A'_i| \geq \min\{m, |S'| - n + 1\}$ unless $N(A', H_a) > 0$ and H_a is a proper, nontrivial subgroup.

Theorem 2. *Let S' be a subsequence of a finite sequence S of terms from an abelian group G, let $A = (A_n, \ldots, A_1)$ be an n-set partition of S', and let $a_i \in A_i$ for $i \in \{1, \ldots, n\}$. Then there exists an n-set partition $A' = (A'_n, \ldots, A'_1)$ of a subsequence S'' of S with sumset H_a-periodic, $|S'| = |S''|$, $\sum_{i=1}^{n} A_i \subseteq \sum_{i=1}^{n} A'_i$, $a_i \in A'_i$ for $i \in \{1, \ldots, n\}$, and*

$$\left| \sum_{i=1}^{n} A'_i \right| \geq (E(A', H_a) + (N(A', H_a) - 1)n + 1)\,|H_a|.$$

Furthermore, if H_a is nontrivial, then $\phi_a(x) \in \phi_a(A'_i)$ for every $i \in \{1, \ldots, n\}$ and every $x \in S \setminus S''$.

Proof. The proof for the case $S' = S$ in [7] easily modifies to prove the more general statement as follows below. In the interest of space, and since so little needs to be added or changed, we refrain from repeating the entirety of the proof given in [7]. We remark that the assumption that the theorem is false with H_k proper and nontrivial is used only once in the original proof, namely in the proof of Lemma 5 where it is used to guarantee the existence of an H_k-doubled H_k-exception, and thus the majority of the modification below simply provides an alternative argument to guarantee the existence of an H_k-doubled H_k-exception when the furthermore statement is included.

Replace, in the definition of an r-maximal partition set of S, both occurrences of 'S' by 'a subsequence of S with length $|S'|$'. For instance, the definition of Λ_0 should read: Λ_0 consists of all ordered n-set partitions, $(Z_n, \ldots, Z_1)$, of a subsequence of S with length $|S'|$, such that $\sum\limits_{i=1}^{n} A_i \subseteq \sum\limits_{i=1}^{n} Z_i$ and $a_i \in Z_i$ for $i \leq n$. Likewise replace S in the definitions of a ρ-factor form and a weak ρ-factor form, and also in the first and fourth sentences of the Proof of Theorem 1. Finally, replace the second sentence in the Proof of Lemma 5 with the following paragraph.

Suppose there does not exist an H_k-doubled H_k-exception. Hence from (II), (I) and Kneser's Theorem it follows, since Theorem 1 does not hold with H_k, that there exists $x \in S \setminus S''$ and a term D of F_ρ such that $\phi_k(x) \notin \phi_k(D)$, where S'' is the subsequence of S that F_ρ partitions. In view of (III) it follows that there exists an index j, with $\rho+1 \leq j < n$, such that $|\sum\limits_{i=j}^{n} Z_i| < |\sum\limits_{i=j+1}^{n} Z_i| + |Z_j| - 1$. Hence from Kneser's theorem it follows that $\sum\limits_{i=j}^{n} Z_j$ is maximally H-periodic with H nontrivial, and that there cannot be an element in Z_j which is the unique element from its H-coset. Consequently, since $H \leq H_k$ follows from (I), it follows that there cannot be an element in Z_j which is the unique element from its H_k-coset. Hence, since there are no H_k-doubled H_k-exceptions, it follows that all elements of $\phi_k(A_j)$ are H_k-nonexceptions and that $|\phi_k^{-1}(\beta) \cap Z_j| \geq 2$ for each H_k-nonexception $\beta \in G/H$. Since $|\sum\limits_{i=j}^{n} Z_i| < |\sum\limits_{i=j+1}^{n} Z_i| + |Z_j| - 1$, it follows in view of Proposition 1 that $\sum\limits_{i=j}^{n} Z_i = \sum\limits_{i=j+1}^{n} Z_i + (Z_j \setminus \{y\})$ for $y \in Z_j$. Hence, since $|\phi_k^{-1}(\beta) \cap Z_j| \geq 2$ for each $\beta \in \phi_a(Z_j)$, it follows that we can choose $y \in Z_j$ such that $a_j \neq y$, such that $|\phi_k(A_j)| = |\phi_k(A_j \setminus \{y\})|$, and such that $\sum\limits_{i=j}^{n} Z_i = \sum\limits_{i=j+1}^{n} Z_i + (Z_j \setminus \{y\})$. Hence it follows that we can remove y from the set partition F_ρ and place x in D to obtain a new ordered n-set partition $F'_\rho = (Z'_n, \ldots, Z'_1)$ of the sequence $S''' = (S' \setminus \{y\}) \cup \{x\}$, yielding a contradiction to the maximality of $\sum\limits_{i=1}^{n} |\phi_k(Z_j)|$ for F_ρ by the arguments used in the proof of Lemma 1. So we may assume there exists an H_k-doubled H_k-exception. $\square$

Note Theorem 3 below, which we will derive from Theorem 2, refines a recent result of [8], and also that Theorem 3(ii) implies $|S| \geq n + |S \setminus S'| + (e+1)|H_a|$.

Theorem 3. *Let S' be a subsequence of a finite sequence S of terms from an abelian group G of order m, let $P = P_1, \ldots, P_n$ be an n-set partition of S', let $a_i \in P_i$ for $i \in \{1, \ldots, n\}$, and let p be the smallest prime divisor of m. If $n \geq \min\{\frac{m}{p} - 1, \frac{|S'|-n+1}{p} - 1\}$, then either:*

(i) there is an n-set partition $A = A_1, \ldots, A_n$ of a subsequence S'' of S with $|S'| = |S''|$, $\sum_{i=1}^{n} P_i \subseteq \sum_{i=1}^{n} A_i$, $a_i \in A_i$ for $i \in \{1, \ldots, n\}$, and

$$\left|\sum_{i=1}^{n} A_i\right| \geq \min\{m, |S'| - n + 1\},$$

(ii) there is a proper, nontrivial subgroup H_a of index a, a coset $\alpha + H_a$ such that all but e terms of S are from $\alpha + H_a$, where

$$e \leq \min\{a - 2, \left\lfloor \frac{|S'| - n}{|H_a|} \right\rfloor - 1\},$$

an n-set partition $A = A_1, \ldots, A_n$ of of subsequence S'' of S with $|S''| = |S'|$, $\sum_{i=1}^{n} P_i \subseteq \sum_{i=1}^{n} A_i$, $a_i \in A_i$ for $i \in \{1, \ldots, n\}$, and $\left|\sum_{i=1}^{n} A_i\right| \geq (e + 1)|H_a|$, and an n-set partition $B = B_1, \ldots, B_n$ of a subsequence S''_0 of S, with all terms of S''_0 from $\alpha + H_a$ and $|S''_0| \leq n + |H_a| - 1$, such that $\sum_{i=1}^{n} B_i = n\alpha + H_a$.

Proof. We use induction on $|S|$ with n fixed. Note that (i) holds trivially with $A = P$ for the base case $|S| = n$. Apply Theorem 2 to the subsequence S' of S with n-set partition P, and let $A = A_1, \ldots, A_n$ be the resulting set partition and H_a the corresponding subgroup. Since $n \geq \min\{\frac{m}{p} - 1, \frac{|S'|-n+1}{p} - 1\}$, then from Theorem 2 we may assume that H_a is a proper, nontrivial subgroup, that $N(A, H_a) = 1$, that $|\sum_{i=1}^{n} A_i| \geq (e + 1)|H_a|$, and that

$$e \leq \min\{a - 2, \left\lfloor \frac{|S'| - n}{|H_a|} \right\rfloor - 1\}, \tag{1}$$

where $e = E(A, H_a)$, since otherwise (i) follows. Thus all but $e \leq \min\{a - 2, \left\lfloor \frac{|S'|-n}{|H_a|} \right\rfloor - 1\}$ terms of S are from the same H_a-coset, say $\alpha + H_a$, where $\phi_a(\alpha)$ is the H_a-nonexception, and $|\sum_{i=1}^{n} A_i| \geq (e + 1)|H_a|$. Hence we may assume $e > 0$, since otherwise in view of Proposition 2 applied to A it follows that (ii) holds with $e = 0$.

Let S_0 be the subsequence of S consisting of all terms from $\alpha + H_a$, let $A' = A'_1, \ldots, A'_n$ where $A'_i = A_i \cap (\alpha + H_a)$, and let S'_0 be the subsequence of S_0 that A' partitions. Note since $N(A, H_a) = 1$, that $|A'_i| > 0$ for all i, and thus A' is an n-set partition of S'_0. From (1) it follows that $|S'_0| \geq n + (e + 1)|H_a| - e \geq n + |H_a|$. Since $e > 0$, it follows that $|S_0| < |S|$.

We may also w.l.o.g. assume $\alpha = 0$. Hence we can apply the induction hypothesis to the subsequence S_0' of S_0 with set partition A' and with $G = H_a$. If (i) holds for S_0, then since $|S_0'| \geq n + |H_a|$, it follows, in view of $|\sum_{i=1}^{n} A_i| \geq (e+1)|H_a|$, (1), and Proposition 2, that (ii) holds for S with subgroup H_a. So assume (ii) holds for S_0 with subgroup $H_{ka} \leq H_a$ of index $k = [H_a : H_{ka}]$, with coset $\beta + H_{ka}$, and with n-set partition $B = B_1, \ldots, B_n$ satisfying $\sum_{i=1}^{n} B_i = n\beta + H_{ka}$. In this case, since by induction hypothesis at most $k-2$ terms of S_0 are not from the coset $\beta + H_{ka}$, and since $|S'| \geq |S_0'| \geq n + |H_a| = n + \frac{m}{a}$, it follows in view of (1) that there are at most

$$k - 2 + \min\{a - 2,\, \frac{|S'| - n}{|H_a|} - 1\} = \min\{k - 2 + a - 2,\, k - 2 + \frac{a(|S'| - n)}{m} - 1\} \leq$$
$$\min\left\{ka - 4,\, \frac{ka(|S'| - n)}{m} - 1 + \left(k - 2 - (k-1)\frac{a(|S'| - n)}{m}\right)\right\} < \min\{ka - 2,\, \frac{(|S'| - n)}{|H_{ka}|} - 1\},$$

terms of S not from the coset $\beta + H_{ka}$. Also,

$$|\sum_{i=1}^{n} A_i| \geq (e+1)|H_a| = k(e+1)|H_{ka}| \geq (k - 1 + e)|H_{ka}| \geq (e' + 1)|H_{ka}|,$$

where e' is the number of terms of S not from $\beta + H_{ka}$. Hence (ii) holds for S with subgroup H_{ka}, coset $\beta + H_{ka}$, and set partitions $A = A_1, \ldots, A_n$ and $B = B_1, \ldots, B_n$. $\square$

We are now ready to begin the proof of Theorem 1. For conceptual convenience the proof has been divided into three sections labelled Steps 1, 2, and 3. The goal of the first is to achieve the conditions needed to apply Theorem 3. The goal of the second is to complete the proof minus the conclusion that $m \wedge S$ is H_a-periodic, which will then be achieved in Step 3 by an extremal argument using the results from Step 2.

Proof of Theorem 1.

Step 1. Since $m \wedge S = |S| \wedge S - (|S| - m) \wedge S$ holds trivially, and since $|1 \wedge S| \geq k$, it follows that (i) holds for $|S| = m + 1$. So assume $|S| \geq m + 2$. Let $\epsilon = \max\{0,\, |S| - (2m - k + 1)\}$, let T be a subsequence of S consisting of k distinct terms including a term of S with greatest multiplicity, let $S_0 = S \setminus T$, let $n = |S| - m$, let $n_0 = |S| - m - 1$, and let $n_1 = m - k + 1 + \epsilon$. Note that

$$\frac{|S_0| - n_1 - 1}{n_0} + 1 = \frac{|S| - m - 2 - \epsilon}{|S| - m - 1} + 1 < 2. \tag{2}$$

If there exists a subset $X \subseteq G$ such that $|X| = 1$ and at least $(n_1 + 1) = m - k + 2 + \epsilon$ terms of S_0 are from X, then, since the multiplicity of every term of S is at most $m - k + 2$, and since T contains a term of S with greatest multiplicity, it follows that $\epsilon = 0$ and that there are two terms of S with multiplicity $m - k + 2$, whence $|S| \geq 2(m - k + 2) + k - 2 = 2m - k + 2$, contradicting $\epsilon = 0$. So we may assume no such subset X exists. Hence, since $|S| \geq m + 2$, then in view of (2) and Proposition 1 applied to S_0, it follows that there exists an n_1-set partition $P_2, P_3, \ldots, P_{n_1+1}$ of S_0 with $|P_i| = 1$ for $i > n_0 + 1 = n$. Letting $P = P_1, \ldots, P_n$, where $P_1 = T$, and letting S' be the subsequence that P partitions, we obtain an n-set partition of the subsequence S' of S with $|S'| = |S| - (n_1 - n_0) = 2|S| - 2m + k - 2 - \epsilon$.

Apply Theorem 2 to the subsequence S' of S with n-set partition P, and let $A = A_1, \ldots, A_n$ be the resulting n-set partition, and H_a the corresponding subgroup. Hence, since $m \wedge S = |S| \wedge S - (|S| - m) \wedge S$, then from Theorem 2 it follows that we may assume,

$$((N-1)(|S|-m)+e+1)|H_a| \leq \min\{|S|-m+k-2,\, m-1\}, \tag{3}$$

where $e = E(A, H_a)$ and $N = N(A, H_a)$, since otherwise (i) holds. Hence H_a is a proper subgroup. Observe that $|S'| - (|S| - m) + 1 \geq \min\{m,\, |S| - m + k - 1\}$. Let l be the number of distinct terms x of S such that $\phi_a(x)$ is an H_a-exception in A. Observe that $e \geq l$ and that

$$\frac{k-l}{|H_a|} \leq N, \tag{4}$$

hold trivially. Since $|S'| - (|S| - m) + 1 \geq \min\{m,\, |S| - m + k - 1\}$, then from (3) it follows that we may assume H_a is nontrivial and $N \geq 1$.

Let $k - |H_a| = l + \delta$, and suppose $\delta \geq 1$. Hence (4) implies $N|H_a| \geq |H_a| + \delta$. Thus, since $|S| \geq m + 1$, since $e \geq l$, and since $\delta \geq 1$, it follows from (3) that

$$k \geq (\delta - 1)(|S| - m) + |H_a|(l+1) + 2 \geq \delta - 1 + |H_a| + l + 2 = |H_a| + l + \delta + 1,$$

contradicting the definition of δ. So we may assume

$$k - |H_a| \leq l. \tag{5}$$

Suppose $N > 1$. Hence (3), $|S| \geq m + 1$, and $e \geq l$ imply

$$(|S| - m)(|H_a| - 1) + (l+1)|H_a| \leq k - 2,$$

which, since (5) implies $|H_a|(l+1) \geq l + |H_a| \geq k$, since $|S| \geq m + 1$, and since $|H_a| \geq 2$, is impossible. So we may assume $N = 1$.

Suppose that $|S| < m + |H_a| + e$. Hence from $N = 1$ and (3) it follows that $e|H_a| - e \leq k - 3$. Thus, since $e \geq l$, it follows from (5) that $e(|H_a| - 2) \leq |H_a| - 3$, which is only possible if $e = 0$. However, if $e = 0$, then every term of S is from the same H_a-coset, say $\alpha + H_a$, and by translation we may w.l.o.g. assume $\alpha = 0$. Hence, since $\sum_{i=1}^{n} A_i$ is H_a-periodic, and since $N = 1$, it follows that $H_a \subseteq (|S| - m) \wedge S$. Since every term of S is from H_a, it follows that $|S| \wedge S \in H_a$. Thus, since $H_a \subseteq (|S| - m) \wedge S$, and since $m \wedge S = |S| \wedge S - (|S| - m) \wedge S$, it follows that $H_a \subseteq m \wedge S$. Hence, since (3) implies that $e \leq \min\{\lfloor \frac{|S|-m+k-2}{|H_a|} \rfloor - 1,\, a - 2\}$, and since $e = 0$ implies $m \wedge S \subseteq H_a$, it follows that (ii) holds. So we may assume that

$$|S| \geq m + |H_a| + e. \tag{6}$$

Since $e \geq l$, then it follows in view of (6) and (5) that

$$|S| \geq m + k. \tag{7}$$

Suppose that $n < \frac{|S'|-n+1}{p} - 1$, where p is the smallest prime divisor of m. Hence, since $n = |S| - m$, and since $|S'| = 2|S| - 2m + k - 2 - \epsilon$, it follows that

$$|S| - m < \frac{|S| - m + k - 1 - \epsilon}{p} - 1. \tag{8}$$

Since $p \geq 2$, and since $|S| \geq m+1$, it follows from (8) that $|S| - m < \frac{|S|-m+k-1-\epsilon}{2} - 1$, implying that $|S| < m + k - 3 - \epsilon$, a contradiction to (7). So we may assume that $n \geq \frac{|S'|-n+1}{p} - 1$.

Step 2. Since $n \geq \frac{|S'|-n+1}{p} - 1$, it follows that we can apply Theorem 3 to the subsequence S' of S with n-set partition A. If Theorem 3(i) holds, then, since $m \wedge S = |S| \wedge S - (|S| - m) \wedge S$, it follows that (i) holds. So assume that Theorem 3(ii) holds with proper, nontrivial subgroup H_b of index b, with coset $\beta + H_b$, with e' terms of S not from $\beta + H_b$, and with n-set partitions $A' = A'_1, \ldots, A'_n$ and $B = B_1, \ldots, B_n$, where $|\sum_{i=1}^{n} A'_i| \geq (e' + 1)|H_b|$ and $\sum_{i=1}^{n} B_i = n\beta + H_b$. Hence the inequality

$$k - |H_b| \leq l', \tag{9}$$

holds trivially, where l' is the number of distinct terms of S not from the coset $\beta + H_b$; and the inequality in Theorem 3(ii) implies

$$e' \leq \min\left\{ \left\lfloor \frac{|S| - m + k - 2}{|H_b|} \right\rfloor - 1,\, b - 2 \right\}. \tag{10}$$

We may w.l.o.g. assume $\beta = 0$. Hence, since $\sum_{i=1}^{n} B_i = H_b$, it follows that $H_b \subseteq (|S| - m) \wedge S$. Thus, if $e' = 0$, then $|S| \wedge S \in H_b$ and $m \wedge S \subseteq H_b$, whence (ii) follows from (10) and $m \wedge S = |S| \wedge S - (|S| - m) \wedge S$. So $e' > 0$. Since there are at most $n + |H_b| - 1$ terms partitioned by the set partition B, it follows in view of (10) that there are at least

$$(e' + 1)|H_b| + m - k + 2 - e' - (n + |H_b| - 1) = 2m - |S| - k + 3 + e'(|H_b| - 1), \tag{11}$$

terms of S from $\beta + H_b$ that are not partitioned by B.

Hence if there are at most $2m - |S| - 1$ terms of S from $\beta + H_b$ that are not partitioned by B, then since $e' > 0$, and since $e' \geq l'$, it follows in view of (11) that $k - 4 \geq e'(|H_b| - 1) \geq e' + |H_b| - 2 \geq l' + |H_b| - 2$, contradicting (9). Consequently we may assume that there are at least $2m - |S| = m - n$ terms of S from $\beta + H_b$ that are not partitioned by B. Thus we can add $m - n$ singleton sets, each containing a term of S from $\beta + H_b$ not partitioned by B, to the set partition B, to obtain an m-set partition whose sumset is H_b. Hence

$$H_b \subseteq m \wedge S. \tag{12}$$

Step 3. In view of $|\sum_{i=1}^{n} A'_i| \geq (e'+1)|H_b|$, (9), (10), and (12), let $H_{b'}$ be a minimal cardinality nontrivial subgroup such that

$$H_{b'} \subseteq m \wedge S, \tag{13}$$

and there exists a coset $\gamma + H_{b'}$ satisfying

$$e'' \leq \min\left\{\left\lfloor \frac{|S|-m+k-2}{|H_{b'}|} \right\rfloor - 1,\, b'-2\right\}, \tag{14}$$

and

$$k - |H_{b'}| \leq l'', \tag{15}$$

and $|m \wedge S| \geq (e''+1)|H_{b'}|$, where b' is the index of $H_{b'}$, and e'' is the number of terms of S not from the coset $\gamma + H_{b'}$, and l'' is the number of distinct terms of S not from $\gamma + H_{b'}$.

Suppose $e'' = 0$. Hence all terms of S are from $\gamma + H_{b'}$. Thus $m \wedge S \subseteq H_{b'}$, and (ii) follows from (13) and (14). So $e'' > 0$.

Suppose $|S| < m + |H_{b'}| + e''$. Hence it follows from (14) that $e''|H_{b'}| - e'' \leq k-3$. Thus, since $e'' \geq l''$, it follows from (15) that $e''(|H_{b'}|-2) \leq |H_{b'}| - 3$, which is only possible if $e'' = 0$, a contradiction. So

$$|S| \geq m + |H_{b'}| + e''. \tag{16}$$

Let $T = (a_1, \ldots, a_m)$ be an m-term subsequence of S. To complete the proof we will show that every element from the same $H_{b'}$-coset as $\sum_{i=1}^{m} a_i$ is contained in $m \wedge S$. By reordering, we may w.l.o.g. assume $a_i \in \gamma + H_{b'}$ for $i \leq n_0$, where e_0 is the number of terms of T not from $\gamma + H_{b'}$, and $n_0 = m - e_0$. Let S_0 be the subsequence of S consisting of terms from $\gamma + H_{b'}$, and let $n_1 = |S| - e'' - |H_{b'}| + 1$. Note $e_0 \leq e''$, and hence in view of (14) and (16) it follows that both n_0 and n_1 are positive integers. Also, since $H_{b'}$ proper, nontrivial implies $m \geq 4$, then it follows in view of (14) that

$$\frac{|S_0| - n_1 - 1}{n_0} + 1 = \frac{|H_{b'}| - 2}{m - e_0} + 1 < \frac{|H_{b'}|}{m - b'} + 1 \leq 2. \tag{17}$$

In view of (16) it follows that $n_1 + 1 = |S| - e'' - |H_{b'}| + 2 \geq m + 2 > m - k + 2$. Hence every term of S_0 has multiplicity at most n_1, and in view of (17) and Proposition 1, it follows that there exists an n_1-set partition $A = A_1, \ldots, A_{n_1}$ of S_0 with $|A_i| = 1$ for $i > n_0$.

Assume A is chosen such that the number of indices $i \leq n_0$ with $a_i \notin A_i$ is minimal. If there exists an index j such that $a_j \notin A_j$, then there will exist an index $j' \neq j$ with $a_j \in A_{j'}$ and, if $j' \leq n_0$, then also with $a_j \neq a_{j'}$, whence the set partition $A' = A'_1, \ldots, A'_{n_1}$ defined by letting $A'_i = A_i$ for $i \neq j,\, j'$, and, if $|A_{j'}| = 1$, letting $A'_j = (A_j \setminus \{y\}) \cup \{a_j\}$ and $A'_{j'} = (A_{j'} \setminus \{a_j\}) \cup \{y\}$, or, if $|A_{j'}| > 1$, then letting $A'_j = A_j \cup \{a_j\}$ and $A'_{j'} = A_{j'} \setminus \{a_j\}$, where $y \in A_j$, will contradict the minimality of A. Hence we may assume $a_i \in A_i$ for all $i \leq n_0$.

Let S_0' be the subsequence of S_0 partitioned by the n_0-set partition $A_1, \ldots, A_{n_0}$. Note $|S_0'| = |S_0| - (n_1 - n_0) = n_0 + |H_{b'}| - 1$. Hence, if $n_0 \leq \frac{|S_0'| - n_0}{p'} - 1$, where p' is the smallest prime divisor of $|H_{b'}|$, then since $e_0 \leq e''$, it follows in view of (14) that $m \leq |H_{b'}| + e_0 - 1 \leq \frac{m}{b'} + b' - 3 \leq \frac{m}{2} - 1$, a contradiction. So assume $n_0 \geq \frac{|S_0'| - n_0 + 1}{p'} - 1$.

We may w.l.o.g. assume $\gamma = 0$. Hence, since $n_0 \geq \frac{|S_0'| - n_0 + 1}{p'} - 1$, it follows that we can apply Theorem 3 to the subsequence S_0' of S_0 with n_0-set partition $A_1, \ldots, A_{n_0}$, with group $G = H_{b'}$, and with fixed elements $a_i \in A_i$ for $i \leq n_0$. If Theorem 3(i) holds with corresponding set partition $A' = A_1', \ldots, A_{n_0}'$, then since $|S_0'| = n_0 + |H_{b'}| - 1$, it follows that $\sum_{i=1}^{n_0} A_i' = H_{b'}$, whence $\left(\sum_{i=n_0+1}^{m} a_i \right) + \sum_{i=1}^{n_0} A_i'$ is $H_{b'}$-periodic, and $\sum_{i=1}^{m} a_i \in \left(\sum_{i=n_0+1}^{m} a_i \right) + \sum_{i=1}^{n_0} A_i'$. Thus every element from the same $H_{b'}$-coset as $\sum_{i=1}^{m} a_i$ is contained in $m \wedge S$, and the proof is complete. So assume that Theorem 3(ii) holds and let $H_{cb'} \leq H_{b'}$ be the corresponding subgroup with $c = [H_{b'} : H_{cb'}]$, let $\gamma' + H_{cb'}$ be the corresponding coset, and let $e_0' \leq c - 2$ be the number of terms of S_0 not from $\gamma + H_{cb'}$. Thus, since $|S| \geq |H_{b'}| + (m - k + 2)$ follows from (14), then it follows from (14) and from $|m \wedge S| \geq (e'' + 1)|H_{b'}|$, as in the proof of Theorem 3, that there are $e''' \leq c - 2 + \min\{\lfloor \frac{|S| - m + k - 2}{|H_{b'}|} \rfloor - 1, b' - 2\} < \min\{\lfloor \frac{|S| - m + k - 2}{|H_{cb'}|} \rfloor - 1, cb' - 2\}$ terms of S not from the coset $\gamma' + H_{cb'}$, and that $|m \wedge S| \geq (e''' + 1)|H_{cb'}|$. Thus (14) holds for S with subgroup $H_{cb'}$. Furthermore, since $H_{cb'} \leq H_{b'}$, then (13) implies that $H_{cb'} \subseteq m \wedge S$. Finally, $k - |H_{cb'}| \leq l_0$, where l_0 is the number of distinct terms not from $\gamma + H_{cb'}$, holds trivially. Consequently, from the conclusions of the last three sentences we see that the minimality of $H_{b'}$ is contradicted by $H_{cb'}$, and the proof is complete. $\square$

We conclude the paper by remarking that the inequality $e \leq \min\{\lfloor \frac{|S| - m + k - 2}{|H_a|} \rfloor - 1, a - 2\}$ from Theorem 1(ii) implies

$$|S| \geq m - k + 2 + (e + 1)|H_a| + \epsilon, \tag{18}$$

where e is the number of terms of S not from the coset $\alpha + H_a$, and $\epsilon = \max\{0, |S| - (2m - k + 1)\}$; also, as seen in the proof of Theorem 1, if $e > 0$, then (18) (which is just the inequality in (3) rearranged with $N = 1$) implies

$$|S| \geq m + |H_a| + e \geq m + |H_a| + l \geq m + k,$$

where l is the number of distinct terms of S not from $\alpha + H_a$.

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