ODD INCONGRUENT RESTRICTED DISJOINT COVERING SYSTEMS

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Received: 6/30/11, Revised: 12/24/11, Accepted: 1/11/12, Published: 10/12/12

Abstract

This paper presents all known information on incongruent restricted disjoint covering systems with only odd moduli. The results in this paper formed part of the author's doctoral dissertation completed at Macquarie University.

–Dedicated to the memory of John Selfridge.

1. Main Results

The paper by Myerson, Poon and Simpson [5] first introduced Incongruent Restricted Disjoint Covering Systems.

Definition 1. An incongruent restricted disjoint covering system of length n is a collection of congruence classes which covers the integers in the interval $[1, n]$, where no modulus is repeated, each integer is contained in exactly one congruence class and each congruence class contains at least two numbers in the interval.

Note: we take the plural of IRDCS to be IRDCS. It will always be clear from context whether we are referring to the singular or the plural.
Example: an IRDCS of length 11 is given by

$$\{0 \pmod 3, 0 \pmod 4, 0 \pmod 5, 1 \pmod 6, 2 \pmod 9\}$$

The alternate notation for an IRDCS is the list of moduli covering each integer from 1 to n; in our example, that would be 6,9,3,4,5,3,6,4,3,5,9. Reversing the alternate notation for an IRDCS gives the alternate notation for another IRDCS, called the reversal of the original. IRDCS always come in such reversal-pairs.

The compact notation for an IRDCS is the list of distinct moduli in the order in which they first appear in the alternate notation; in our example, 6,9,3,4,5. Clearly any IRDCS can be reconstructed in its entirety from its compact notation.

Two useful invariants of an IRDCS are the heft, which is the sum of the reciprocals of the moduli, and the order, which is the number of moduli. Our length 11 example has heft 1.0611 and order 5.

The paper [5] asks the question of whether there exists an odd IRDCS, that is, an IRDCS with only odd moduli. In the classical setting of covering systems Erdős and the late John Selfridge asked whether there exists a covering system with only odd moduli in [2]. This question has proven famously difficult to decide and was proposed in the negative by Selfridge [2] who offered $2000 for an explicit odd covering, according to [3].

In this paper we present summary information about all known odd IRDCS. By exhaustive search, we have found that the shortest odd IRDCS is of length 83; it is unique, up to reversal. In alternate notation, it is

$$61, 41, 21, 65, 9, 43, 53, 59, 11, 15, 37, 17, 13, 9, 23, 19, 27, 33, 55, 11, 35,$$
$$25, 9, 21, 15, 13, 31, 29, 17, 51, 11, 9, 49, 45, 19, 47, 39, 23, 13, 15, 9, 11,$$
$$41, 27, 21, 17, 25, 37, 43, 9, 33, 13, 11, 19, 15, 35, 29, 31, 9, 53, 23, 61, 17,$$
$$11, 13, 21, 59, 9, 65, 15, 27, 25, 19, 55, 11, 39, 9, 13, 45, 17, 51, 49, 47.$$

The compact notation for the above IRDCS is

$$61, 41, 21, 65, 9, 43, 53, 59, 11, 15, 37, 17, 13, 23,$$
$$19, 27, 33, 55, 35, 25, 31, 29, 51, 49, 45, 47, 39.$$

This IRDCS has heft 1.02042, order 27 and uses all of the available odd moduli from 9 to 65 excluding 57 and 63.

Theorem 2. *There exist odd IRDCS for lengths 83 to 101 inclusive.*

We have computed exhaustive data for lengths 83 to 90 inclusive, and will present the results in what follows. Once we reach length 90, the computation time required to find all odd IRDCS for a given length becomes unfeasible. As such, for lengths 91 and higher we have used Knuth's Dancing Links implementation of Algorithm X [4] to find a single solution and thus prove the existence of at least 2 odd IRDCS for the given length. We have computed odd IRDCS of lengths up to and including 101 using this algorithm.

1.1. Length 84 Odd IRDCS

There are exactly 7 odd IRDCS of length 84, up to reversal. We present them, in alternate notation, with some summary statistics:

$$57, 13, 59, 47, 17, 21, 9, 29, 11, 39, 49, 35, 37, 15, 13, 9, 23, 27, 19, 11, 25, 17, 45,$$
$$31, 9, 55, 21, 13, 15, 53, 11, 33, 51, 9, 43, 41, 29, 19, 17, 23, 13, 11, 9, 15, 27, 25,$$
$$35, 21, 39, 37, 47, 9, 11, 13, 31, 17, 19, 57, 15, 49, 9, 59, 23, 11, 33, 29, 13, 45, 21,$$
$$9, 25, 27, 17, 15, 11, 19, 41, 43, 9, 13, 55, 35, 53, 51;$$

49, 13, 59, 47, 17, 21, 9, 29, 11, 39, 57, 35, 45, 15, 13, 9, 23, 27, 19, 11, 25, 17, 37,
31, 9, 55, 21, 13, 15, 53, 11, 33, 51, 9, 43, 41, 29, 19, 17, 23, 13, 11, 9, 15, 27, 25,
35, 21, 39, 49, 47, 9, 11, 13, 31, 17, 19, 45, 15, 37, 9, 59, 23, 11, 33, 29, 13, 57, 21,
9, 25, 27, 17, 15, 11, 19, 41, 43, 9, 13, 55, 35, 53, 51;

59, 13, 55, 51, 17, 21, 9, 29, 11, 41, 19, 35, 37, 15, 13, 9, 23, 27, 43, 11, 25, 17, 53,
57, 9, 31, 21, 13, 15, 19, 11, 33, 45, 9, 49, 47, 29, 39, 17, 23, 13, 11, 9, 15, 27, 25,
35, 21, 19, 37, 41, 9, 11, 13, 51, 17, 31, 55, 15, 59, 9, 43, 23, 11, 33, 29, 13, 19, 21,
9, 25, 27, 17, 15, 11, 53, 39, 45, 9, 13, 57, 35, 47, 49;

49, 13, 59, 53, 17, 21, 9, 29, 11, 41, 19, 35, 45, 15, 13, 9, 23, 27, 57, 11, 25, 17, 37,
31, 9, 55, 21, 13, 15, 19, 11, 33, 51, 9, 43, 47, 29, 39, 17, 23, 13, 11, 9, 15, 27, 25,
35, 21, 19, 49, 41, 9, 11, 13, 31, 17, 53, 45, 15, 37, 9, 59, 23, 11, 33, 29, 13, 19, 21,
9, 25, 27, 17, 15, 11, 57, 39, 43, 9, 13, 55, 35, 47, 51;

59, 13, 55, 53, 17, 21, 9, 29, 11, 41, 19, 35, 37, 15, 13, 9, 23, 27, 57, 11, 25, 17, 39,
31, 9, 51, 21, 13, 15, 19, 11, 33, 45, 9, 49, 47, 29, 43, 17, 23, 13, 11, 9, 15, 27, 25,
35, 21, 19, 37, 41, 9, 11, 13, 31, 17, 53, 55, 15, 59, 9, 39, 23, 11, 33, 29, 13, 19, 21,
9, 25, 27, 17, 15, 11, 57, 51, 45, 9, 13, 43, 35, 47, 49;

59, 13, 55, 51, 17, 21, 9, 29, 11, 41, 19, 35, 37, 15, 13, 9, 23, 27, 57, 11, 25, 17, 39,
53, 9, 31, 21, 13, 15, 19, 11, 33, 45, 9, 49, 47, 29, 43, 17, 23, 13, 11, 9, 15, 27, 25,
35, 21, 19, 37, 41, 9, 11, 13, 51, 17, 31, 55, 15, 59, 9, 39, 23, 11, 33, 29, 13, 19, 21,
9, 25, 27, 17, 15, 11, 57, 53, 45, 9, 13, 43, 35, 47, 49.

All of these IRDCS have heft 1.00619 and order 26. They all use every available odd modulus from 9 to 59 inclusive.

The final two length 84 odd IRDCS are the length 83 odd IRDCS extended in length by, in this instance, covering the position before the start of the previously presented length 83 IRDCS with the modulus 13, and the other is its reversal. The IRDCS in alternate notation is

13, 61, 41, 21, 65, 9, 43, 53, 59, 11, 15, 37, 17, 13, 9, 23, 19, 27, 33, 55, 11, 35, 25,
9, 21, 15, 13, 31, 29, 17, 51, 11, 9, 49, 45, 19, 47, 39, 23, 13, 15, 9, 11, 41, 27, 21,
17, 25, 37, 43, 9, 33, 13, 11, 19, 15, 35, 29, 31, 9, 53, 23, 61, 17, 11, 13, 21, 59, 9,
65, 15, 27, 25, 19, 55, 11, 39, 9, 13, 45, 17, 51, 49, 47,

and again has heft 1.02042 and order 27.

1.2. Lengths 85 to 94 Odd IRDCS

For odd IRDCS with lengths 85 to 90, we give the following summary table.

Length	Number Of Odd IRDCS	Heft Range	Order Range
85	80	1.00504 - 1.02091	26 - 27
86	382	1.00373 - 1.0291	26 - 27
87	474	1.00563 - 1.0196	26 - 28
88	152	1.00988 - 1.01884	27
89	80	1.00619 - 1.01739	25 - 26
90	4	1.00504 - 1.00511	26

There are only 4 odd IRDCS of length 90, made up of two IRDCS and their reversals. The two unique IRDCS up to reversals are presented below in their compact notations.

$$55, 15, 23, 51, 31, 17, 9, 19, 11, 49, 13, 61, 35, 27, 39, 21,$$
$$25, 37, 29, 59, 53, 41, 33, 43, 47, 45.$$

$$55, 15, 23, 51, 31, 17, 9, 19, 11, 63, 13, 47, 35, 27, 39, 21,$$
$$25, 37, 29, 57, 53, 41, 33, 43, 49, 45.$$

These IRDCS use all of the moduli from 9 to 55 inclusive, with the first presented example above and its reversal additionally using the moduli 59 and 61, and the second example and its reversal using additional moduli 57 and 63. This same compact notation also gives IRDCS of lengths 91 to 94 inclusive. I thank the referee for pointing this out.

1.3. Larger Lengths Using Dancing Links

For length 95 and higher, Knuth's dancing links implementation of Algorithm X [4] is used to compute a single example of an IRDCS for the particular lengths. For length 95, the IRDCS has heft 1.014 and order 30. The IRDCS in compact notation is:

$$57, 77, 47, 63, 59, 37, 23, 13, 9, 61, 11, 45, 19, 17, 27,$$
$$43, 29, 33, 21, 71, 25, 31, 35, 67, 39, 49, 41, 55, 53, 51.$$

For length 96, the IRDCS has heft 1.0108 and order 28. The IRDCS in compact notation is:

$$49, 57, 15, 17, 61, 13, 9, 33, 11, 47, 19, 35, 63, 21,$$
$$81, 45, 29, 23, 41, 59, 27, 39, 31, 37, 53, 55, 43, 51.$$

For length 97, the IRDCS has heft 1.0184 and order 31. The IRDCS in compact notation is:

$$69, 59, 53, 15, 63, 23, 65, 9, 19, 11, 29, 43, 83, 49, 31, 17,$$
$$21, 39, 35, 25, 27, 33, 61, 67, 55, 47, 37, 57, 41, 51, 45.$$

For lengths 98 and 99, the computed IRDCS for length 98 can be extended to one of length 99. The length 98 IRDCS has heft 1.0194 and order 32. The length 99 example is found by moving the modulus 47 to the first position in the compact notation of the length 98 example, which is:

$$9, 25, 65, 59, 55, 77, 23, 61, 13, 15, 67, 19, 51, 43, 17, 21,$$
$$27, 29, 71, 31, 33, 37, 69, 35, 63, 39, 57, 41, 53, 45, 49, 47.$$

Below is an independent IRDCS of length 99 which has heft 1.018 and order 32. In compact notation this IRDCS is:

$$63, 59, 49, 53, 69, 9, 65, 47, 19, 13, 15, 17, 81, 45, 21, 33,$$
$$75, 29, 25, 23, 31, 67, 27, 35, 37, 57, 39, 41, 61, 43, 51, 55.$$

In searching for this length 99 odd IRDCS I assumed that the modulus covering the middle position was 9, in order to produce input for our implementation of that algorithm that was of an acceptable size. Excluding length 90, all of the lengths for which exhaustive data has been computed had at least one solution with middle modulus 9.

For lengths 100 and 101 assume once more that the middle modulus is 9. For length 100, the IRDCS has heft 1.0187 and order 32. The IRDCS of length 100 can also be modified to be an IRDCS of length 101 by letting the modulus 41 cover position 0. The IRDCS in compact notation is:

$$63, 55, 69, 57, 17, 9, 47, 19, 31, 13, 75, 23, 21, 15, 67, 77, 45,$$
$$33, 25, 29, 43, 27, 65, 37, 35, 59, 61, 39, 41, 49, 53, 51.$$

2. Open Questions

All of the odd IRDCS that we have found so far have minimum modulus 9. Is it true that all odd IRDCS must use moduli at least as large as 9? Moreover all of the odd IRDCS found, excluding those of length 90, use the modulus 9 to cover the middle position. Is there a good reason for this, or is it just chance?

The number of odd IRDCS for a given length decays from length 87 to length 90. All length 90 IRDCS also form IRDCS of lengths 91 to 94, and there exist at least two others for these lengths as shown in [1], so we do not know what is happening for larger lengths. This may be similar to the standard IRDCS case where there is a solution of length 11 and then no solutions until length 17. The numbers in the standard case also don't grow all the time, for example from length 23 to 24 and length 27 to 28.

Every odd IRDCS found to date has heft at least 1.003. We know from [1] that there are IRDCS with heft as low as 0.987952. Are there any odd IRDCS with heft less than 1?

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