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FIVE GUIDELINES FOR PARTITION ANALYSIS WITH APPLICATIONS TO LECTURE HALL-TYPE THEOREMS

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Abstract

Five simple guidelines are proposed to compute the generating function for the nonnegative integer solutions of a system of linear inequalities. In contrast to other approaches, the emphasis is on deriving recurrences. We show how to use the guidelines strategically to solve some nontrivial enumeration problems in the theory of partitions and compositions. This includes a strikingly different approach to lecture hall-type theorems, with new q-series identities arising in the process. For completeness, we prove that the guidelines suffice to find the generating function for any system of homogeneous linear inequalities with integer coefficients. The guidelines can be viewed as a simplification of MacMahon's partition analysis with ideas from matrix techniques, Elliott reduction, and "adding a slice."

1. Introduction

This continues our work in [18, 19] studying nonnegative integer solutions to linear inequalities as they relate to the enumeration of integer partitions and compositions. Define the *weight* of a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers to be $|\lambda| = \lambda_1 + \cdots + \lambda_n$. If sequence λ of weight N has all parts nonnegative, we call it a *composition* of N; if, in addition, λ is a

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nonincreasing sequence, we call it a *partition* of N.

Given an $r \times n$ integer matrix $C = [c_{i,j}]$, we consider the set S_C of nonnegative integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the constraints

$$c_{i,0} + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots + c_{i,n}\lambda_n \geq 0, \quad 1 \leq i \leq r. \tag{1}$$

We seek the full generating function

$$F_C(x_1, x_2, \ldots, x_n) = \sum_{\lambda \in S_C} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}, \tag{2}$$

which can be viewed as an encapsulation of the solution set S_C: the coefficient of q^N in $F_C(qx_1, qx_2, \ldots, qx_n)$ is a listing (as the terms of a polynomial) of all nonnegative integer solutions to (1) of weight N and the number of such solutions is the coefficient of q^N in $F_C(q, q, \ldots, q)$.

Variations of this problem arise in other areas of mathematics, e.g., solving systems of linear equations, finding volume of polytopes, as well as in enumeration. In the papers [18, 19] we demonstrated that in the area of partition and composition enumeration many familiar sets of linear constraints can be easily handled a matrix inversion: for homogeneous systems, if the constraint matrix C is an $n \times n$ invertible matrix, and if all entries of $C^{-1} = B = [b_{i,j}]$ are *nonnegative integers* then by Theorem 1 in [19]:

$$F_C(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n} \frac{1}{(1 - x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})}.$$

This theorem (in its full generality) suffices to handle the enumeration of such families as Hickerson partitions [22], Santos' interpretation of Euler's family [28], Sellers' generalization of Santos [29, 30], partitions with nonnegative second differences [3], super-concave partitions [31], partitions with r-th differences nonnegative [3, 14, 33], partitions with mixed difference conditions [3], and examples (0-5) of Pak in [27]. The theorem provides bijections as well as generating functions.

However, it is easy to find simple examples where the "C matrix" technique fails. In Section 2, we propose five simple guidelines for computing the generating function $F_{\mathcal{C}}$ of a system $\mathcal{C}$ of linear diophantine inequalities. The guidelines can be viewed as a simplification of MacMahon's partition analysis [25], with ideas from matrix methods, Elliott reduction [20], and "adding a slice" (e.g. [23]).

Our focus is on the use of the guidelines to *derive a recurrence* for the generating function $F_{\mathcal{C}_n}$ of an *infinite family* $\{\mathcal{C}_n | n \geq 1\}$ of constraint systems. This is in contrast to the focus of the Omega package [6], a software implementation of partition analysis, well-designed to compute the generating function of a given fixed, finite system of linear constraints. The advantage of a recurrence for $F_{\mathcal{C}_n}$ is a program which computes $F_{\mathcal{C}_n}$ for any given n. But more significantly, if the recurrence can be solved, it provides a closed form for the generating function for the infinite family.

In Sections 3-6, we show how to use the guidelines of Section 2 strategically to solve some nontrivial enumeration problems in the theory of partitions and compositions. Sections 3 and 4 address well-studied problems, included as "warm-up" exercises to illustrate the approach and the handling of the recurrences that result. Sections 5 and 6 apply the method to the problem of enumerating anti-lecture hall compositions [16] and truncated lecture hall partitions [17], giving a simpler approach than in [16, 17]. For completeness, in Section 7 we prove that the guidelines suffice to find the generating function for the nonnegative integer solutions of any homogeneous system of linear inequalities with integer coefficients.

This work was inspired by the the work of Andrews, Paule, and Riese in the sequence of papers [2, 3, 6, 4, 12, 7, 8, 9, 5, 10, 11], which illustrate many applications of partition analysis. The Omega Package software [6] was an invaluable tool in our early investigations. As illustrated in papers such as [2, 3, 12, 9, 10], recurrences can certainly be derived using partition analysis. However, we found that the task became easier with a simpler set of tools which appear to be no less powerful. In Section 8 we discuss MacMahon's partition analysis and show how the proposed guidelines can be be viewed as essential ideas underlying his theory.

2. The Five Guidelines

Let $\mathcal{C}$ be a set of linear constraints in n variables, $\lambda_1, \ldots, \lambda_n$, each constraint $c \in \mathcal{C}$ of the form

$$c: \quad [a_0 + \sum_{i=1}^{n} a_i \lambda_i \geq 0],$$

for integer values $a_0, a_1, \ldots, a_n$.

Let $S_{\mathcal{C}}$ be the set of nonnegative integer sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying all constraints in $\mathcal{C}$. Since we are only interested here in *nonnegative* integer solutions, *we will always assume that $\mathcal{C}$ contains the constraints $[\lambda_i \geq 0]$ for $1 \leq i \leq n$.* Define the *full generating function of $\mathcal{C}$* to be:

$$F_{\mathcal{C}}(x_1, \ldots, x_n) \triangleq \sum_{\lambda \in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

If c is the constraint: $[a_0 + \sum_{i=1}^{n} a_i \lambda_i \geq 0]$ define the *negation* of c, $\neg c$, to be the constraint $[-a_0 - \sum_{i=1}^{n} a_i \lambda_i \geq 1]$. Then any sequence $(\lambda_1, \ldots, \lambda_n)$ satisfies c or $\neg c$, but not both. A constraint c is *implied by* the set of constraints $\mathcal{C}$ if $S_{\mathcal{C} \cup \{\neg c\}} = \emptyset$. A constraint c is *redundant* if $S_{\mathcal{C} \cup \{c\}} = S_{\mathcal{C}}$.

Let $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ denote the set of constraints which results from replacing λ_i by $\lambda_i + a\lambda_j$ in every constraint in $\mathcal{C}$. Note that if constraint c is implied by $\mathcal{C}$ then $c_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ is implied by $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$. Thus observe that if $\mathcal{C}$ contains the constraints $[\lambda_k \geq 0], 1 \leq k \leq n$, and if

$[\lambda_i - a\lambda_j \geq 0]$ is implied by $\mathcal{C}$, then all of the constraints $[\lambda_k \geq 0], 1 \leq k \leq n$, are also implied by $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$.

Lemma 1 *Let $\mathcal{C}$ be a set of linear constraints on variables $\lambda_1, \ldots, \lambda_n$ which contains the constraints $[\lambda_k \geq 0], 1 \leq k \leq n$. Let a be any integer (possibly negative). Suppose $[\lambda_i - a\lambda_j \geq 0]$ is implied by $\mathcal{C}$ and let $\mathcal{C}' = \mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$. Then*

$$\beta = (\beta_1, \ldots, \beta_n) \in S_{\mathcal{C}} \quad \text{iff} \quad \beta' = (\beta_1, \ldots, \beta_{i-1}, \beta_i - a\beta_j, \beta_{i+1}, \ldots, \beta_n) \in S_{\mathcal{C}'}.$$

Proof. By the remarks preceding the lemma, the constraints $\mathcal{C}$ and $\mathcal{C}'$ guarantee that $S_{\mathcal{C}}$ and $S_{\mathcal{C}'}$ contain only nonnegative integer solutions. So, it suffices to show that β satisfies a constraint in $\mathcal{C}$ iff β' satisfies the corresponding constraint in $\mathcal{C}'$.

Let $c(\lambda) = c_0 + \sum_{t=1}^{n} c_t\lambda_t$ and assume $[c(\lambda) \geq 0] \in \mathcal{C}$. Under the substitution $\lambda_i \leftarrow \lambda_i + a\lambda_j$, $c(\lambda)$ becomes $c'(\lambda)$ defined by

$$c'(\lambda) = c_0 + \sum_{t=1}^{n} c_t\lambda_t + c_i a\lambda_j = c(\lambda) + c_i a\lambda_j$$

and $[c'(\lambda) \geq 0] \in \mathcal{C}'$. Thus

$$c(\beta) = c'(\beta) - c_i a\beta_j = c'(\beta'),$$

so $c(\beta) \geq 0$ iff $c'(\beta') \geq 0$. □

Finally, to simplify notation, we will let X_n refer to the parameter list $x_1, \ldots, x_n$, so that $F(X_n)$ denotes $F(x_1, \ldots, x_n)$. Let $F(X_n; x_i \leftarrow x_i x_j^a)$ denote the function $F(X_n)$ with all occurrences of x_i replaced by $x_i x_j^a$.

Theorem 1 *(The Five Guidelines)*

1. If $\mathcal{C} = \{[\lambda_1 \geq t]\}$, for integer $t \geq 0$, then

$$F_{\mathcal{C}}(x_1) = \frac{x_1^t}{1 - x_1}.$$

2. If $\mathcal{C}_1$ is a set of constraints on variables $\lambda_1, \ldots, \lambda_j$ and $\mathcal{C}_2$ is a set of constraints on variables $\lambda_{j+1}, \ldots, \lambda_n$, then

$$F_{\mathcal{C}_1 \cup \mathcal{C}_2}(x_1, \ldots, x_n) = F_{\mathcal{C}_1}(x_1, \ldots x_j) F_{\mathcal{C}_2}(x_{j+1}, \ldots, x_n).$$

3. Let $\mathcal{C}$ be a set of linear constraints on variables $\lambda_1, \ldots, \lambda_n$ and assume $\mathcal{C}$ contains the constraints $[\lambda_i \geq 0], 1 \leq i \leq n$. Let a be any integer (possibly negative). If $[\lambda_i - a\lambda_j \geq 0]$ is implied by $\mathcal{C}$,

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}}(X_n; x_j \leftarrow x_j x_i^a).$$

4. Let c be any constraint with the same variables as the set $\mathcal{C}$. Then

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}\cup\{c\}}(X_n) + F_{\mathcal{C}\cup\{\neg c\}}(X_n).$$

5. Let $c \in \mathcal{C}$. Then

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}-\{c\}}(X_n) - F_{\mathcal{C}-\{c\}\cup\{\neg c\}}(X_n).$$

Proof.

1. This is clear since $F_{\mathcal{C}}(x_1) = x_1^t + x_1^{t+1} + \cdots$.
2. The sequence $(\lambda_1, \ldots, \lambda_n) \in S_{\mathcal{C}_1\cup\mathcal{C}_2}$ iff $(\lambda_1, \ldots, \lambda_j) \in S_{\mathcal{C}_1}$ and $(\lambda_{j+1}, \ldots, \lambda_n) \in S_{\mathcal{C}_2}$.
3. Let $\mathcal{C}' = \mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$. By Lemma 1,

$$(\lambda_1, \ldots, \lambda_n) \in S_{\mathcal{C}'} \quad \text{iff} \quad (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + a\lambda_j, \lambda_{i+1}, \ldots, \lambda_n) \in S_{\mathcal{C}}.$$

So,

$$\begin{aligned}
F_{\mathcal{C}'}(X_n; x_j \leftarrow x_j x_i^a) &= \sum_{\lambda \in S_{\mathcal{C}'}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{j-1}^{\lambda_{j-1}} (x_j x_i^a)^{\lambda_j} x_{j+1}^{\lambda_{j+1}} \cdots x_n^{\lambda_n} \\
&= \sum_{\lambda \in S_{\mathcal{C}'}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{i-1}^{\lambda_{i-1}} x_i^{(\lambda_i + a\lambda_j)} x_{i+1}^{\lambda_{i+1}} \cdots x_n^{\lambda_n} \\
&= \sum_{\lambda \in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_i^{\lambda_i} \cdots x_n^{\lambda_n} \\
&= F_{\mathcal{C}}(X_n).
\end{aligned}$$

4. $S_{\mathcal{C}}$ can be partitioned into those λ that satisfy c and those that do not.
5. By guideline 4, $F_{\mathcal{C}-\{c\}}(X_n) = F_{\mathcal{C}-\{c\}\cup\{c\}}(X_n) + F_{\mathcal{C}-\{c\}\cup\{\neg c\}}(X_n)$. Then $\mathcal{C}-\{c\}\cup\{c\} = \mathcal{C}$, since $c \in \mathcal{C}$. □

3. Minc's Partition Function and Cayley Compositions

Minc's partition function $\nu(d, N)$ is the number of compositions of N in which the first part is d and each part is at most twice the size of the preceding part [26]. For example, in the special case $d = 1$, these are called Cayley compositions [15, 1, 12]. In this section we compute the generating function $\nu(q) = \sum_{d,N\geq 0} \nu(d, N) q^N = q + 2q^2 + 4q^3 + 7q^4 + 13q^5 + 24q^6 + \cdots$. For example, the coefficient of q^5 is 13, since of the 16 compositions of 5, only these three violate the constraints: $(1, 4)$, $(1, 3, 1)$, and $(1, 1, 3)$.

Let $\mathcal{C}_n$ be the set of constraints $\mathcal{C}_n = \{\lambda_i \geq \frac{1}{2}\lambda_{i+1} > 0 \mid 1 \leq i < n\}$ and let $C_n(x_1, \ldots, x_n)$ be the generating function of $\mathcal{C}_n$. Focusing on the constraint $c = [\lambda_{n-1} \geq \frac{1}{2}\lambda_n]$, after noting that $[\lambda_{n-1} > 0]$ is redundant, we can write $\mathcal{C}_n$ as

$$\mathcal{C}_n = \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{1}{2}\lambda_3 \\ & \vdots & \\ \lambda_{n-2} & \geq & \frac{1}{2}\lambda_{n-1} \\ \lambda_{n-1} & \geq & \frac{1}{2}\lambda_n \\ \lambda_n & > & 0 \end{array}\right] = \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{1}{2}\lambda_3 \\ & \vdots & \\ \lambda_{n-2} & \geq & \frac{1}{2}\lambda_{n-1} \\ \lambda_{n-1} & \geq & \frac{1}{2}\lambda_n \\ \lambda_{n-1} & > & 0 \\ \lambda_n & > & 0 \end{array}\right]$$

$$= \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{1}{2}\lambda_3 \\ & \vdots & \\ \lambda_{n-2} & \geq & \frac{1}{2}\lambda_{n-1} \\ \lambda_{n-1} & > & 0 \\ \lambda_n & > & 0 \end{array}\right] - \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{1}{2}\lambda_3 \\ & \vdots & \\ \lambda_{n-2} & \geq & \frac{1}{2}\lambda_{n-1} \\ \lambda_n & > & 2\lambda_{n-1} \\ \lambda_{n-1} & > & 0 \end{array}\right],$$

where c has been removed from the next-to-last system, making it $\mathcal{C}_{n-1} \cup [\lambda_n > 0]$, and c has been replaced by $\neg c$ in the last system. By guidelines 1 and 2, $x_n C_{n-1}(x_1, \ldots, x_{n-1})/(1-x_n)$ is the generating function for $\mathcal{C}_{n-1} \cup [\lambda_n > 0]$. Note further that the substitution $\lambda_n \leftarrow \lambda_n + 2\lambda_{n-1}$ in the last system results $\mathcal{C}_{n-1} \cup [\lambda_n > 0]$, so by guideline 3, the last system has generating function $x_n C_{n-1}(x_1, \ldots, x_{n-1}x_n^2)/(1 - x_n)$. Putting this together with guideline 5 and the initial condition $C_1(x_1) = x_1/(1 - x_1)$ gives the recurrence

$$C_n(x_1, \ldots, x_n) = \frac{x_n}{1 - x_n}(C_{n-1}(x_1, \ldots, x_{n-1}) - C_{n-1}(x_1, \ldots, x_{n-2}, x_{n-1}x_n^2)).$$

Let $C_n(q, s) = C_n(q, q, \ldots, q, s)$. Then the above recurrence gives $C_1(q, s) = s/(1 - s)$ and for $n \geq 2$,

$$C_n(q, s) = \frac{s}{1 - s}(C_{n-1}(q, q) - C_{n-1}(q, qs^2)).$$

Set $C(q, s) = \sum_{n=1}^{\infty} C_n(q, s)$ and use the recurrence for $C_n(q, s)$ to get

$$C(q, s) = \sum_{n=1}^{\infty} C_n(q, s) = \frac{s}{1 - s} + \sum_{n=2}^{\infty} C_n(q, s) = \frac{s}{1 - s}(1 + C(q, q) - C(q, qs^2)).$$

Iterating the recurrence for $C(q, s)$ gives

$$C(q, s) = (1 + C(q, q)) \sum_{i=1}^{\infty} (-1)^{i-1} \prod_{j=0}^{i-1} \frac{q^{2^j-1}s^{2^j}}{(1 - q^{2^j-1}s^{2^j})}.$$

Let $C(q) = C(q, q)$, then

$$\nu(q) = 1 + C(q) = \frac{1}{1 + \sum_{i=1}^{\infty} \frac{(-1)^i q^{2^{i+1}-i-2}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2^i-1})}}.$$

4. Two-Rowed Plane Partitions

This example illustrates the advantage of guideline 3 of Theorem 1 when $a < 0$. The *two-rowed plane partitions* are those integer sequences $(a_1, b_1, \ldots, a_n, b_n)$ satisfying the constraints

$$\mathcal{P}_n \quad = \quad [a_i \geq b_i \geq 0, \quad 1 \leq i \leq n; \qquad a_i \geq a_{i+1}, \quad b_i \geq b_{i+1}, \quad 1 \leq i \leq n-1].$$

It is well-known that the generating function for $\mathcal{P}_n$ is [24]

$$P_n(q) = \frac{1}{(q;q)_n (q^2;q)_n}. \tag{3}$$

In [3], Andrews shows how MacMahon's *partition analysis* can be used to compute $P_n(q)$ by considering an intermediate family $\mathcal{G}_n$. We will use this approach, but with a slight twist, to show how the generating function for $\mathcal{P}_n$, can be computed via $\mathcal{G}_n$ from the guidelines of Theorem 1.

We will use the convention that when a constraint system is represented by a calligraphic letter, its generating function is represented by the corresponding roman letter. Also, to keep notation simple, when the meaning is clear from context, we will use the same letter to refer to multivariable and single variable forms of the generating function.

Define $\mathcal{G}_n$ to be the set of constraints below:

$$\mathcal{G}_n \quad = \quad \begin{bmatrix} a_1 + a_2 + \cdots + a_n & \geq & b_1 + b_2 + \cdots + b_n \\ a_2 + \cdots + a_n & \geq & b_2 + \cdots + b_n \\ \vdots & \vdots & \vdots \\ a_{n-1} + a_n & \geq & b_{n-1} + b_n \\ a_n & \geq & b_n \\ a_i, b_i \geq 0, & & i = 1, \ldots, n \end{bmatrix}.$$

Denote the full generating functions for $\mathcal{P}_n$ and $\mathcal{G}_n$ by

$$P_n(x_1, y_1, \ldots, x_n, y_n) \triangleq \sum_{(a_1,b_1,\ldots,a_n,b_n) \in S_{\mathcal{P}_n}} x_1^{a_1} y_1^{b_1} \ldots, x_n^{a_n} y_n^{b_n},$$

$$G_n(x_1, y_1, \ldots, x_n, y_n) \triangleq \sum_{(a_1,b_1,\ldots,a_n,b_n) \in S_{\mathcal{G}_n}} x_1^{a_1} y_1^{b_1} \ldots, x_n^{a_n} y_n^{b_n}.$$

Note that $\mathcal{P}_n$ can be transformed into $\mathcal{G}_n$ by the sequence of substitutions:

$$a_i \leftarrow a_i + a_{i+1}; \qquad b_i \leftarrow b_i + b_{i+1}; \qquad i = 1, 2, \ldots n-1.$$

We focus on G_n. Since for $1 \le i \le n-1$, $a_i - a_{i+1} \ge 0$ and $b_i - b_{i+1} \ge 0$ in $\mathcal{P}$, by guideline 3 of Theorem 1, P_n is obtained from G_n by the sequence of substitutions:

$$x_i \leftarrow x_i x_{i-1}; \qquad y_i \leftarrow y_i y_{i-1} \qquad i = n, n-1, n-2, \ldots, 2.$$

Thus

$$P_n(x_1, y_1, \ldots, x_n, y_n) = G_n(x_1, y_1, x_1x_2, y_1y_2, \ldots, x_1x_2\cdots x_n, y_1y_2\cdots y_n).$$

In particular, the generating function (3) for two-rowed plane partitions is obtained by setting $x_i = y_i = q$ in P_n for $i = 1, \ldots, n$:

$$P_n(q, q, q, \ldots, q) = G_n(q, q, q^2, q^2, \ldots, q^n, q^n). \tag{4}$$

Since $a_n - b_n \ge 0$ in $\mathcal{G}_n$, by guideline 3, we can do the substitution $a_n \leftarrow a_n + b_n$ in $\mathcal{G}_n$ to get $\mathcal{F}_n$ and recover G_n from F_n as shown below.

$$\mathcal{F}_n \quad = \quad \left[\begin{array}{rcr} a_1 + a_2 + \cdots + a_n & \ge & b_1 + b_2 + \cdots + b_{n-1} \\ a_2 + \cdots + a_n & \ge & b_2 + \cdots + b_{n-1} \\ \vdots & \vdots & \vdots \\ a_{n-1} + a_n & \ge & b_{n-1} \\ a_i, b_i \ge 0, & & i = 1, \ldots, n \end{array} \right],$$

$$G_n(x_1, y_1, \ldots, x_n, y_n) = F_n(x_1, y_1, \ldots, x_n, y_n; y_n \leftarrow x_n y_n).$$

Since $a_{n-1} + a_n \ge 0$ in $\mathcal{F}_n$, by guideline 3, we can substitute $a_{n-1} \leftarrow a_{n-1} - a_n$ in $\mathcal{F}_n$ to get $\mathcal{H}_n$ and recover F_n from H_n as shown.

$$\mathcal{H}_n \quad = \quad \left[\begin{array}{rcr} a_1 + a_2 + \cdots + a_{n-1} & \ge & b_1 + b_2 + \cdots + b_{n-1} \\ a_2 + \cdots + a_{n-1} & \ge & b_2 + \cdots + b_{n-1} \\ \vdots & \vdots & \vdots \\ a_{n-1} & \ge & b_{n-1} \\ a_{n-1} & \ge & a_n \\ a_i, b_i \ge 0 & & i = 1, \ldots, n \end{array} \right],$$

$$F_n(x_1, y_1, \ldots, x_n, y_n) = H_n(x_1, y_1, \ldots, x_n, y_n; x_n \leftarrow x_n/x_{n-1}).$$

Summarizing to this point, we have

$$G_n(x_1, y_1, \ldots, x_n, y_n) = H_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n/x_{n-1}, x_n y_n). \tag{5}$$

Now apply guideline 5 to $\mathcal{H}_n$ using the constraint $c = [a_{n-1} \geq a_n]$. Then $\mathcal{H}_n = \mathcal{K}_n - \mathcal{L}_n$, where $\mathcal{K}_n = \mathcal{H}_n - \{[a_{n-1} \geq a_n]\}$ and $\mathcal{L}_n = \mathcal{H}_n - \{[a_{n-1} \geq a_n]\} \cup \{[a_n \geq a_{n-1} + 1]\}$, that is,

$$\mathcal{K}_n = \left[\begin{array}{rcr} a_1 + a_2 + \cdots + a_{n-1} & \geq & b_1 + b_2 + \cdots + b_{n-1} \\ a_2 + \cdots + a_{n-1} & \geq & b_2 + \cdots + b_{n-1} \\ \vdots & \vdots & \vdots \\ a_{n-1} & \geq & b_{n-1} \\ a_i, b_i \geq 0 & & i = 1, \ldots, n \end{array}\right]$$

and

$$\mathcal{L}_n = \left[\begin{array}{rcr} a_1 + a_2 + \cdots + a_{n-1} & \geq & b_1 + b_2 + \cdots + b_{n-1} \\ a_2 + \cdots + a_{n-1} & \geq & b_2 + \cdots + b_{n-1} \\ \vdots & \vdots & \vdots \\ a_{n-1} & \geq & b_{n-1} \\ a_n & \geq & a_{n-1} + 1 \\ a_i, b_i \geq 0 & & i = 1, \ldots, n \end{array}\right],$$

so that

$$H_n(x_1, y_1, \ldots, x_n, y_n) = K_n(x_1, y_1, \ldots, x_n, y_n) - L_n(x_1, y_1, \ldots, x_n, y_n). \tag{6}$$

Now observe that

$$\mathcal{K}_n = \mathcal{G}_{n-1} \cup \{[a_n \geq 0], [b_n \geq 0]\},$$

so by guidelines 1 and 2,

$$K_n(x_1, y_1, \ldots, x_n, y_n) = \frac{G_{n-1}(x_1, y_1, \ldots, x_{n-1}, y_{n-1})}{(1 - x_n)(1 - y_n)}. \tag{7}$$

Returning to $\mathcal{L}_n$, since $a_n - a_{n-1} \geq 0$ in $\mathcal{L}_n$, we can do the substitution $a_n \leftarrow a_n + a_{n-1}$, resulting in

$$(\mathcal{L}_n)_{a_n \leftarrow a_n + a_{n-1}} = \mathcal{G}_{n-1} \cup \{[a_n \geq 1], [b_n \geq 0]\},$$

so by guidelines 1, 2, and 3,

$$L_n(x_1, y_1, \ldots, x_n, y_n) = \frac{x_n G_{n-1}(x_1, y_1, \ldots, x_{n-1}, y_{n-1}; x_{n-1} \leftarrow x_{n-1} x_n)}{(1 - x_n)(1 - y_n)}. \tag{8}$$

Combining (6),(7), and (8), we have

$$H_n(x_1, y_1, \ldots, x_n, y_n) = \frac{G_{n-1}(x_1, y_1, \ldots, x_{n-1}, y_{n-1})}{(1 - x_n)(1 - y_n)} - \frac{x_n G_{n-1}(x_1, y_1, \ldots, x_{n-2}, y_{n-2}, x_{n-1} x_n, y_{n-1})}{(1 - x_n)(1 - y_n)}.$$

Finally, substituting this expression for H_n into (5) gives a recurrence for G_n:

$$G_n(x_1, y_1, \ldots, x_n, y_n) = \frac{G_{n-1}(x_1, y_1, \ldots, x_{n-1}, y_{n-1}) - \frac{x_n}{x_{n-1}} G_{n-1}(x_1, y_1, \ldots, x_{n-2}, y_{n-2}, x_n, y_{n-1})}{(1 - x_n / x_{n-1})(1 - x_n y_n)}, \tag{9}$$

with initial condition $G_1(x_1, y_1) = 1/(1 - x_1)/(1 - x_1 y_1)$.

Let $G_n^*(q, s) = G_n(q, q, q^2, q^2, \dots, s, q^n)$. Then from the recursion (9),

$$G_n^*(q, s) = \frac{G_{n-1}^*(q, q^{n-1}) - (s/q^{n-1})G_{n-1}^*(q, s)}{(1 - s/q^{n-1})(1 - sq^n)}.$$

It is straightforward to show by induction that $G_n^*(q, s)$ satisfies

$$G_n^*(q, s) = \frac{1}{(1 - s)(1 - sq)(q; q)_{n-1}(q^2; q)_{n-1}}.$$

Substituting $s = q^n$ gives

$$P_n(q) = G_n(q, q, q^2, q^2, \dots, q^n, q^n) = G_n^*(q, q^n) = \frac{1}{(q; q)_n (q^2; q)_n},$$

the desired generating function for $2 \times n$ plane partitions.

5. Anti-Lecture Hall Compositions

In [16], we considered the set of sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ satisfying the constraints

$$\mathcal{A}_n \quad = \quad \left[\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_n}{n} \geq 0 \right].$$

We referred to these as *anti-lecture hall compositions* and showed that the generating function is

$$A_n(q) \triangleq \sum_{\lambda \in A_n} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1 + q^i}{1 - q^{i+1}}. \tag{10}$$

Here we show how to apply the guidelines of Theorem 1 to get a recurrence for the full generating function $A_n(x_1, x_2, \dots x_n)$ and use it to give an “easy” proof of (10). The idea is easily extended to the *truncated anti-lecture hall compositions* studied in [17]. We start with $\mathcal{B}_n$, a slight variation of $\mathcal{A}_n$.

Lemma 2 *The full generating function for the integer sequences defined by the constraints*

$$\mathcal{B}_n \quad = \quad \left[\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_{n-1}}{n-1} \geq \frac{\lambda_n}{1} \geq 0 \right]. \tag{11}$$

satisfies $B_n(x_1, \dots, x_n) = \dfrac{A_{n-1}(x_1 \dots, x_{n-1})}{1 - x_1 x_2^2 x_3^3 \cdots x_{n-1}^{n-1} x_n}$.

Proof. The following sequence of substitutions transforms $\mathcal{B}_n$ into $\mathcal{A}_{n-1} \cup \{[\lambda_n \geq 0]\}$, as illustrated in Figure 1: $\lambda_i \leftarrow \lambda_i + i\lambda_n, \quad i = n-1, \ldots, 1$. Note that the constraint $\lambda_{i-1} \geq (i-1)\lambda_n$ is implied at each stage, so by guidelines 1,2, and 3, B_n is recovered from A_n by performing the sequence of substitutions on $\frac{A_{n-1}(x_1,\ldots,x_{n-1})}{(1-x_n)}$: $x_n \leftarrow x_n x_i^i, \quad i = 1, \ldots, n-1$. □

$$\begin{bmatrix} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{2}{3}\lambda_3 \\ & \vdots & \\ \lambda_{n-3} & \geq & \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} & \geq & \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} & \geq & (n-1)\lambda_n \\ \lambda_n & \geq & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{2}{3}\lambda_3 \\ & \vdots & \\ \lambda_{n-3} & \geq & \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} & \geq & \frac{n-2}{n-1}\lambda_{n-1} + (n-2)\lambda_n \\ \lambda_{n-1} & \geq & 0 \\ \lambda_n & \geq & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{2}{3}\lambda_3 \\ & \vdots & \\ \lambda_{n-3} & \geq & \frac{n-3}{n-2}\lambda_{n-2} + (n-3)\lambda_n \\ \lambda_{n-2} & \geq & \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} & \geq & 0 \\ \lambda_n & \geq & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{2}{3}\lambda_3 + 2\lambda_n \\ & \vdots & \\ \lambda_{n-3} & \geq & \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} & \geq & \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} & \geq & 0 \\ \lambda_n & \geq & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{1}{2}\lambda_2 + \lambda_n \\ \lambda_2 & \geq & \frac{2}{3}\lambda_3 \\ & \vdots & \\ \lambda_{n-3} & \geq & \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} & \geq & \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} & \geq & 0 \\ \lambda_n & \geq & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{1}{2}\lambda_2 \\ \lambda_2 & \geq & \frac{2}{3}\lambda_3 \\ & \vdots & \\ \lambda_{n-3} & \geq & \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} & \geq & \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} & \geq & 0 \\ & & \\ \lambda_n & \geq & 0 \end{bmatrix}$$

Figure 1. Transformation of $\mathcal{B}_n$ into $\mathcal{A}_{n-1} \cup \{[\lambda_n \geq 0]\}$ in proof of Lemma 2.

Proposition 1 *The full generating function for anti-lecture hall compositions satisfies:*

$$A_n(x_1, \ldots x_n) = \frac{A_{n-1}(x_1, \ldots, x_{n-1})}{1 - x_n} - A_{n-1}(x_1, \ldots, x_{n-2}, x_n x_{n-1}) \left(\frac{1}{1-x_n} - \frac{1}{1 - x_1 x_2^2 x_3^3 \cdots x_n^n} \right)$$

with initial condition $A_1(x_1) = 1/(1 - x_1)$.

Proof. Using guideline 5 with $c = [\lambda_{n-1} \geq \frac{n-1}{n}\lambda_n]$,

$$A_n(x_1, \ldots, x_n) = C_n(x_1, \ldots, x_n) - D_n(x_1, \ldots, x_n),$$

where

$$\mathcal{C}_n = \left[\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_{n-1}}{n-1} \geq 0; \quad \lambda_n \geq 0\right]; \tag{12}$$

$$\mathcal{D}_n = \left[\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_{n-1}}{n-1} \geq 0; \quad \frac{\lambda_n}{n} > \frac{\lambda_{n-1}}{n-1}\right]. \tag{13}$$

Note that $\mathcal{C}_n = \mathcal{A}_{n-1} \cup \{[\lambda_n \geq 0]\}$, so by guideline 2, $\mathcal{C}_n$ has generating function

$$C_n(x_1 \ldots, x_n) = \frac{A_{n-1}(x_1 \ldots, x_{n-1})}{1 - x_n}. \tag{14}$$

Since $\lambda_n \geq \lambda_{n-1}$ is implied by $\mathcal{D}_n$ in (13), by guideline 3, substituting $\lambda_n \leftarrow \lambda_n + \lambda_{n-1}$ in $\mathcal{D}_n$ gives

$$\mathcal{E}_n = \left[\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_{n-1}}{n-1} \geq 0; \quad \lambda_n > \frac{\lambda_{n-1}}{n-1}\right] \tag{15}$$

and

$$D_n(x_1, \ldots, x_n) = E_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n),$$

where X_n represents the argument list $x_1, \ldots, x_n$. Using guideline 5 again, with $c = [\lambda_n > \frac{\lambda_{n-1}}{n-1}]$, gives

$$E_n(X_n) = C_n(X_n) - B_n(X_n),$$

where $\mathcal{C}_n$ is (12) and where $\mathcal{B}_n$ is (11). Putting this all together, we have

$$\begin{aligned} A_n(X_n) &= C_n(X_n) - D_n(X_n) \\ &= C_n(X_n) - E_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n) \\ &= C_n(X_n) - C_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n) + B_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n) \end{aligned}$$

Substituting from (14) and Lemma 2 gives the result. □

In order to make use of the recurrence of Proposition 1 to prove the generating function (10) for anti-lecture hall compositions, let $A_n(q,s) \triangleq A_n(q,q,q,\ldots,q,s)$. Then the recurrence of Proposition 1 becomes

$$A_n(q,s) = \frac{A_{n-1}(q,q)}{1-s} - A_{n-1}(q,qs)\frac{s(1-s^{n-1}q^{\binom{n}{2}})}{(1-s)(1-s^n q^{\binom{n}{2}})}, \tag{16}$$

with initial condition $A_0(q,s) = 1$. If we were to proceed as with two-rowed plane partitions, we would (i) "guess" the form of $A_n(q,s)$, (ii) prove by induction that it satisfies (16), and then (iii) show that setting $s = q$ gives (10). This would be the easiest proof and it would give a refinement of the anti-lecture hall generating function, enumerating solutions according to both the weight and the size of the last part:

$$\sum_{\lambda \in S_{\mathcal{A}_n}} q^{|\lambda|} s^{\lambda_n} = A_n(q,qs).$$

Since we have *not* succeeded in guessing $A_n(q,s)$, we follow a different approach. Iterating the recurrence of (16) gives:

$$A_n(q,s) = \sum_{i=0}^{n-1} (-1)^i A_{n-1-i}(q,q) s^i q^{\binom{i}{2}} \frac{1 - s^{n-i} q^{\binom{n}{2}-\binom{i}{2}}}{(s;q)_{i+1}(1 - s^n q^{\binom{n}{2}})}. \tag{17}$$

Now, setting $s = q$ gives a recurrence independent of s:

$$A_n(q,q) = \sum_{i=0}^{n-1} (-1)^i A_{n-1-i}(q,q) \frac{q^{\binom{i+1}{2}} - q^{\binom{n+1}{2}}}{(q;q)_{i+1}(1 - q^{\binom{n+1}{2}})}. \tag{18}$$

We show by induction that the solution to (18) is

$$A_n(q,q) = \frac{(-q)_n}{(q^2)_n}.$$

Assume inductively that $A_{n-1-i} = (-q)_{n-1-i}/(q^2)_{n-1-i}$. Then we need to prove that

$$\frac{B_n(q) - q^{\binom{n+1}{2}} C_n(q)}{1 - q^{\binom{n+1}{2}}} = \frac{(-q)_n}{(q^2)_n};$$

with

$$C_n(q) = \sum_{i=0}^{n-1} (-1)^i \frac{(-q)_{n-1-i}}{(q^2)_{n-1-i}(q)_{i+1}}$$

and

$$B_n(q) = \sum_{i=0}^{n-1} (-1)^i q^{i(i+1)/2} \frac{(-q)_{n-1-i}}{(q^2)_{n-1-i}(q)_{i+1}}$$

We will prove that

$$B_{2n+1}(q) = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}} \quad C_{2n+1}(q) = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}}$$

$$B_{2n}(q) = \frac{(-q)_{2n}}{(q^2)_{2n}} - \frac{q^{\binom{2n+1}{2}}}{(q^2)_{2n}}; \qquad C_{2n}(q) = \frac{(-q)_{2n}}{(q^2)_{2n}} - \frac{1}{(q^2)_{2n}}$$

Therefore, we need to prove the following identities for C_n :

$$\sum_{i=0}^{2n} (-1)^i \frac{(-q)_{2n-i}}{(q^2)_{2n-i}(q)_{i+1}} = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}}. \tag{19}$$

$$\sum_{i=0}^{2n-1} (-1)^i \frac{(-q)_{2n-1-i}}{(q^2)_{2n-1-i}(q)_{i+1}} = \frac{(-q)_{2n}}{(q^2)_{2n}} - \frac{1}{(q^2)_{2n}}. \tag{20}$$

A few q-series manipulations show that the two previous equations are equivalent to:

$$\sum_{j=0}^{n}(-1)^j(-1;q)_j \begin{bmatrix} n \\ j \end{bmatrix}_q = (-1)^n \tag{21}$$

Recalling that

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(q^{-n})_j(-1)^j q^{nj-j(j-1)/2}}{(q)_j},$$

we see that the identity follows from the case $a = -1, c \to \infty$ of q-Chu Vandermonde summation (1.5.2 in [21]),

$$\sum_{j=0}^{n} \frac{(a)_j(q^{-n})_j(cq^n/a)^j}{(c)_j(q)_j} = \frac{(c/a)_n}{(c)_n}. \tag{22}$$

Now we need

$$\sum_{i=0}^{2n}(-1)^i q^{\binom{i+1}{2}} \frac{(-q)_{2n-i}}{(q^2)_{2n-i}(q)_{i+1}} = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}}. \tag{23}$$

$$\sum_{i=0}^{2n-1}(-1)^i q^{i(i+1)/2} \frac{(-q)_{2n-1-i}}{(q^2)_{2n-1-i}(q)_{i+1}} = \frac{(-q)_{2n}}{(q^2)_{2n}} + \frac{q^{\binom{2n+1}{2}}}{(q^2)_{2n}}. \tag{24}$$

The same q-series manipulations show that the two previous equations are equivalent to:

$$\sum_{j=0}^{n}(-1)^j(-1;q)_j \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{n-j}{2}} = (-1)^n q^{\binom{n}{2}} \tag{25}$$

This follows in a similar way from the "other" q-Chu Vandermonde summation (1.5.3 in [21]),

$$\sum_{j=0}^{n} \frac{(a)_j(q^{-n})_j q^j}{(c)_j(q)_j} = \frac{a^n(c/a)_n}{(c)_n}, \tag{26}$$

under the substitutions $a = -1, c = 0$. □

6. Lecture Hall Partitions

In [13], Bousquet-Mélou and Eriksson studied the set of integer sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying the constraints

$$\mathcal{L}_n = \left[\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_n}{1} \geq 0\right].$$

They referred to these as *lecture hall partitions* and showed that the generating function is

$$L_n(q) \triangleq \sum_{\lambda \in S_{\mathcal{L}_n}} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}. \tag{27}$$

In [2], Andrews showed how to use partition analysis to derive a recurrence for the full generating function of $\mathcal{L}_n$. However, substantial new ideas, outside of partition analysis, were required to move from this to the solution (27).

In this section, we show that by strategic application of Theorem 1, we can derive a recurrence for the full generating function of a generalization of $\mathcal{L}_n$ that will reduce the proof of (27) to a q-series calculation (albeit nontrivial). Our derivation here via the five guidelines is both simpler and more elementary than the approach in [17] (at the expense of a more challenging q-series calculation).

In [17], we defined *truncated lecture hall partitions* to be the integer sequences satisfying:

$$\mathcal{L}_{n,k} \quad = \quad \left[\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} \geq 0\right].$$

We showed that if

$$\bar{\mathcal{L}}_{n,k} \quad = \quad \left[\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} > 0\right], \tag{28}$$

that is, all parts must be positive, the generating function is

$$\bar{L}_{n,k}(q) = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2n-k+1};q)_k}. \tag{29}$$

It can be checked that setting $k = n$ and dividing by $q^{\binom{n+1}{2}}$ gives (27).

Proposition 2 *The generating function for truncated lecture hall partitions* (28) *satisfies*

$$\begin{aligned} \bar{L}_{n,k}(x_1,\ldots,x_k) \quad = \quad & \frac{x_k \bar{L}_{n,k-1}(x_1,\ldots,x_{k-1})}{1-x_k} - \frac{\bar{L}_{n,k-1}(x_1,\ldots,x_{k-2},x_{k-1}x_k)}{1-x_k} \\ & - \frac{z_{n,k}\bar{L}_{n,k-1}(x_1,\ldots,x_{k-2},x_{k-1}x_k)}{1-z_{n,k}}. \end{aligned}$$

with $z_{n,k} = x_1^n x_2^{n-1} \ldots x_k^{n-k+1}$.

Proof. Note that $\lambda_{k-1} > \lambda_k$ is implied by $\bar{\mathcal{L}}_{n,k}$, so by guideline 4, $\bar{\mathcal{L}}_{n,k} = \bar{\mathcal{L}}_{n,k} \cup \{[\lambda_{k-1} > \lambda_k]\}$.

Now apply guideline 5 with $c = [\lambda_{k-1} \geq \frac{n-k+2}{n-k+1}\lambda_k]$ to get $\bar{\mathcal{L}}_{n,k} = \mathcal{D} - \mathcal{E}$:

$$\bar{\mathcal{L}}_{n,k} = \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} & \geq & \frac{n-k+2}{n-k+1}\lambda_k \\ \lambda_{k-1} & > & \lambda_k \\ \lambda_k & > & 0 \end{array}\right] = \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ & & \\ \lambda_{k-1} & > & \lambda_k \\ \lambda_k & > & 0 \end{array}\right] - \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_k & > & \frac{n-k+1}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} & > & \lambda_k \\ \lambda_k & > & 0 \end{array}\right] \quad (30)$$

The first system on the right, $\mathcal{D}$, implies the constraint $\lambda_{k-1} > 0$, so it can be added. Now apply guideline 5 to $\mathcal{D}$ using $c = [\lambda_{k-1} > \lambda_k]$ to get:

$$\mathcal{D} = \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} & > & \lambda_k \\ \lambda_{k-1} & > & 0 \\ \lambda_k & > & 0 \end{array}\right] = \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ & & \\ \lambda_{k-1} & > & 0 \\ \lambda_k & > & 0 \end{array}\right] - \left[\begin{array}{rcl} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_k & \geq & \lambda_{k-1} \\ \lambda_{k-1} & > & 0 \\ \lambda_k & > & 0 \end{array}\right]. \quad (31)$$

The first system on the right of (31) is just $\bar{L}_{n,k-1} \cup \{[\lambda_k > 0]\}$. The second system on the right becomes $\bar{L}_{n,k-1} \cup \{[\lambda_k \geq 0]\}$ after the substitution $\lambda_k \leftarrow \lambda_k + \lambda_{k-1}$. So, by Theorem 1 and summarizing so far, we have

$$\bar{L}_{n,k}(x_1,\ldots,x_k) = \frac{x_k \bar{L}_{n,k-1}(x_1,\ldots,x_{k-1})}{(1-x_k)} - \frac{\bar{L}_{n,k-1}(x_1,\ldots,x_{k-1}x_k)}{(1-x_k)} - E(x_1,\ldots,x_k), \quad (32)$$

where $E(x_1,\ldots,x_k)$ is the generating function for the last constraint system, $\mathcal{E}$, in (30). Apply $\lambda_{k-1} \leftarrow \lambda_{k-1} + \lambda_k$ to $\mathcal{E}$ followed by $\lambda_k \leftarrow \lambda_k + (n-k+1)\lambda_{k-1}$ as illustrated below $\mathcal{E} \to \mathcal{E}' \to \mathcal{F}$:

(33)

$$\begin{bmatrix} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_k & > & \frac{n-k+1}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} & > & \lambda_k \\ \lambda_k & > & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}(\lambda_{k-1}+\lambda_k) \\ \lambda_k & > & (n-k+1)\lambda_{k-1} \\ \lambda_{k-1} & > & 0 \\ \lambda_k & > & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_k \\ & & +(n-k+3)\lambda_{k-1} \\ \lambda_k & > & 0 \\ \lambda_{k-1} & > & 0 \end{bmatrix}.$$

By guideline 3,

$$E(x_1,\ldots,x_k) = E'(x_1,\ldots,x_{k-1},x_{k-1}x_k),$$
$$E'(x_1,\ldots x_k) = F(x_1,\ldots,x_{k-2},x_{k-1}x_k^{n-k+1},x_k),$$

so

$$E(x_1,\ldots,x_k) = F(x_1,\ldots,x_{k-2},x_{k-1}^{n-k+2}x_k^{n-k+1},x_{k-1}x_k). \tag{34}$$

Finally, starting from $\mathcal{F}$, the last set of constraints in (33), perform the following sequence of substitutions

$$\lambda_i \leftarrow \lambda_i + (n-i+1)\lambda_{k-1}; \quad i = k-2,\ldots 1,$$

as illustrated below:

$$\mathcal{F} \rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} + (n-k+4)\lambda_{k-1} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k & > & 0 \\ \lambda_{k-1} & > & 0 \end{bmatrix} \rightarrow \ldots \rightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 + (n-1)\lambda_{k-1} \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k & > & 0 \\ \lambda_{k-1} & > & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 + n\lambda_{k-1} \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k & > & 0 \\ \lambda_{k-1} & > & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \lambda_1 & \geq & \frac{n}{n-1}\lambda_2 \\ \lambda_2 & \geq & \frac{n-1}{n-2}\lambda_3 \\ & \vdots & \\ \lambda_{k-3} & \geq & \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} & \geq & \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k & > & 0 \\ \lambda_{k-1} & > & 0 \end{bmatrix} = \mathcal{G}. \tag{35}$$

The resulting system of constraints, $\mathcal{G}$, in (35) can be viewed as $\mathcal{L}_{n,k-1}$, where λ_{k-1} has been replaced by λ_k, together with the constraint $[\lambda_{k-1} > 0]$. Thus,

$$G(x_1,\ldots,x_k) = \frac{x_{k-1}L_{n,k-1}(x_1,\ldots,x_{k-2},x_k)}{(1-x_{k-1})}. \tag{36}$$

By guideline 3, the generating function for $\mathcal{F}$ is obtained from G by the sequence of substitutions

$$x_{k-1} \leftarrow x_{k-1}x_i^{n-i+1}; \quad i = 1 \ldots k-2,$$

giving

$$F(x_1, \ldots, x_k) = G(x_1, \ldots, x_{k-2}, x_1^n x_2^{n-1} \cdots x_{k-2}^{n-k+3} x_{k-1}, x_k). \tag{37}$$

Returning to E in (34) and using (36) and (37),

$$\begin{aligned} E(x_1, \ldots, x_k) &= F(x_1, \ldots, x_{k-2}, x_{k-1}^{n-k+2} x_k^{n-k+1}, x_{k-1}x_k) \\ &= G(x_1, \ldots, x_{k-2}, x_1^n x_2^{n-1} \cdots x_{k-2}^{n-k+3} x_{k-1}^{n-k+2} x_k^{n-k+1}, x_{k-1}x_k) \\ &= \frac{x_1^n x_2^{n-1} \cdots x_k^{n-k+1} L_{n,k-1}(x_1, \ldots, x_{k-2}, x_{k-1}x_k)}{1 - x_1^n x_2^{n-1} \cdots x_k^{n-k+1}}. \end{aligned} \tag{38}$$

Combining (38) with (32) gives the result. □

Let $\bar{L}_{n,k}(q, s) = \bar{L}_{n,k}(q, q, \ldots, q, s)$. Setting $x_k = s$ and $x_i = q$ for $i < k$ in Proposition 2 gives

$$\bar{L}_{n,k}(q, s) = \frac{s}{1-s} \bar{L}_{n,k-1}(q, q) - \bar{L}_{n,k-1}(q, sq) \left(\frac{1}{1-s} + \frac{z_{n,k}}{1 - z_{n,k}} \right), \tag{39}$$

where $z_{n,k} = s^{n-k+1} q^{\binom{n+1}{2} - \binom{n-k+2}{2}}$. One would hope to prove (29) now by finding a closed form for $\bar{L}_{n,k}(q, s)$, proving that it satisfies the recurrence (39) and then setting $s = q$ to get (29). Since we were unable to guess $\bar{L}_{n,k}(q, s)$, we proceed as for anti-lecture hall compositions to iterate the recurrence (39) and get

$$\bar{L}_{n,k}(q, s) = \sum_{j \geq 1} (-1)^{j-1} \frac{sq^{j-1}}{(s;q)_j} \cdot \frac{1 - s^{n-k+j} q^{(n-k+j)(j-2) + \binom{n+1}{2} - \binom{n-k+j}{2}}}{1 - s^{n-k+1} q^{\binom{n+1}{2} - \binom{n-k+2}{2}}} \cdot \bar{L}_{n,k-j}(q, q).$$

Now setting $s = q$ we need only a single argument:

$$\bar{L}_{n,k}(q) = \sum_{j \geq 1} (-1)^{j-1} \frac{q^j}{(q;q)_j} \cdot \frac{1 - q^{k(n-k+j) + \binom{k-j+1}{2}}}{1 - q^{\binom{n+1}{2} - \binom{n-k+1}{2}}} \cdot \bar{L}_{n,k-j}(q).$$

It remains to prove that this recurrence is satisfied by (29). We defer the details until a later report; our main point was to show that strategic application of the guidelines reduce the truncated lecture hall theorem to a q-series computation.

7. The Five Guidelines Suffice

Let $\mathcal{C}$ be the set of inequalities

$$c_{i,0} + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots + c_{i,n}\lambda_n \geq 0, \quad 1 \leq i \leq r. \tag{40}$$

and let $S_{\mathcal{C}}$ be the set of of nonnegative integer sequences satisfying all constraints in $\mathcal{C}$. In this section we show that the five guidelines of Theorem 1 are powerful enough to find the generating function of $S_{\mathcal{C}}$ for any integers $c_{i,j}$. We will assume that all constraints are *homogeneous*, i.e., that $c_{i,0} = 0$. Otherwise, introduce a new variable λ_0 and let $\mathcal{C}'$ be the same as $\mathcal{C}$, except that for every i, the ith constraint is now:

$$c_{i,0}\lambda_0 + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots + c_{i,n}\lambda_n \geq 0.$$

Then $F_{\mathcal{C}}(x_1, \ldots x_n)$ is the coefficient of x_0 in $F_{\mathcal{C}'}(x_0, x_1, \ldots x_n)$. We also generalize the claim a bit to allow any of the constraints of $\mathcal{C}$ to be equalities.

Theorem 2 *The five guidelines of Theorem 1 are sufficient to find the full generating function for any homogeneous system of linear inequalities and equalities.*

Proof. Let $\mathcal{C}$ be a homogeneous system of linear inequalities and equalities with variables $\lambda_1, \ldots, \lambda_n$. Since we require nonnegative integer solutions, we can assume that for each variable λ_i, $\mathcal{C}$ contains a constraint b_i of the form $[\lambda_i \geq 0]$ or $[\lambda_i = 0]$. Call these constraints b_i *basic*. Write $\mathcal{C}$ in the form

$$\mathcal{C} = [c_1, c_2, \ldots, c_r; b_1, b_2, \ldots, b_n],$$

where $c_1, c_2, \ldots, c_r$ is an ordered list of the non-basic constraints in $\mathcal{C}$. If $r = 0$, all constraints are basic and the generating function follows from guidelines 1 and 2 (and $F_{[\lambda_i=0]}(x_i) = 1$).

Otherwise, define:

M: the largest positive coefficient of c_1 (0, if none);

e_{max}: the number of occurrences of M among the coefficients of c_1;

m: the smallest negative coefficient of c_1 (0, if none);

e_{min}: the number of occurrences of m among the coefficients of c_1.

When $r > 0$ we show that we can use the guidelines to reduce the computation of the generating function of $\mathcal{C}$ to the computation of the generating function of one or more systems $\mathcal{C}'$ in which at least one of the statistics $\{r, M, e_{max}, |m|, e_{min}\}$ has been reduced.

If $m = 0$, all coefficients of c_1 are nonnegative, so c_1 is redundant and can be deleted. Otherwise, if $M = 0$, all coefficients of c_1 are nonpositive and so we get an equivalient system replacing λ_j by 0 in $c_1, \ldots, c_r$ and setting $b_j = [\lambda_j = 0]$. In so doing we have decreased $|m|$ or e_{min}.

Otherwise, $m < 0$ and $M > 0$; we do a version of *Elliott reduction* [20]. Let i and j be such that m is the coefficient of λ_i in c_1 and M is the coefficient of λ_j. We would like to use guideline 3 and reduce to a system with smaller M or e_{max} or $|m|$ or e_{min}. First use guideline 4 with $c = [\lambda_i \geq \lambda_j]$:

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}\cup[\lambda_i \geq \lambda_j]}(X_n) + F_{\mathcal{C}\cup[\lambda_j > \lambda_i]}(X_n).$$

For the first term, $F_{\mathcal{C}\cup[\lambda_i\geq\lambda_j]}(X_n)$, do the substitution $\lambda_i \leftarrow \lambda_i+\lambda_j$ into constraints $c_1, c_2, \ldots, c_r$ in $\mathcal{C}$. This decreases the coefficient of λ_j, thereby decreasing M or e_{max}. By guideline 3, the substitution $x_j \leftarrow x_j x_i$ in the generating function of the resulting constraint system gives $F_{\mathcal{C}\cup[\lambda_i\geq\lambda_j]}(X_n)$.

For the second term, $F_{\mathcal{C}\cup[\lambda_j>\lambda_i]}(X_n)$, if $b_j = [\lambda_j = 0]$, there are no solutions and the generating function is 0. Otherwise, $b_j = [\lambda_j \geq 0]$. Substitute $\lambda_j \leftarrow \lambda_j + \lambda_i$ into constraints $c_1, c_2, \ldots, c_r$ in $\mathcal{C}$ to get $\mathcal{C}'$. This increases the coefficient of λ_i, thereby decreasing $|m|$ or e_{min}. Substituting $\lambda_j \leftarrow \lambda_j + \lambda_i$ into $[\lambda_j > \lambda_i]$ gives $[\lambda_j > 0]$. By guideline 3,

$$F_{\mathcal{C}\cup[\lambda_j>\lambda_i]}(X_n) = F_{\mathcal{C}'\cup[\lambda_j>0]}(X_n; x_i \leftarrow x_i x_j).$$

However, we disallow strict inequalities. So, use guideline 5 with $c = [\lambda_j > 0]$ and observe that $b_j = [\lambda_j \geq 0] \in \mathcal{C}'$. Let $\mathcal{C}''$ denote $\mathcal{C}'$ with $b_j \leftarrow [\lambda_j = 0]$. Then

$$F_{\mathcal{C}'\cup[\lambda_j>0]}(X_n) = F_{\mathcal{C}'}(X_n) - F_{\mathcal{C}'\cup[\lambda_j\leq 0]}(X_n) = F_{\mathcal{C}'}(X_n) - F_{\mathcal{C}''}(X_n).$$

□

(Note that we have optimized the proof for simplicity at the expense of algorithmic efficiency.)

It follows from Theorem 2 and its proof that the full generating function of (40) can be built up from the functions $1/(1 - x_i)$ by a finite number of additions, subtractions, and substitutions. We get then as a corollary the following well-known result: The full generating function for the nonnegative integer solutions to any system of linear inequalities in n variables with integer coefficients has the form

$$\frac{p(x_1, \ldots, x_n)}{(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_t)},$$

where $t \geq 0$, p is a polynomial in $x_1, \ldots, x_n$ and each α_i is a monomial in $x_1, \ldots, x_n$.

8. Relationship to MacMahon's Partition Analysis

We give a brief introduction to partition analysis in order to highlight the fact that the guidelines of Theorem 1 underlie the work of MacMahon. Indeed, they were distilled from partition analysis by a study of MacMahon's work in [25] and its application by Andrews, Paule and Riese in the series of papers [2, 3, 6, 4, 12, 7, 8, 9, 5, 10, 11].

Consider the set of constraints $\mathcal{C} = \{c_1, \ldots c_r\}$ where

$$c_i = [a_{i,1}\lambda_1 + \cdots + a_{i,n}\lambda_n \geq 0].$$

We seek the full generating function for the set $S_{\mathcal{C}}$ of nonnegative integer sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying c_i, $1 \leq i \leq r$:

$$F_{\mathcal{C}}(x_1, \ldots, x_n) = \sum_{\lambda\in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

The method of *partition analysis*, developed by MacMahon in [25] is to view the problem as follows. Let

$$P(x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_n) = \prod_{j=1}^{n} \frac{1}{1 - x_j c_1^{a_{1,j}} c_2^{a_{2,j}} \cdots c_r^{a_{r,j}}}. \tag{41}$$

Expanding P gives

$$\begin{aligned} P(x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_n) &= \sum_{\lambda_1, \ldots \lambda_n \geq 0} \prod_{j=1}^{n} (x_j c_1^{a_{1,j}} c_2^{a_{2,j}} \cdots c_r^{a_{r,j}})^{\lambda_j} \\ &= \sum_{\lambda_1, \ldots \lambda_n \geq 0} \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \prod_{i=1}^{r} c_i^{a_{i,1}\lambda_1 + \cdots a_{i,n}\lambda_n} \right). \end{aligned} \tag{42}$$

Observe that $\lambda \in S_{\mathcal{C}}$ iff in the term corresponding to λ in the sum (42) every c_i, $1 \leq i \leq r$, has nonnegative exponent. Thus $F_{\mathcal{C}}(x_1, \ldots, x_n)$ is recovered from $P(x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_n)$ by deleting all terms in which some c_i has a negative exponent and then setting $c_1 = c_2 = \cdots = c_r = 1$. MacMahon uses the *Omega operator* to express this process:

$$F_{\mathcal{C}}(x_1, \ldots, x_n) = \underset{\geq}{\Omega} \; P(x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_n).$$

The core of partition analysis is a system of *Omega*-rules designed to be applied strategically to transform $P(x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_n)$ step-by-step into $F_{\mathcal{C}}(x_1, \ldots, x_n)$. This view converts the combinatorial problem into an algebraic one, opening the possibility, for example, of a partial fraction decomposition of (41) to assist in the transformation from P to F. A list of basic Omega-rules appears in [25](pp. 103-106) and [2].

This approach has proven both powerful and systematic in the computer solution of systems of inequalities. However, for deriving recurrences for infinite families, we found that a return to some of the basic underlying ideas simplified the process. We note the roots of guidelines 3-5 of Theorem 1 in the work of MacMahon [25].

Our use of guideline 3 (which performs limited column operations on the constraint matrix) is used to much the same effect as the following Omega-rule:

$$\underset{\geq}{\Omega} \; \frac{1}{(1 - x_i c)(1 - \frac{x_j}{c^a})} = \frac{1}{(1 - x_i)(1 - x_j x_i^a)}.$$

The utility of guideline 4 was recognized by MacMahon. He writes in [25], p. 103, "A very useful principle is that of adding an inequality which is afortiori true." It is also used in a decomposition shown at the beginning of [25], Section 379, p. 131. Guideline 5 *is* one of MacMahon's Omega-rules, found in [25], Section 351, p. 104 (slightly transformed):

$$\underset{\geq}{\Omega} \; P(c) = P(1) - \underset{\geq}{\Omega} \; P(1/c).$$

9. Concluding Remarks

The "five guidelines" of Theorem 1 provide a unified setting for computing the full generating function for many challenging families of constraints. However, even though they are guaranteed to be sufficient to find the generating function for any homogeneous linear system, we are not necessarily guaranteed to be able to use them to devise a *recurrence* for a parametrized family of constraint sets.

In continuing work we consider the case when all constraints have the form $\lambda_i \geq \lambda_j$ or $\lambda_i > \lambda_j$, forming a directed graph. We show how to get a recurrence by strategically manipulating the diagrams. Many examples are presented, including two- and three-rowed plane partitions, plane partitions with diagonals, plane partition diamonds, and hexagonal plane partitions.

Finally, we note that in [32], Xin offers a speed-up to the Omega package for implementing MacMahon's partition analysis. Xin's method uses the theory of iterated Laurent series and partial fraction decompositions.

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