FRAENKEL'S PARTITION AND BROWN'S DECOMPOSITION

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Abstract

We give short proofs of Fraenkel's Partition Theorem and Brown's Decomposition. Denote the sequence $(\lfloor (n-\alpha')/\alpha \rfloor)_{n=1}^{\infty}$ by $\mathcal{B}(\alpha,\alpha')$, a so-called Beatty sequence. Fraenkel's Partition Theorem gives necessary and sufficient conditions for $\mathcal{B}(\alpha,\alpha')$ and $\mathcal{B}(\beta,\beta')$ to tile the positive integers, i.e., for $\mathcal{B}(\alpha,\alpha')\cap\mathcal{B}(\beta,\beta')=\emptyset$ and $\mathcal{B}(\alpha,\alpha')\cup\mathcal{B}(\beta,\beta')=\mathbb{N}$. Fix $\alpha\in(0,1)$, and let $c_k=1$ if $k\in\mathcal{B}(\alpha,0)$, and $c_k=0$ otherwise, i.e., $c_k=\lfloor (k+1)\alpha\rfloor-\lfloor k\alpha\rfloor$. For a positive integer m let C_m be the binary word $c_1c_2c_3\cdots c_m$. Brown's Decomposition gives integers $q_1,q_2,\ldots,$ *independent of m and growing at least exponentially*, and integers $t,z_0,z_1,z_2,\ldots,z_t$ (depending on m) such that $C_m=C_{q_t}^{z_t}C_{q_{t-1}}^{z_{t-1}}\cdots C_{q_1}^{z_1}C_{q_0}^{z_0}$. In other words, Brown's Decomposition gives a sparse set of initial segments of C_∞ and an explicit decomposition of C_m (for every m) into a product of these initial segments.

1 Introduction

In 1894, Rayleigh [10] observed that

> "If x be an incommensurable number less than unity, one of the series of quantities $m/x, m/(1-x)$, where m is a whole number, can be found which shall lie between any given consecutive integers, and but one such quantity can be found."

S. Beatty [2] posed this as a Monthly problem in 1926, and it has come to be known as Beatty's Theorem.

The Beatty sequence with density α and offset α' is defined by

$$\mathcal{B}(\alpha,\alpha') := \left(\left\lfloor \frac{n-\alpha'}{\alpha}\right\rfloor\right)_{n=1}^{\infty}, \tag{1}$$

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where $\lfloor x \rfloor$ is the floor of x. When the second argument is 0 we omit it from our notation, i.e., $\mathcal{B}(\alpha) := \mathcal{B}(\alpha, 0)$. We write $\{x\} := x - \lfloor x \rfloor$ for the fractional part of x, and $\lceil x \rceil$ for the smallest integer larger than or equal to x. We say that two sequences *tile* a set S if they are disjoint and their union is S.

For example, we now state Beatty's Theorem in this language.

Beatty's Theorem: *The sequences $\mathcal{B}(\alpha)$ and $\mathcal{B}(\beta)$ tile $\mathbb{N}$ if and only if $0 < \alpha < 1$, $\alpha + \beta = 1$, and α is irrational.*

Beatty sequences arise in a number of areas, including Computer Graphics, Signal Processing, Automata, Quasicrystals, Combinatorial Games, and Diophantine Approximation. They are natural counterparts to Kronecker's fractional part sequences. There is the obvious connection of $\mathcal{B}(\alpha, \alpha')$ to $\left(\left\{\frac{n-\alpha'}{\alpha}\right\}\right)_{n=1}^{\infty}$, but also a more subtle connection to $(\{n\alpha + \alpha'\})_{n=1}^{\infty}$ which we exploit here to give simple proofs of Fraenkel's Partition and Brown's Decomposition.

Beatty sequences generalize arithmetic progressions, which correspond to the special case $\alpha^{-1} \in \mathbb{Z}$. Most work on Beatty sequences has the aim of extending some known result on APs. For example, the Chinese Remainder Theorem identifies precisely when APs are disjoint; Fraenkel's Partition (stated precisely in Section 2) identifies precisely when two Beatty sequences are disjoint and have union $\mathbb{N}$. The situation for more than 2 sequences is inadequately understood; see [15] and [14] for an up-to-date survey of knowledge in that direction.

Fraenkel proved his elegant generalization of Beatty's Theorem in 1969. Although his argument is well motivated and geometric, it is rather long and hampered by unfortunate notation. Skolem [12] attempted to deal with the special case of Beatty sequences with irrational densities. Alas, both his statement of the theorem and his proof are incorrect (this is discussed in [5]). Borwein & Borwein [3] give "a new proof of a theorem of Fraenkel". They write, "Our proof, once we have developed the other machinery of this paper, is considerably shorter." Their proof is indeed short and the "other machinery" consists only of two straightforward functional equations relating a pair of generating functions. Unfortunately, the "theorem of Fraenkel" that they prove is a *very* special case of Fraenkel's Partition. In Section 2, we give the statements put forward by Skolem, Fraenkel, and Borwein & Borwein.

Skolem's work is incorrect, Fraenkel's proof is long, and the Borwein brothers shied away from proving the full theorem. For these reasons, the author feels that there is room in the literature for another proof of Fraenkel's Partition, provided that it is correct, short, and complete. We state Fraenkel's Partition in Section 2, define some relevant notation in Section 4, and give the proof in Section 5. Because of the theorem's history of incorrect proofs and inadequate statements, we have perhaps erred on the side of including too much detail. To ease the work of a casual reader, the main ideas are given in Section 5.1.

We now introduce Brown's Decomposition. Fix an $\alpha \in (0,1)$, and set

$$c_k := \begin{cases} 1, & k \in \mathcal{B}(\alpha); \\ 0, & \text{otherwise}, \end{cases}$$

and let C_m be the binary word $c_1 c_2 \dots c_m$. The word C_∞ is called the characteristic word with density α. If $\alpha^{-1} = q \in \mathbb{N}$, then $c_k = 1$ if and only if $k \equiv \lfloor -q\alpha' \rfloor \pmod q$. In particular, $c_k = c_{q+k}$ for every k, whence $C_m = C_q^{\lfloor m/q \rfloor} C_{m-q\lfloor m/q \rfloor}$. In fact, if $\alpha = \frac{a}{q}$, then the sequence $c_1, c_2, \dots$ is periodic with period q and so $C_m = C_q^{\lfloor m/q \rfloor} C_{m-q\lfloor m/q \rfloor}$.

If α is irrational, then the sequence $c_1, c_2, \dots$ is not periodic. But α is near to rationals, and so an initial segment of the sequence will appear to be periodic. Brown's Decomposition (stated precisely in Section 3) is a quantitative description of this, given in terms of the convergents of the continued fraction of α.

We comment that this 'almost-periodicity' is precisely what makes Beatty sequences interesting to quasicrystallographers. If $\alpha \notin \mathbb{Q}$, then C_∞ is the prototypical example of a Sturmian Word. Much of the literature on Beatty sequences is couched in the equivalent (and often more convenient) language of Sturmian Words, especially the literature analyzing quasicrystallographic properties.

The first steps toward Brown's Decomposition were made in the oft-cited work of Stolarsky [13]. He was studying functions h for which $c_1 c_2 \cdots = h(c_1) h(c_2) \cdots$. A nontrivial example is worth a thousand words: set $\alpha = \frac{\sqrt{5}-1}{2}$, in which case

$$c_1 c_2 c_3 \cdots = 101\,101\,011\,011\,010 \cdots$$

is the "Fibonacci word". With h defined by $h(1) = 10, h(0) = 1$ we have

$$\begin{aligned} h(c_1)h(c_2)h(c_3)\cdots &= h(1)h(0)h(1)\cdots \\ &= 10\,1\,10\cdots \\ &= c_1 c_2 c_3 c_4 c_5 \cdots . \end{aligned}$$

In the early 1990s, T. C. Brown found the useful and succinct decomposition of C_m using morphisms. His proof is nicely exposited in the new book of Allouche & Shallit [1]. We remark that the properties of morphisms received a great deal of attention throughout the 1990s. Excellent accounts of the current theory are given in both [1] and in the recent book of Lothaire [8, Chapter 2].

Here, we give a short direct proof of Brown's Decomposition. Our proof relies on the same characterization of $\mathcal{B}(\alpha)$ in terms of the fractional part sequence $\{k\alpha\}$ that we use in our proof of Fraenkel's Partition. We also need a well-known theorem from continued fractions (which is also used in Brown's proof). The Decomposition is stated precisely in Section 3, some notation is introduced in Section 4, and the proof is given in Section 6.

2 Statement of Fraenkel's Partition

Fraenkel's Partition Theorem: *The sequences $\mathcal{B}(\alpha, \alpha')$ and $\mathcal{B}(\beta, \beta')$ tile $\mathbb{N}$ if and only if the following five conditions are satisfied.*

1. $0 < \alpha < 1$.
2. $\alpha + \beta = 1$.
3. $0 \leq \alpha + \alpha' \leq 1$.
4. *If α is irrational, then $\alpha' + \beta' = 0$ and $k\alpha + \alpha' \notin \mathbb{Z}$ for $2 \leq k \in \mathbb{N}$.*
5. *If α is rational (say $q \in \mathbb{N}$ is minimal with $q\alpha \in \mathbb{N}$), then $\frac{1}{q} \leq \alpha + \alpha'$ and $\lceil q\alpha' \rceil + \lceil q\beta' \rceil = 1$.*

We note first Conditions *1–5* are symmetric in α and β. At first glance this is not the case; for example, it is not clear that Conditions *1–5* imply that $0 \leq \beta + \beta' \leq 1$. Our proof of Fraenkel's Partition begins by proving the claimed symmetry.

Note also that if one divides the equation in Condition *5* by q, and then takes the limit as $q \to \infty$, one obtains the equation $\alpha' + \beta' = 0$ of Condition *4*. This hints that the irrational case can be derived as a limit of the rational case. However, the presence of the additional clause "$k\alpha + \alpha' \notin \mathbb{Z}$" of Condition *4* indicates that this approach is not trivial. The proof given here handles the two cases separately. In Section 5.5, we derive the irrational case from the rational case using nonstandard analysis.

If one wishes to consider tilings of $\{N+1, N+2, \dots\}$ instead of $\{1, 2, \dots\}$, it is not difficult to adapt our statement. Indeed, $\mathcal{B}(\alpha, \alpha')$ and $\mathcal{B}(\beta, \beta')$ tile $\mathbb{N}$ if and only if $\mathcal{B}(\alpha, \alpha' - N\alpha)$ and $\mathcal{B}(\beta, \beta' - N\beta)$ tile $\{N+1, N+2, \dots\}$. A similar adjustment allows one to easily change the range of n in the definition of $\mathcal{B}$.

Skolem [12] stated (incorrectly) that if α, β are positive irrationals, then $\mathcal{B}(\alpha, \alpha')$ and $\mathcal{B}(\beta, \beta')$ tile $\{n \in \mathbb{Z} \colon n \geq \min\{\lfloor \frac{1-\alpha'}{\alpha} \rfloor, \lfloor \frac{1-\beta'}{\beta} \rfloor\}\}$ if and only if $\alpha + \beta = 1$ and $\alpha' + \beta' \in \mathbb{Z}$. Borwein & Borwein [3] assume that $0 < \alpha < 1$, α irrational, $0 < \alpha + \alpha' \leq \alpha$, and $\frac{n-\alpha'}{\alpha}$ is never integral. Under these hypotheses, they prove (correctly) that $\mathcal{B}(\alpha, \alpha')$ and $\mathcal{B}(\beta, \beta')$ tile $\mathbb{N}$ if and only if $\alpha + \alpha' = 1$ and $\alpha' + \beta' = 0$.

Fraenkel's statement (using a *simplified* form of his notation) is as follows. Let α and β be positive real numbers, either both rational or both irrational, and let γ and δ be arbitrary real numbers. Let S and T be the sets of integers of the form $\phi_n = [n\alpha + \gamma]$ and $\psi_n = [n\beta + \delta]$, respectively, where n ranges over $\mathbb{N}$. Further, assume that $\phi_1 \leq \psi_1$. If α, β are irrational, then S and T tile $\{n : n \geq \phi_1\}$ if and only if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_1 - 1$, and

$$n\beta + \delta = K, \quad n, K \text{ integral implies } n < 1.$$

If α, β are rational, then S and T tile $\{n : n \geq \phi_1\}$ if and only if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_1 - 1 - a^{-1} + \eta + \rho,$$

where $\alpha = \frac{a}{c}$, $\gcd(a, c) = 1$, $\frac{\gamma}{\alpha} \equiv \eta \pmod{a^{-1}}$, $0 \leq \eta < a^{-1}$, $\beta = \frac{b}{d}$, $\gcd(b, d) = 1$, $\frac{\delta}{\beta} \equiv \rho \pmod{b^{-1}}$, $0 \leq \rho < b^{-1}$. It is remarkable how much simplification we purchase by considering $\left\lfloor \frac{n-\alpha'}{\alpha} \right\rfloor$ in place of $\lfloor n\alpha + \gamma \rfloor$. Our less-obvious definition leads to a somewhat simpler statement, and a much simpler proof. (Note: our additional restriction to $0 \leq \alpha + \alpha' \leq 1$ in the irrational case and $\frac{1}{q} \leq \alpha + \alpha' \leq 1$ in the rational case correspond to insisting that $\phi_1 = 1$.)

3 Statement of Brown's Decomposition

Before stating Brown's Decomposition, we must define the *continued fraction expansion* of a natural number. Let $[0; a_1, a_2, \ldots]$ be the continued fraction expansion of α, and denote the continuants (the denominators of the convergents to α) by $q_0, q_1, \ldots$, i.e., $q_0 := 1$, $q_1 := a_1$, and $q_i := a_i q_{i-1} + q_{i-2}$. For a positive integer m, define $z_0, z_1, \ldots$ by writing m greedily as a sum of q_i (that is, always use the largest q_i possible):

$$m = z_t q_t + z_{t-1} q_{t-1} + \ldots z_1 q_1 + z_0 q_0.$$

We call $(z_t z_{t-1} \cdots z_1 z_0)_\alpha$ the continued fraction expansion of m with respect to α. The standard reference for this and other systems of numeration is [6].

We can now state Brown's Decomposition.

Brown's Decomposition: *Let* $\alpha = [0; a_1, a_2, \ldots]$ *have continuants* $q_0, q_1, q_2, \ldots,$ *and let* $m = (z_t \cdots z_1 z_0)_\alpha$. *Then for* $i \geq 2$

$$C_{q_i} = C_{q_{i-1}}^{a_i} C_{q_{i-2}} \quad \textit{and} \quad C_m = C_{q_t}^{z_t} C_{q_{t-1}}^{z_{t-1}} \cdots C_{q_1}^{z_1} C_{q_0}^{z_0}.$$

An avid reader may enjoy proving that $z_0, z_1, \ldots$ is the unique sequence of nonnegative integers such that $m = \sum_{i \geq 0} z_i q_i$, $0 \leq z_i \leq a_i$, and $(z_i = a_i) \Rightarrow (i \geq 1$ and $z_{i-1} = 0)$. We remark that this is sometimes referred to as the Ostrowski expansion, and sometimes as the Zeckendorf expansion, especially when $q_0, q_1, \ldots$ are Fibonacci numbers.

4 Preliminaries

Much of our work will take place in $\mathbb{R}/\mathbb{Z}$. While there is no natural linear ordering of $\mathbb{R}/\mathbb{Z}$, there is a natural 'ternary' order: we say that real numbers w, x, y are *in order* if there is a nondecreasing function $f : [0,1] \to \mathbb{R}$ with $\{f(0)\} = \{w\}$, $\left\{f(\frac{1}{2})\right\} = \{x\}$, $\{f(1)\} = \{y\}$, and $f(1) - f(0) < 1$.

This definition is precise but awkward; there is a geometric description that is conceptually simpler. Let $\tau : \mathbb{R} \to \mathbb{C}$ be defined by $\tau(z) = e^{2\pi i z}$. The range of τ is the circle $D := \{z \colon |z| = 1\}$, and the group $(D, \cdot)$ is isomorphic to $\mathbb{R}/\mathbb{Z}$ (in fact, τ is one isomorphism). We say that w, x, y are in order if $\tau(x)$ is on the counter-clockwise arc from $\tau(w)$ to $\tau(y)$. We write $x \equiv y$ when $\tau(x) = \tau(y)$, i.e., when $x - y \in \mathbb{Z}$.

We define the arcs $\overline{(w,y)}, \overline{(w,y]}$, and $\overline{[w,y)}$ to be $\emptyset$ if $w \equiv y$, and otherwise define the arcs through

$$\begin{aligned}
\overline{(w,y)} &:= \{x \in \mathbb{R} \colon x \not\equiv w, x \not\equiv y, \text{ and } w, x, y \text{ are in order.}\} \\
\overline{(w,y]} &:= \{x \in \mathbb{R} \colon x \not\equiv w, \text{ and } w, x, y \text{ are in order.}\} \\
\overline{[w,y)} &:= \{x \in \mathbb{R} \colon x \not\equiv y, \text{ and } w, x, y \text{ are in order.}\}
\end{aligned}$$

Our proofs of both Fraenkel's Theorem and Brown's Decomposition rely heavily on the following lemma.

Lemma 1. *Let k be an integer, and $0 < \alpha < 1$. Then $k \in \mathcal{B}(\alpha, \alpha')$ if and only if*

$$(k\alpha + \alpha' > 0) \quad \texttt{AND} \quad \left(k\alpha \in \overline{(-\alpha - \alpha', -\alpha']}\right).$$

Proof.

$$\begin{aligned}
k \in \mathcal{B}(\alpha, \alpha') &\Leftrightarrow \exists n \in \mathbb{N} \left(k \le \tfrac{n-\alpha'}{\alpha} < k+1\right) \\
&\Leftrightarrow \exists n \in \mathbb{N} \left(k\alpha + \alpha' \le n < (k+1)\alpha + \alpha'\right) \\
&\Leftrightarrow (k\alpha + \alpha' > 0) \quad \texttt{AND} \quad \left((k+1)\alpha + \alpha' \in \overline{(0, \alpha]}\right) \\
&\Leftrightarrow (k\alpha + \alpha' > 0) \quad \texttt{AND} \quad \left(k\alpha \in \overline{(-\alpha - \alpha', -\alpha']}\right)
\end{aligned}$$

∎

5 Proof of Fraenkel's Theorem

5.1 Spirit of Proof

In this section we prove a theorem whose statement and proof are similar to Fraenkel's Theorem, but for which the technical details are considerably reduced. We note that Fraenkel [5] also proved this result. Set

$$S_\alpha := \left(\left\lfloor \tfrac{n-\alpha'}{\alpha} \right\rfloor\right)_{n=-\infty}^{\infty} \quad \text{and} \quad S_\beta := \left(\left\lfloor \tfrac{n-\beta'}{\beta} \right\rfloor\right)_{n=-\infty}^{\infty}.$$

Theorem 2. *Let α, β be positive irrationals. The sequences S_α and S_β tile $\mathbb{Z}$ if and only if $\alpha+\beta=1$, $\alpha'+\beta' \in \mathbb{Z}$, and $k\alpha+\alpha'$ is never an integer (for $k \in \mathbb{Z}$).*

The additional difficulty of proving Fraenkel's Partition is in dealing with the 'edge effects' introduced by restricting the index n in the definition of S_α and S_β to $n \geq 1$, and in dealing with rational α. The proof of Fraenkel's Theorem given below is self-contained; this subsection is included only to give the look-and-feel of our approach.

Proof. First, we note that the density of S_α is α, and that of S_β is β; it is clearly necessary that $\alpha+\beta=1$. From this point forward, we assume that $\alpha+\beta=1$.

Next, observe that $k \in S_\alpha$ exactly if there is an $n \in \mathbb{Z}$ with $k \leq \frac{n-\alpha'}{\alpha} < k+1$, which is the same as

$$k\alpha+\alpha' \leq n < k\alpha+\alpha'+\alpha.$$

This, in turn, is the same as $k\alpha+\alpha' \in \overline{(-\alpha,0]}$. Thus (arguing identically for β),

$$S_\alpha = \left\{k \colon k\alpha \in \overline{(-\alpha-\alpha',-\alpha']}\right\} \quad \text{and} \quad S_\beta = \left\{k \colon k\beta \in \overline{(-\beta-\beta',-\beta']}\right\}.$$

But $k\beta = k(1-\alpha) \equiv -k\alpha$, so that S_β is also given by

$$S_\beta = \left\{k \colon k\alpha \in \overline{[\beta',\beta+\beta')}\right\}.$$

Thus, $k \in S_\alpha$ if $k\alpha \in \overline{(-\alpha-\alpha',-\alpha']} =: A$ and $k \in S_\beta$ if $k\alpha \in \overline{[\beta',\beta+\beta')} =: B$. Since $(\{k\alpha\})_{k=-\infty}^{\infty}$ is dense in $[0,1)$, if A and B intersect in an arc, there are infinitely many k in both S_α and S_β; and if A and B both omit some arc, there are infinitely many k in neither S_α nor S_β. It follows that the right endpoint of A is the left endpoint of B, i.e., $-\alpha' \equiv \beta'$, which is the same as $\alpha'+\beta' \in \mathbb{Z}$. The only point in $A \cap B$ is $-\alpha' \equiv \beta$, so that if $k\alpha \equiv -\alpha' \equiv \beta$, then k is in both S_α and S_β. This happens if and only if $k\alpha+\alpha'$ is an integer. ∎

5.2 Without loss of generality ...

5.2.1 Fraenkel's Partition is symmetric in α and β.

We first note that the Theorem is symmetric in α and β. Obviously, "The sequences $\mathcal{B}(\alpha,\alpha')$ and $\mathcal{B}(\beta,\beta')$ tile $\mathbb{N}$" has the claimed symmetry. Combining Conditions *2* and *3*, we find $0<\beta<1$, the symmetric counterpart to Condition *1*. Condition *2* is symmetric as stated. We return to Condition *3* in the next paragraph. In Condition *4*, the equation $\alpha'+\beta'=0$ is already symmetric; and "$k\alpha+\alpha' \notin \mathbb{Z}$" implies that $k\beta+\beta' = k(1-\alpha)+(-\alpha) = k-(k\alpha+\alpha') \notin \mathbb{Z}$ (using Condition *2* and $\alpha'+\beta'=0$). Thus Condition *4* implies its symmetric counterpart. If α is rational, then $\beta = 1-\alpha$ (from Condition *2*) is also, and moreover both α and β have the same denominator. Thus Condition *5* implies its symmetric counterpart also.

If α is irrational, then $\alpha = 1-\beta$ (from Condition *2*) and $\alpha' = -\beta'$ (from Condition *4*), so $0 \le \alpha + \alpha' \le 1$ (from Condition *3*) implies $0 \le \beta + \beta' \le 1$, the symmetric twin of Condition *3*. Now suppose that α is rational, say $q\alpha \in \mathbb{N}$. Then the inequalities $\frac{1}{q} \le \alpha + \alpha' \le 1$ (from Conditions *3* and *5*) yield $1 \le q\alpha + q\alpha' \le q$, and since $1, q\alpha, q$ are all integers, this yields $1 \le q\alpha + \lceil q\alpha' \rceil \le q$. Now from Condition *2*, $\alpha = 1 - \beta$, and from Condition *5*, $\lceil q\alpha' \rceil = 1 - \lceil q\beta' \rceil$, so the inequalities imply $1 \le q(1-\beta) + 1 - \lceil q\beta' \rceil \le q$, which simplifies to $1 \le q\beta + \lceil q\beta' \rceil \le q$. Now, since $q\beta = q - q\alpha \in \mathbb{N}$, and $1, q$ are integers also, the inequalities become $1 \le q\beta + q\beta' \le q$, or more simple $\frac{1}{q} \le \beta + \beta' \le 1$. This gives the symmetric counterpart to Condition *3* (since $\frac{1}{q} > 0$) and to Condition *5*.

5.2.2 If α is rational, then $\alpha, \alpha', \beta, \beta'$ are all rational with the same denominator.

We now observe that if α is rational (with denominator q), we can assume without loss of generality that α' is also rational with denominator q. For $0 \le \alpha + \alpha' \le 1$ if and only if $0 \le \alpha + \frac{\lceil q\alpha' \rceil}{q} \le 1$ and clearly $\left\lceil q\frac{\lceil q\alpha' \rceil}{q} \right\rceil + \left\lceil q\frac{\lceil q\alpha' \rceil}{q} \right\rceil = \lceil q\alpha' \rceil + \lceil q\beta' \rceil$, and so the Conditions *1*–*5* are unaffected by replacing α' with $\frac{\lceil q\alpha' \rceil}{q}$. By Lemma 3 below, the sequence $\mathcal{B}(\alpha, \alpha')$ is also unaffected. Thus, we assume from this point on that if α is rational with denominator q, then so is α'. Further, if β is rational with denominator q, we assume that β' is too. The equation in Condition *5* now simplifies to $\alpha' + \beta' = \frac{1}{q}$.

When α, β are known to be positive rationals with $\alpha + \beta = 1$, we define natural numbers q, a, a', b, b' by the conditions

$$\alpha = \frac{a}{q}, \quad \gcd(a, q) = 1, \quad \alpha' = \frac{a'}{q}, \quad \beta = \frac{b}{q}, \quad \beta' = \frac{b'}{q}.$$

Lemma 3. *For any $a, q \in \mathbb{N}$ and $\alpha' \in \mathbb{R}$, $\mathcal{B}(\frac{a}{q}, \alpha') = \mathcal{B}(\frac{a}{q}, \frac{\lceil q\alpha' \rceil}{q})$.*

Proof. Set $\alpha = \frac{a}{q}$. We have $k\alpha + \alpha' > 0 \Leftrightarrow ka + q\alpha' > 0$, and since $ka \in \mathbb{Z}$, we have $ka + q\alpha' > 0 \Leftrightarrow ka + \lceil q\alpha' \rceil > 0 \Leftrightarrow k\alpha + \frac{\lceil q\alpha' \rceil}{q} > 0$.

Now $k\alpha$ is rational with denominator q, and there are no such numbers in $\overline{(-\frac{\lceil q\alpha' \rceil}{q}, -\alpha']}$ or in $\overline{(-\alpha - \frac{\lceil q\alpha' \rceil}{q}, -\alpha - \alpha']}$. Thus

$$k\alpha \in \overline{(-\alpha, -\alpha - \alpha']} \Leftrightarrow k\alpha \in \overline{(-\alpha - \frac{\lceil q\alpha' \rceil}{q}, -\frac{\lceil q\alpha' \rceil}{q}]}.$$

Apply Lemma 1 to finish the proof. ∎

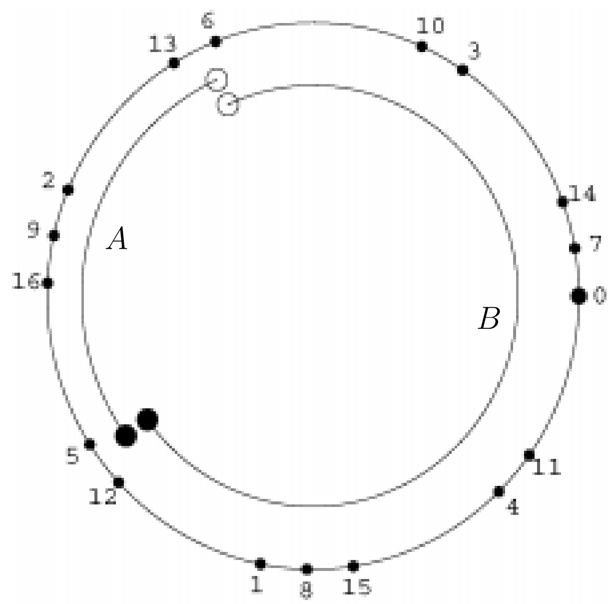

Figure 1: The partition of $\mathbb{N}$ into $\mathcal{B}(3-e, \frac{2}{5}) = \{2, 5, 9, 13, 16, \ldots\}$ and $\mathcal{B}(e-2, -\frac{2}{5}) = \{1, 3, 4, 6, 7, 8, \ldots\}$.

5.2.3 The sets A and B are important.

We define

$$A := \overline{(-\alpha - \alpha', -\alpha']} \text{ and } B := \overline{[\beta', \beta + \beta')}.$$

Lemma 4. *Suppose $\alpha + \beta = 1$. Then $k \in \mathcal{B}(\alpha, \alpha')$ if and only if*

$$(k\alpha + \alpha' > 0) \quad \texttt{AND} \quad (k\alpha \in A)\,.$$

Further, $k \in \mathcal{B}(\beta, \beta')$ if and only if

$$(k\beta + \beta' > 0) \quad \texttt{AND} \quad (k\alpha \in B)\,.$$

Figure 1 gives an example of Fraenkel's Partition with $\alpha = e - 2$, $\alpha' = \frac{2}{3}$, $\beta = 3 - e$ and $\beta' = -\frac{2}{3}$. Shown in the figure are the circle $|z| = 1$ and the angles that correspond to the sets A and B. Also shown are the points $\tau(k\alpha)$ $(0 \le k \le 16)$, labelled "k".

Proof. The condition for $k \in \mathcal{B}(\alpha, \alpha')$ is simply a restatement of Lemma 1. To prove the condition for $k \in \mathcal{B}(\beta, \beta')$, we must show that

$$k\beta \in \overline{(-\beta - \beta', -\beta']} \Leftrightarrow k\alpha \in \overline{[\beta', \beta + \beta')}.$$

This is obvious since $k\beta = k(1-\alpha) \equiv -k\alpha$, and from the observation that $-w, -k\alpha, -y$ are in order if and only if $y, k\alpha, w$ are too. ∎

5.3 Conditions *1–5* are Sufficient

Suppose that $\alpha, \alpha', \beta, \beta'$ satisfy Conditions *1–5*. We have $\frac{n-\alpha'}{\alpha} \geq \frac{1-\alpha'}{\alpha} \geq 1$ (using Condition *3*), so $B(\alpha, \alpha') \subseteq \mathbb{N}$, and likewise (by the symmetry proved in Section 5.2.1) $B(\beta, \beta') \subseteq \mathbb{N}$.

We now show the equivalence (for $k \in \mathbb{N}$)

$$k \in \mathcal{B}(\alpha, \alpha') \Leftrightarrow k \notin \mathcal{B}(\beta, \beta'),$$

i.e., we show that the two sequences tile $\mathbb{N}$. We break the work into four cases: $k = 1$ or $k > 1$, and $\alpha \in \mathbb{Q}$ or $\alpha \notin \mathbb{Q}$.

5.3.1 $k = 1$.

Since $\alpha \in (0, 1)$ and $\beta = 1 - \alpha \in (0, 1)$, the two sequences are strictly increasing. Since $B(\alpha, \alpha')$ and $B(\beta, \beta')$ are in $\mathbb{N}$ (as proved above), we have for $x \in \{\alpha, \beta\}$

$$1 \in B(x, x') \Leftrightarrow \frac{1 - x'}{x} < 2.$$

Specifically, we have

$$1 \in \mathcal{B}(\alpha, \alpha') \Leftrightarrow \frac{1 - \alpha'}{\alpha} < 2 \Leftrightarrow 1 < 2\alpha + \alpha' \tag{2}$$

and

$$1 \notin \mathcal{B}(\beta, \beta') \Leftrightarrow \frac{1 - \beta'}{\beta} \geq 2 \Leftrightarrow 1 \geq 2\beta + \beta' \tag{3}$$

If α is irrational then, by Condition *4*, $2\alpha + \alpha' \neq 1$ and so $1 < 2\alpha + \alpha' \Leftrightarrow 1 \leq 2\alpha + \alpha'$. By Condition *2*, $\alpha = 1 - \beta$, and by Condition *4*, $\alpha' = -\beta'$. Thus,

$$1 < 2\alpha + \alpha' \Leftrightarrow 1 \leq 2\alpha + \alpha' \Leftrightarrow 1 \geq 2\beta + \beta',$$

connecting the equivalences in Lines (2) and (3).

Now suppose that α is rational, and recall that we may assume that $\alpha, \alpha', \beta, \beta'$ are all rationals with denominator q, as per the discussion in Section 5.2.2. Using $\alpha = 1 - \beta$ and $\alpha' = \frac{1}{q} - \beta'$, we have

$$1 < 2\alpha + \alpha' \Leftrightarrow 2\beta + \beta' < 1 + \frac{1}{q} \Leftrightarrow 2\beta + \beta' \leq 1,$$

the last equivalence following from β, β' being rationals with denominator q. This connects the equivalences in Lines (2) and (3).

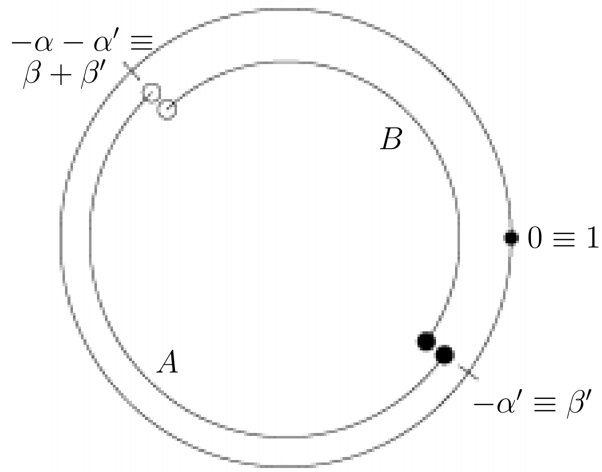

Figure 2: A and B, with α irrational

5.3.2 $k > 1$.

Suppose that $k > 1$, so that $k\alpha + \alpha' > \alpha + \alpha' \geq 0$ (by Condition *3*), and by symmetry $k\beta + \beta' > 0$.

Lemma 4 reduces to

$$k \in \mathcal{B}(\alpha, \alpha') \Leftrightarrow k\alpha \in A \quad \text{and} \quad k \in \mathcal{B}(\beta, \beta') \Leftrightarrow k\alpha \in B.$$

Thus, to prove $k \in \mathcal{B}(\alpha, \alpha') \Leftrightarrow k \notin \mathcal{B}(\beta, \beta')$ it will suffice to show that $k\alpha \notin A \cap B$ and $k\alpha \notin A^c \cap B^c$.

The next two paragraphs are accompanied by Figures 2 and 3. In these figures the point $\tau(z)$ is labelled "z". These figures show the circle $\tau(\mathbb{R})$ (the outer circle with the point $\tau(0) = \tau(1)$), and display the angles corresponding to $\tau(A)$ and $\tau(B)$ as arcs inside the circle. Also labelled are the points $\tau(-\alpha'), \tau(\beta'), \tau(-\alpha - \alpha')$, and $\tau(\beta + \beta')$.

If α is irrational (see Figure 2), then $B := \overline{[\beta', \beta + \beta')} = \overline{[-\alpha', -\alpha - \alpha')}$. We have $A \cap B = \{y \colon y \equiv -\alpha'\}$. Since $k\alpha + \alpha' \not\equiv 0$ by Condition *4*, we know that $k\alpha \notin A \cap B$. We have $A^c \cap B^c = \{y \colon y \equiv -\alpha - \alpha'\}$. If $k\alpha \in A^c \cap B^c$ then $k\alpha \equiv -\alpha - \alpha'$. Thus $(k+1)\alpha + \alpha' \in \mathbb{Z}$, but this is forbidden by Condition *4*.

If α is rational (see Figure 3), then $B := \overline{[\beta', \beta + \beta')} = \overline{[\frac{1}{q} - \alpha', \frac{1}{q} - \alpha - \alpha')}$. We have $A \cap B = \overline{(-\alpha - \alpha', \beta + \beta')} = \overline{(\frac{-a-a'}{q}, \frac{-a-a'+1}{q})}$ and $A^c \cap B^c = \overline{(-\alpha', \beta')} = \overline{(\frac{-a}{q}, \frac{-a+1}{q})}$, neither of which contain a rational with denominator q. Since $k\alpha$ is rational with denominator q, we know that $k\alpha \notin A \cap B$ and $k\alpha \notin A^c \cap B^c$.

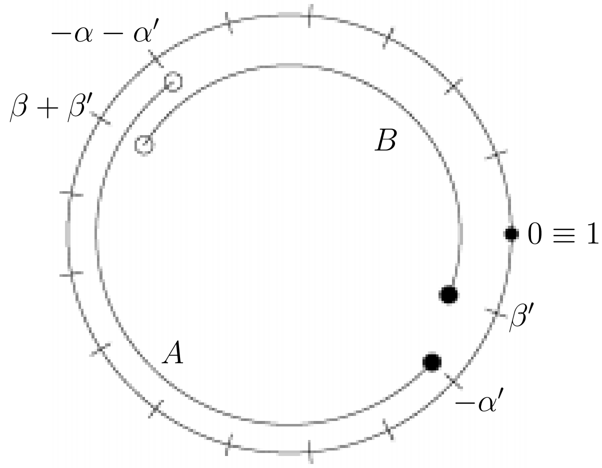

Figure 3: A and B, with α rational

5.4 Conditions *1–5* are Necessary

We now assume that $\mathcal{B}(\alpha,\alpha')$ and $\mathcal{B}(\beta,\beta')$ tile $\mathbb{N}$, and prove Conditions *1–5*.

We define $k_0 := \max\{0, -\alpha'/\alpha, -\beta'/\beta\}$. For $k > k_0$, then both $k\alpha+\alpha'$ and $k\beta+\beta'$ are positive, and this will simplify our work tremendously.

The sequences $\mathcal{B}(\alpha,\alpha'), \mathcal{B}(\beta,\beta')$ contain infinitely many positive integers, so α and β are nonnegative.

We can count the number of elements of $\mathcal{B}(\alpha,\alpha')$ which are less than an integer k:

$$\begin{aligned}|\mathcal{B}(\alpha,\alpha')\cap(-\infty,k)| &= \left|\left\{n\ge 1\colon \left\lfloor\frac{n-\alpha'}{\alpha}\right\rfloor < k\right\}\right|\\ &= \left|\left\{n\ge 1\colon \frac{n-\alpha'}{\alpha} < k\right\}\right|\\ &= \max\{0, \lceil k\alpha+\alpha'\rceil - 1\}\end{aligned}$$

and likewise

$$|\mathcal{B}(\beta,\beta')\cap(-\infty,k)| = \max\{0, \lceil k\beta+\beta'\rceil - 1\}.$$

Since $\mathcal{B}(\alpha,\alpha')$ and $\mathcal{B}(\beta,\beta')$ tile $\mathbb{N}$, we have (for $k > k_0$)

$$\begin{aligned}k-1 &= |(\mathcal{B}(\alpha,\alpha')\cup\mathcal{B}(\beta,\beta'))\cap(-\infty,k)|\\ &= |\mathcal{B}(\alpha,\alpha')\cap(-\infty,k)| + |\mathcal{B}(\beta,\beta')\cap(-\infty,k)|\\ &= \lceil k\alpha+\alpha'\rceil + \lceil k\beta+\beta'\rceil - 2. \end{aligned}\tag{4}$$

Divide by k, and let k go to infinity to find $1 = \alpha+\beta$ (Condition *2*).

As α and β are positive, and $\alpha+\beta = 1$, we find $0 < \alpha < 1$ (Condition *1*).

We will use the following short lemma several times.

Lemma 5. *If $k > k_0$, $\{k\alpha + \alpha'\} \neq 0$, and $\{k\beta + \beta'\} \neq 0$, then*

$$\{k\alpha + \alpha'\} + \{k\beta + \beta'\} = \alpha' + \beta' + 1.$$

Proof. Since $k\alpha + \alpha'$ is not an integer, we have $\lceil k\alpha + \alpha' \rceil = k\alpha + \alpha' + 1 - \{k\alpha + \alpha'\}$, and likewise $\lceil k\beta + \beta' \rceil = k\beta + \beta' + 1 - \{k\beta + \beta'\}$. Equation (4) now simplifies to finish the proof. ∎

The hypothesis that $k \in \mathcal{B}(\alpha, \alpha') \Leftrightarrow k \notin \mathcal{B}(\beta, \beta')$ and Lemma 5 imply that (for $k > k_0$)

$$k\alpha \in A \Leftrightarrow k\alpha \notin B.$$

This, in turn, is equivalent to the assertion for $k > k_0$

$$k\alpha \notin A \cap B \quad \text{and} \quad k\alpha \notin A^c \cap B^c. \tag{5}$$

5.4.1 If α is irrational ...

The set $\{\{k\alpha\} : k > k_0\}$ is dense in $[0, 1)$; Line (5) implies that there is no nontrivial interval contained in $A \cap B$, nor in $A^c \cap B^c$. Thus, the right endpoint of B is congruent (modulo 1) to the left endpoint of A, i.e., $\beta' \equiv -\alpha'$.

Suppose (by way of contradiction) that both $s_1\alpha + \alpha'$ and $s_2\alpha + \alpha'$ are integers (with $s_i \in \mathbb{Z}$). Then so is $(s_1 - s_2)\alpha = (s_1\alpha + \alpha') - (s_2\alpha + \alpha')$, contrary to our hypothesis that α is irrational. Thus, there is at most one integer s such that $s\alpha + \alpha' \in \mathbb{Z}$. Consequently, we may choose k_1 so that neither $k_1\alpha + \alpha'$ nor $k_1\beta + \beta'$ are integers. We know from the previous paragraph that $\beta' + \alpha'$ is an integer, so from Lemma 5, $\{k_1\alpha + \alpha'\} + \{k_1\beta + \beta'\}$ is an integer. By the definition of fractional part, $0 \leq \{k_1\alpha + \alpha'\} + \{k_1\beta + \beta'\} < 2$, and by the choice of k_1, $\{k_1\alpha + \alpha'\} + \{k_1\beta + \beta'\} > 0$. Thus $\{k_1\alpha + \alpha'\} + \{k_1\beta + \beta'\} = 1$, and Equation (5) reduces to $\alpha' + \beta' = 0$.

Now suppose that $k\alpha + \alpha' \in \mathbb{Z}$ (with $k \geq 2$, not necessarily larger than k_0). Note that $k\beta + \beta' = k(1 - \alpha) - \alpha' = k - (k\alpha + \alpha') \in \mathbb{Z}$ also. We have $k\alpha \equiv -\alpha' \equiv \beta' \in A \cap B$, so Lemma 5 implies that $k\alpha + \alpha' \leq 0$ or $k\beta + \beta' \leq 0$. Without loss of generality, suppose that $k\alpha + \alpha' \leq 0$. Then $(k - 1)\alpha + \alpha' \leq 0$, so by Lemma 1, $k - 1 \notin \mathcal{B}(\alpha, \alpha')$. Also $(k - 1)\alpha \equiv -\alpha - \alpha' \equiv \beta + \beta' \notin B$, so $k - 1 \notin \mathcal{B}(\beta, \beta')$. That is, $k - 1$ is in neither $\mathcal{B}(\alpha, \alpha')$ nor $\mathcal{B}(\beta, \beta')$, contradicting the hypothesis that these sets tile $\mathbb{N}$. This establishes Condition *4*.

Since $\mathcal{B}(\alpha, \alpha') \subseteq \mathbb{N}$, we have $\frac{1-\alpha'}{\alpha} \geq 1$, whence $\alpha + \alpha' \leq 1$. Likewise, $1 \geq \beta + \beta' = 1 - \alpha - \alpha'$, whence $0 \leq \alpha + \alpha'$, establishing Condition *3*.

5.4.2 If α is rational ...

If $\alpha = \frac{1}{2}$,then $\beta = 1-\alpha = \frac{1}{2}$. We have $\mathcal{B}(\alpha,\alpha') = (2n-a')_{n=1}^{\infty}$ and $\mathcal{B}(\beta,\beta') = (2n-b')_{n=1}^{\infty}$. This is the even-odd tiling of $\mathbb{N}$; we have $\{a',b'\} = \{0,1\}$. Conditions *3* and *5* are now easily verified. From this point on, we can assume that one of α,β is strictly less than $\frac{1}{2}$.

The set of fractional parts $\{\{k\alpha\} : k > k_0\} = \{\frac{i}{q} : 0 \le i < q\}$, so Line (5) says that there is no multiple of $\frac{1}{q}$ in $A \cap B$ or in $A^c \cap B^c$. Consider the multiples of $\frac{1}{q}$

$$-\alpha-\alpha', -\alpha-\alpha'+\tfrac{1}{q}, -\alpha-\alpha'+\tfrac{2}{q}, \dots, -\alpha', -\alpha'+\tfrac{1}{q}.$$

Since $\alpha \le 1-\frac{1}{q}$, the first and the last (which may be congruent modulo 1) are clearly not in $A := \overline{(-\alpha-\alpha',-\alpha']}$. Therefore, they must be in $B := \overline{[\beta+\beta',\beta')}$. That is $B = \overline{[-\alpha'+\frac{1}{q}, -\alpha-\alpha'+\frac{1}{q})}$. In particular, $\beta' \equiv -\alpha'+\frac{1}{q}$.

If $x \notin \mathbb{Z}$ then for any k at most one of $sx+x'$, $(s+1)x+x'$ is integral (where $s \in \mathbb{Z}$), since their difference is not integral. If $0 < x < \frac{1}{2}$, then at most one of $sx+x'$, $(s+1)x+x'$, $(s+2)x+x'$ is integral (if two were, then their difference would be also, but their difference is < 1). Thus, since one of α,β is strictly less than $\frac{1}{2}$, we may choose $k_1 \in \{\lfloor k_0+1\rfloor, \lfloor k_0+2\rfloor, \lfloor k_0+3\rfloor\}$ with neither $k_1\alpha+\alpha'$ nor $k_1\beta+\beta'$ integral. In particular $\{k_1\alpha+\alpha'\}+\{k_1\beta+\beta'\} \ge \frac{2}{q}$.

Choose $k_1 > k_0$ so that $\{k_1\alpha+\alpha'\}$ and $\{k_1\beta+\beta'\}$ are positive. By Lemma 5 we have $\{k_1\alpha+\alpha'\}+\{k_1\beta+\beta'\}-\frac{1}{q} \in \mathbb{Z}$, and by the definition of fractional part, $\{k_1\alpha+\alpha'\}+\{k_1\beta+\beta'\}-\frac{1}{q} \in [-\frac{1}{q}, 2-\frac{1}{q})$. Thus, $\{k_1\alpha+\alpha'\}+\{k_1\beta+\beta'\}-\frac{1}{q} = 1$. Plugging this into Lemma 5, we find $\alpha'+\beta' = \frac{1}{q}$.

Since $\mathcal{B}(\alpha,\alpha') \subseteq \mathbb{N}$, we have $\frac{1-\alpha'}{\alpha} \ge 1$, whence $\alpha+\alpha' \le 1$. Likewise, $1 \ge \beta+\beta' = 1-\alpha+\frac{1}{q}-\alpha'$, so that $\alpha+\alpha' \ge \frac{1}{q} \ge 0$. This establishes Condition *3* and the last piece of Condition *5*.

5.5 Using Nonstandard Analysis to Derive Irrational Case

We can use nonstandard analysis to easily derive the irrational case of Fraenkel's Partition from the rational case. Specifically, we now show that if Conditions *1–3*, *5* are sufficient for rational α, then Conditions *1–4* are sufficient for irrational α.

Suppose that α is irrational and $\alpha,\alpha',\beta,\beta'$ satisfy Conditions *1–4*. Since Conditions *1–4* are symmetric in α,β (as per the comment in Section 5.2.1), we may label α and β so that $\alpha+\alpha' \ne 0$ and $\beta+\beta' \ne 1$.

Let $q \in {}^*\mathbb{N} \setminus \mathbb{N}$ be an infinite prime number. Set

$$\gamma := \frac{\lfloor q\alpha\rfloor}{q}, \quad \gamma' := \frac{\lfloor q\alpha'\rfloor}{q}, \quad \delta := \frac{\lceil q\beta\rceil}{q}, \quad \delta' := \frac{\lceil q\beta\rceil+1}{q},$$

all $*$-rationals with denominator q. It is straightforward to verify that $\gamma, \gamma', \delta, \delta'$ satisfy Conditions *1*, *2*, *3*, and *5*, so $\mathcal{B}(\gamma, \gamma')$ and $\mathcal{B}(\delta, \delta')$ tile ${}^*\mathbb{N}$.

We have $\frac{n-\gamma'}{\gamma} \geq \frac{n-\alpha'}{\alpha}$ and for n finite $st(\frac{n-\gamma'}{\gamma}) = \frac{n-\alpha'}{\alpha}$, so $\left\lfloor \frac{n-\gamma'}{\gamma} \right\rfloor = \left\lfloor \frac{n-\alpha'}{\alpha} \right\rfloor$. Thus $B(\alpha, \alpha') = \mathbb{N} \cap \mathcal{B}(\gamma, \gamma')$. We also have $st(\frac{n-\delta'}{\delta}) = \frac{n-\beta'}{\beta}$, so $\left\lfloor \frac{n-\delta'}{\delta} \right\rfloor = \left\lfloor \frac{n-\beta'}{\beta} \right\rfloor$ unless $\frac{n-\beta'}{\beta}$ is an integer. If $\frac{n-\beta'}{\beta} = k \geq 2$, then $k - n = k - (k\beta + \beta') = k\alpha + \alpha' \in \mathbb{Z}$, contrary to Condition *4*. If $\frac{n-\beta'}{\beta} = 1$, then $\beta + \beta' = 1$, contrary to our assumption that $\beta + \beta' \neq 1$.

6 Proof of Brown's Decomposition

Fix an irrational $\alpha = [0; a_1, a_2, \ldots]$, and let $q_0, q_1, \ldots$ be its continuants. Specifically, let q_t be the largest continuant strictly less than m. We denote the distance to the nearest integer by

$$\|x\| := \min\left\{x - \lfloor x \rfloor, \lceil x \rceil - x\right\} = \min\left\{\, \{x\}, 1 - \{x\} \,\right\}.$$

Lemma 6. $C_m = C_{q_t} C_{m-q_t}$.

Brown's Decomposition follows immediately from Lemma 6 by induction. The proof of Lemma 6 relies on the following well-known result from the theory of continued fractions. It can be found as Theorem 10.15 (page 370) of [11], for example, or on page 163 of [9].

Lemma 7. *If* $|s| < q_{t+1}$ *then* $\|s\alpha\| > \|q_t\alpha\|$.

Proof of Lemma 6. First, if $\alpha = [0; a_1, \ldots, a_n]$ is rational, then we can replace it with an irrational x between α and $\alpha' = [0; a_1, \ldots, a_n, m+1]$. The continuants of α' that are less than m are the same as the continuants of α, so the continued fraction expansion of m looks the same. And if the irrational x is sufficiently close to α, then $\mathcal{B}(\alpha) \cap [1, m] = \mathcal{B}(x) \cap [1, m]$ and in particular C_m does not change when α is replaced with x. Thus, we may assume that α is irrational.

If $q_t = 1$, then $t < 2$ since $q_2 = a_2 q_1 + q_0 \geq 2 > q_t$. Thus either $m = (m)_\alpha$ or $m = (m0)_\alpha$. In the first case ($t = 0$), we have $m < q_1 = a_1$, and so $\alpha < \frac{1}{a_1} < \frac{1}{m}$. Therefore $C_m = 0^m = C_0 C_{m-1}$, proving the Lemma. In the second case ($t = 1$), we have $a_1 = 1$ and $1 \leq m < q_2 = a_2 + 1$, so $\alpha > \frac{a_2}{a_2+1}$. Now $k/\alpha < k\frac{a_2+1}{a_2} = k + \frac{k}{a_2}$, so $\lfloor k/\alpha \rfloor = k$ for $k = m \leq a_2$. Thus $C_m = 1^m = C_1 C_{m-1}$, proving the Lemma. Thus we may assume from this point on that $q_t > 1$.

It is obvious that C_{q_t} is an initial segment of C_m; the content of Lemma 6 is that $c_{q_t+k} = c_k$ for $1 \leq k \leq m - q_t \leq q_{t+1} - q_t$. In particular both $|k|$ and $|k+1|$ are strictly less than q_{t+1}.

By definition

$$c_k = 1 \quad \Leftrightarrow \quad k \in \mathcal{B}(\alpha)$$

and by Lemma 1 (with $\alpha' = 0$ and $k \geq 1$)

$$k \in \mathcal{B}(\alpha) \quad \Leftrightarrow \quad k\alpha \in \overline{(-\alpha, 0]}.$$

Likewise,

$$c_{q_t+k} = 1 \Leftrightarrow q_t + k \in \mathcal{B}(\alpha) \Leftrightarrow (q_t + k)\alpha \in \overline{(-\alpha, 0]} \Leftrightarrow k\alpha \in \overline{(-\alpha - q_t\alpha, -q_t\alpha]}.$$

We need to show only that $k\alpha$ is not in the symmetric difference of $\overline{(-\alpha, 0]}$ and $\overline{(-\alpha - q_t\alpha, -q_t\alpha]}$. This symmetric difference is contained in

$$\overline{[-\|q_t\alpha\|, \|q_t\alpha\|]} \cup \overline{[-\|q_t\alpha\| - \alpha, \|q_t\alpha\| - \alpha]}.$$

Now $|k| < q_{t+1}$, so by Lemma 7, $\|k\alpha\| > \|q_t\alpha\|$ and consequently

$$k\alpha \notin \overline{[-\|q_t\alpha\|, \|q_t\alpha\|]}.$$

Also $|k+1| < q_{t+1}$, so $\|(k+1)\alpha\| > \|q_t\alpha\|$, which is the same as $(k+1)\alpha \notin \overline{[-\|q_t\alpha\|, \|q_t\alpha\|]}$. Thus

$$k\alpha \notin \overline{[-\|q_t\alpha\| - \alpha, \|q_t\alpha\| - \alpha]}.$$

■

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