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GENERALIZATION OF A BINOMIAL IDENTITY OF SIMONS

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Abstract

We give a generalization of a binomial identity due to S. Simons using Cauchy's integral formula.

In [2] Simons proved a binomial identity which can be equivalently written as

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} (-1)^{n-k}(1+x)^k \,. \tag{1}$$

In [1] and [3] this identity is proved in different short ways. In this note, using Cauchy's integral formula as in [3], we give a generalization of (1). Consider the term

$$f_n = [t^n]\frac{(1+yt)^\alpha}{(1-xt)^\beta} = \sum_{k=0}^{n} \binom{\alpha}{n-k}\left(\!\!\binom{\beta}{k}\!\!\right) x^k y^{n-k}$$

where $\left(\!\binom{\beta}{k}\!\right) = \beta(\beta+1)\cdots(\beta+k-1)/k!$ and α, β, x, y are indeterminates. By Cauchy's integral formula we have

$$f_n = [t^n]\frac{(1+yt)^\alpha}{(1-xt)^\beta} = \frac{1}{2\pi \mathrm{i}} \oint \frac{(1+yz)^\alpha}{(1-xz)^\beta}\frac{\mathrm{d}z}{z^{n+1}} \,.$$

With the substitution $z = w/(1-sw)$, where s is an indeterminate, we have $\mathrm{d}z = \mathrm{d}w/(1-sw)^2$ and

$$\begin{aligned} f_n &= \frac{1}{2\pi \mathrm{i}} \oint \frac{(1+(y-s)w)^\alpha}{(1-(x+s)w)^\beta}\,(1-sw)^{\beta-\alpha+n-1}\frac{\mathrm{d}w}{w^{n+1}} \\ &= [t^n]\frac{(1+(y-s)t)^\alpha}{(1-(x+s)t)^\beta}(1-st)^{\beta-\alpha+n-1} \,. \end{aligned}$$

We now distinguish some cases. First let $s = y$. Then

$$f_n = [t^n]\frac{(1-yt)^{\beta-\alpha+n-1}}{(1-(x+y)t)^{\beta}} = \sum_{k=0}^{n} \binom{\beta-\alpha+n-1}{n-k} \left(\!\!\binom{\beta}{k}\!\!\right) (-1)^{n-k}(x+y)^k y^{n-k}.$$

Hence we have the identity

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \left(\!\!\binom{\beta}{k}\!\!\right) x^k y^{n-k} = \sum_{k=0}^{n} \binom{\beta-\alpha+n-1}{n-k} \left(\!\!\binom{\beta}{k}\!\!\right) (-1)^{n-k}(x+y)^k y^{n-k}. \qquad (2)$$

Substituting β with $\beta+1$ identity (2) becomes

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\beta+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{\beta-\alpha+n}{n-k} \binom{\beta+k}{k} (-1)^{n-k}(x+y)^k y^{n-k}. \qquad (3)$$

In particular for $\alpha = \beta$ identity (3) becomes

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\alpha+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+k}{k} (-1)^{n-k}(x+y)^k y^{n-k}. \qquad (4)$$

In particular, when $\alpha = n$ we have Simons's identity (1).

Suppose now that $s = -x$. Then

$$f_n = [t^n](1+(x+y)t)^{\alpha}(1+xt)^{\beta-\alpha+n-1} = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta-\alpha+n-1}{n-k} (x+y)^k x^{n-k}$$

and hence

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \left(\!\!\binom{\beta}{k}\!\!\right) x^k y^{n-k} = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta-\alpha+n-1}{n-k} (x+y)^k x^{n-k}. \qquad (5)$$

Substituting β with $\beta+1$, we have

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\beta+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta-\alpha+n}{n-k} (x+y)^k x^{n-k}. \qquad (6)$$

Finally for $\alpha = \beta$ we get

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\alpha+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha}{k} (x+y)^k x^{n-k}. \qquad (7)$$

Let now $y = 2s$ and $\beta = 2\alpha - n + 1$. Then

$$f_n = [t^n]\frac{(1-s^2t^2)^{\alpha}}{(1-(x+s)t)^{2\alpha-n+1}} = \sum_{k\geq 0} \binom{\alpha}{k} \binom{2\alpha-2k}{n-2k} (-1)^k s^{2k}(x+s)^{n-2k}.$$

Hence

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{2\alpha - k}{n-k} (2s)^k x^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{2\alpha - 2k}{n-2k} (-1)^k s^{2k} (x+s)^{n-2k} \,. \qquad (8)$$

Similarly, for $x = -2s$ and $\alpha = 2\beta + n - 1$, after the substitution of β with $\beta + 1$, we have

$$f_n = [t^n] \frac{(1 + (y-s)t^2)^{2\beta+n+1}}{(1 - s^2 t^2)^{\beta+1}} = \sum_{k \geq 0} \binom{2\beta + n + 1}{n - 2k} \binom{\beta + k}{k} s^{2k} (y-s)^{n-2k}$$

and thus

$$\sum_{k=0}^{n} \binom{2\beta + n + 1}{n-k} \binom{\beta + k}{k} (-2s)^k y^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2\beta + n + 1}{n - 2k} \binom{\beta + k}{k} s^{2k} (y-s)^{n-2k} \,. \qquad (9)$$

References

[1] R. Chapman, *A curious identity revised*, The Mathematical Gazette **87** (2003), 139–141.

[2] S. Simons, *A curious identity*, The Mathematical Gazette **85** (2001), 296–298.

[3] H. Prodinger, *A curious identity proved by Cauchy's integral formula*, The Mathematical Gazette, to appear.