A NOTE ON BOOLEAN LATTICES AND FAREY SEQUENCES

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Abstract

We establish monotone bijections between the Farey sequences of order m and the halfsequences of Farey subsequences associated with the rank m elements of the Boolean lattice of subsets of a $2m$-set. We also present a few related combinatorial identities.

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1. Introduction

The *Farey sequence of order* n, denoted by $\mathcal{F}_n$, is the ascending sequence of irreducible fractions $\frac{h}{k} \in \mathbb{Q}$ with $0 \le h \le k \le n$, see, e.g., [2, Chapter 27], [3, §3], [4, Chapter 4], [5, Chapter III], [8, Chapter 6], [9, Chapter 5]; their numerators and denominators are presented in sequences A006842 and A006843 in Sloane's *On-Line Encyclopedia of Integer Sequences*. For example, $\mathcal{F}_6 = \left(\frac{0}{1} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \frac{1}{1}\right)$.

For any integer m, $0 < m < n$, the ascending sets

$$\left(\tfrac{h}{k} \in \mathcal{F}_n : \; h \le m\right) \tag{1}$$

are interesting Farey subsequences [1].

Let C be a finite set of cardinality $n := |C|$ greater than or equal to two, and A its proper subset; $m := |A|$. Denote the Boolean lattice of subsets of C by $\mathbb{B}(n)$; the empty set is denoted by $\hat{0}$, and the family of l-element subsets of C is denoted by $\mathbb{B}(n)^{(l)}$. Let $\gcd(\cdot,\cdot)$ denote the greatest common divisor of two integers. The ascending sequence of fractions

$$\begin{aligned}\mathcal{F}\big(\mathbb{B}(n), m\big) :&= \left(\tfrac{|B\cap A|}{\gcd(|B\cap A|,|B|)}\Big/\tfrac{|B|}{\gcd(|B\cap A|,|B|)} : \; B \subseteq C, \; |B| > 0\right)\\ &= \left(\tfrac{h}{k} \in \mathcal{F}_n : \; h \le m, \; k-h \le n-m\right) \; ,\end{aligned}$$

considered in [7], has the properties very similar to those of the standard Farey sequence $\mathcal{F}_n$ and of Farey subsequence (1).

The Farey subsequences $\mathcal{F}\big(\mathbb{B}(2m), m\big) := \big(\frac{h}{k} \in \mathcal{F}_{2m} :\ h \leq m,\ k-h \leq m\big)$ arise in analysis of decision-making problems [6]. One such subsequence is

$$\mathcal{F}\big(\mathbb{B}(12),6\big) = \big(\tfrac{0}{1} < \tfrac{1}{7} < \tfrac{1}{6} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{2}{7} < \tfrac{1}{3} < \tfrac{3}{8} < \tfrac{2}{5} < \tfrac{3}{7} < \tfrac{4}{9} < \tfrac{5}{11} < \tfrac{1}{2} < \tfrac{6}{11} < \tfrac{5}{9} < \tfrac{4}{7} < \tfrac{3}{5} < \tfrac{5}{8} < \tfrac{2}{3} < \tfrac{5}{7} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{5}{6} < \tfrac{6}{7} < \tfrac{1}{1}\big) .$$

The fractions in the above-mentioned Farey (sub)sequence are indexed starting with zero.

In Theorem 5 of this note we establish the connection between the standard Farey sequence $\mathcal{F}_m$ and the halfsequences of $\mathcal{F}\big(\mathbb{B}(2m), m\big)$.

2. The Farey Subsequence $\mathcal{F}\big(\mathbb{B}(n), m\big)$

Recall that the map $\mathcal{F}_n \to \mathcal{F}_n$, which sends a fraction $\frac{h}{k}$ to $\frac{k-h}{k}$, is order-reversing and bijective. The sequences $\mathcal{F}\big(\mathbb{B}(n), m\big)$ and $\mathcal{F}\big(\mathbb{B}(n), n-m\big)$ have an analogous property:

Lemma 1 [7] *The map*

$$\mathcal{F}\big(\mathbb{B}(n), m\big) \to \mathcal{F}\big(\mathbb{B}(n), n-m\big) , \quad \tfrac{h}{k} \mapsto \tfrac{k-h}{k} , \tag{2}$$

is order-reversing and bijective.

If we write the fractions $\frac{h}{k} \in \mathbb{Q}$ as the column vectors $\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \in \mathbb{Z}^2$, then map (2) can be thought of as the map

$$\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] .$$

Let $a' \in \mathbb{B}(n)$ and $0 < m := \rho(a') < n$, where $\rho(a')$ denotes the poset rank of a' in $\mathbb{B}(n)$. For a subset $A \subset \mathbb{B}(n)$, let $\mathfrak{I}(A)$ and $\mathfrak{F}(A)$ denote the order ideal and filter in $\mathbb{B}(n)$, generated by A, respectively. The subposet $\mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(1)}\big)$, of cardinality $2^n - 2^{n-m}$, can be partitioned in the following way:

$$\mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(1)}\big) = \big(\mathfrak{I}(a') - \{\hat{0}\}\big) \ \dot{\cup} \dot{\bigcup_{\substack{\frac{h}{k} \in \mathcal{F}(\mathbb{B}(n),m): \\ \frac{0}{1} < \frac{h}{k} < \frac{1}{1}}}} \dot{\bigcup_{1 \leq s \leq \left\lfloor \min\left\{ \frac{m}{h},\ \frac{n-m}{k-h} \right\} \right\rfloor}} \Big(\mathbb{B}(n)^{(s\cdot k)} \cap \big(\mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h)}\big) - \mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h+1)}\big)\big)\Big) ,$$

where $\big|\mathbb{B}(n)^{(s\cdot k)} \cap \big(\mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h)}\big) - \mathfrak{F}\big(\mathfrak{I}(a') \cap \mathbb{B}(n)^{(s\cdot h+1)}\big)\big)\big| = \binom{m}{s\cdot h}\binom{n-m}{s\cdot(k-h)}$.

Since $|\mathfrak{I}(a') - \{\hat{0}\}| = 2^m - 1$, we obtain

$$2^n - 2^{n-m} = 2^m - 1 + \sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(n),m):\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\leq s\leq\left\lfloor\min\left\{\frac{m}{h},\ \frac{n-m}{k-h}\right\}\right\rfloor} \binom{m}{s\cdot h}\binom{n-m}{s\cdot(k-h)} \ .$$

If $a'' \in \mathbb{B}(n)$ and $\rho(a'') = n - m$, then Lemma 1 implies

$$2^n - 2^m = 2^{n-m} - 1 + \sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(n),n-m):\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\leq s\leq\left\lfloor\min\left\{\frac{n-m}{h},\ \frac{m}{k-h}\right\}\right\rfloor} \binom{n-m}{s\cdot h}\binom{m}{s\cdot(k-h)} \ ,$$

and we come to the following conclusion:

Proposition 2 *Fractions from the Farey subsequences $\mathcal{F}(\mathbb{B}(n), m)$ and $\mathcal{F}(\mathbb{B}(n), n-m)$ satisfy the equality:*

$$\begin{aligned}&\sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(n),m):\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\leq s\leq\left\lfloor\min\left\{\frac{m}{h},\ \frac{n-m}{k-h}\right\}\right\rfloor} \binom{m}{s\cdot h}\binom{n-m}{s\cdot(k-h)}\\ =&\sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(n),n-m):\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\leq s\leq\left\lfloor\min\left\{\frac{n-m}{h},\ \frac{m}{k-h}\right\}\right\rfloor} \binom{n-m}{s\cdot h}\binom{m}{s\cdot(k-h)}\\ =&\ 2^n - 2^m - 2^{n-m} + 1 \ .\end{aligned}$$

3. The Farey Subsequence $\mathcal{F}\big(\mathbb{B}(2m), m\big)$

Denote the left and right halfsequences of $\mathcal{F}\big(\mathbb{B}(2m), m\big)$ by

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) := \big(f \in \mathcal{F}\big(\mathbb{B}(2m), m\big) :\ f \leq \tfrac{1}{2}\big)$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) := \big(f \in \mathcal{F}\big(\mathbb{B}(2m), m\big) :\ f \geq \tfrac{1}{2}\big) \ ,$$

respectively.

Lemma 3 [6] *The maps*

$$\begin{aligned}\mathcal{F}\big(\mathbb{B}(2m), m\big) &\to \mathcal{F}\big(\mathbb{B}(2m), m\big) \ , & \tfrac{h}{k} &\mapsto \tfrac{k-h}{k} \ , & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right] &\mapsto \left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right] \ ,\\ \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) &\to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \ , & \tfrac{h}{k} &\mapsto \tfrac{k-2h}{2k-3h} \ , & \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right] &\mapsto \left[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right] \ ,\end{aligned}$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big) \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\ , \qquad \tfrac{h}{k} \mapsto \tfrac{h}{3h-k}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

are order-reversing and bijective.

Corollary 4 *The maps*

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big) \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\ , \qquad \tfrac{h}{k} \mapsto \tfrac{k-h}{2k-3h}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big) \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\ , \qquad \tfrac{h}{k} \mapsto \tfrac{2h-k}{3h-k}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2 & -1 \\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

are order-preserving and bijective.

Let $f_{t_3^1}$, $f_{t_2^1}$, $f_{t_3^2}$, $f_{t_1^1} \in \mathcal{F}\big(\mathbb{B}(2m),m\big)$, $m > 1$, where

$$f_{t_3^1} := \tfrac{1}{3}\ , \quad f_{t_2^1} := \tfrac{1}{2}\ , \quad f_{t_3^2} := \tfrac{2}{3}\ , \quad f_{t_1^1} := \tfrac{1}{1}\ ;$$

then Lemma 3 and Corollary 4 imply that $t_2^1 = 2t_3^1$, $t_3^2 = 3t_3^1$, and $t_1^1 = 4t_3^1$. This in particular means that the number $\big|\mathcal{F}\big(\mathbb{B}(2m),m\big)\big| - 1 =: t_1^1$ is divisible by four.

4. The Farey Sequence $\mathcal{F}_m$ and the Farey Subsequence $\mathcal{F}\big(\mathbb{B}(2m),m\big)$

Let h be a positive integer, and $[i,l] := \{j :\ i \leq j \leq l\}$ an interval of positive integers. Let

$$\phi(h;[i,l]) := \big|\big\{j \in [i,l] :\ \gcd(h,j) = 1\big\}\big|\ ;$$

thus, $\phi(h;[1,h])$ is the *Euler ϕ-function*. Recall that for a nonempty interval of positive integers $[i'+1,i'']$ it holds $\phi(h;[i'+1,i'']) = \sum_{d\in[1,\min\{i'',h\}]:\ d|h} \overline{\mu}(d) \cdot \big(\lfloor \frac{i''}{d} \rfloor - \lfloor \frac{i'}{d} \rfloor\big)$, where $d|h$ means that d divides h, and $\overline{\mu}(\cdot)$ stands for the *Möbius function*: $\overline{\mu}(1) := 1$; if $p^2|d$, for some prime p, then $\overline{\mu}(d) := 0$; if $d = p_1p_2\cdots p_s$ is the product of distinct primes $p_1, p_2, \ldots, p_s$, then $\overline{\mu}(d) := (-1)^s$.

Let m be an integer, $m > 1$. For every integer h, $1 \leq h \leq m$, we have

$$\begin{aligned} \big|\big\{\tfrac{h}{k} \in \mathcal{F}\big(\mathbb{B}(2m),m\big) :\ \tfrac{h}{k} < \tfrac{1}{2}\big\}\big| &= \phi(h;[2h+1,h+m]) \\ &= \sum_{d\in[1,h]:\ d|h} \overline{\mu}(d) \cdot \big(\lfloor \tfrac{h+m}{d} \rfloor - \tfrac{2h}{d}\big) = \sum_{d\in[1,h]:\ d|h} \overline{\mu}(d) \cdot \big(\lfloor \tfrac{m}{d} \rfloor - \tfrac{h}{d}\big) \\ &= \phi(h;[h+1,m]) = \big|\big\{\tfrac{h}{k} \in \mathcal{F}_m :\ \tfrac{h}{k} < \tfrac{1}{1}\big\}\big|\ ; \end{aligned}$$

hence, the sequences $\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)$ and $\mathcal{F}_m$ are of the same cardinality. Noticing that fractions $\frac{h_j}{k_j}$ and $\frac{h_{j+1}}{k_{j+1}}$ are consecutive in $\mathcal{F}_m$ if and only if the fractions $\frac{h_j}{k_j+h_j}$ and $\frac{h_{j+1}}{k_{j+1}+h_{j+1}}$ are consecutive in $\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)$, we arrive, with the help of Lemma 3 and Corollary 4, at the following conclusion:

Theorem 5 *Let m be an integer, $m > 1$. The maps*

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}_m\ , \qquad \tfrac{h}{k} \mapsto \tfrac{h}{k-h}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

and

$$\mathcal{F}_m \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big)\ , \qquad \tfrac{h}{k} \mapsto \tfrac{h}{k+h}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

are order-preserving and bijective; the maps

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}_m\ , \qquad \tfrac{h}{k} \mapsto \tfrac{k-h}{h}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

and

$$\mathcal{F}_m \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big)\ , \qquad \tfrac{h}{k} \mapsto \tfrac{k}{k+h}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

are order-reversing and bijective.

Direct counting gives $\big|\mathcal{F}\big(\mathbb{B}(2), 1\big)\big| = 3$ and $\big|\mathcal{F}\big(\mathbb{B}(4), 2\big)\big| = 5$.

Since $|\mathcal{F}_m| - 1 = \frac{1}{2}\sum_{d\geq 1} \overline{\mu}(d) \cdot \lfloor \frac{m}{d} \rfloor \cdot \lfloor \frac{m}{d} + 1 \rfloor$ (see, e.g., [4, §4.9]), Theorem 5 implies that for $m > 1$ we have

$$\big|\mathcal{F}\big(\mathbb{B}(2m), m\big)\big| - 1 = \sum_{d\geq 1} \overline{\mu}(d) \cdot \lfloor \tfrac{m}{d} \rfloor \cdot \lfloor \tfrac{m}{d} + 1 \rfloor\ .$$

By means of Theorem 5, the descriptions of sequences $\mathcal{F}_m$ and $\mathcal{F}\big(\mathbb{B}(2m), m\big)$ supplement each other. For example, consider a fraction $\frac{h}{k} \in \mathcal{F}_m - \big\{\frac{0}{1}, \frac{1}{1}\big\}$. If x_0 is the integer such that $hx_0 \equiv -1 \pmod{k}$ and $m - k + 1 \leq x_0 \leq m$, then it is known (see, e.g., [2, §27.1]) that the fraction $\frac{hx_0+1}{k}\Big/x_0$ succeeds the fraction $\frac{h}{k}$ in $\mathcal{F}_m$. Similarly, if x_0 is the integer such that $hx_0 \equiv 1 \pmod{k}$ and $m - k + 1 \leq x_0 \leq m$, then the fraction $\frac{hx_0-1}{k}\Big/x_0$ precedes $\frac{h}{k}$ in $\mathcal{F}_m$. Theorem 5 leads to an analogous statement:

Remark 6 *Let m be an integer, $m > 1$.*

(i) *Let $\frac{h}{k} \in \mathcal{F}\big(\mathbb{B}(2m), m\big)$. Suppose that $\frac{0}{1} < \frac{h}{k} \leq \frac{1}{2}$. Let x_0 be the integer such that $hx_0 \equiv 1 \pmod{(k-h)}$ and $m - k + h + 1 \leq x_0 \leq m$. The fraction $\frac{hx_0-1}{k-h}\Big/\frac{kx_0-1}{k-h}$ precedes $\frac{h}{k}$ in $\mathcal{F}\big(\mathbb{B}(2m), m\big)$.*

(ii) *Let $\frac{h}{k} \in \mathcal{F}\big(\mathbb{B}(2m), m\big)$. Suppose that $\frac{0}{1} \leq \frac{h}{k} < \frac{1}{2}$. Let x_0 be the integer such that $hx_0 \equiv -1 \pmod{(k-h)}$ and $m - k + h + 1 \leq x_0 \leq m$. The fraction $\frac{hx_0+1}{k-h}\Big/\frac{kx_0+1}{k-h}$ succeeds $\frac{h}{k}$ in $\mathcal{F}\big(\mathbb{B}(2m), m\big)$.*

Proposition 2 can be reformulated in the case where $n := 2m$, with the help of the bijections mentioned in Lemma 3 and Corollary 4, in several ways which we now summarize:

Proposition 7 *Let m be an integer, $m > 1$. The following combinatorial identities hold for fractions from the Farey subsequence $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$:*

(i)

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(2m),m):\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\le s\le\left\lfloor\min\left\{\frac{m}{h},\ \frac{m}{k-h}\right\}\right\rfloor} \binom{m}{s\cdot h}\binom{m}{s\cdot(k-h)} = 2^{2m} - 2^{m+1} + 1\ .$$

(ii)

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(2m),m):\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{2}}} \sum_{1\le s\le\left\lfloor\frac{m}{k-h}\right\rfloor} \binom{m}{s\cdot h}\binom{m}{s\cdot(k-h)} = \sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(2m),m):\\ \frac{1}{2}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\le s\le\left\lfloor\frac{m}{h}\right\rfloor} \binom{m}{s\cdot h}\binom{m}{s\cdot(k-h)}$$
$$= 2^{2m-1} - 2^m - \tfrac{1}{2}\binom{2m}{m} + 1\ .$$

(iii)

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(2m),m):\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{3}}} \sum_{1\le s\le\left\lfloor\frac{m}{k-h}\right\rfloor} \binom{m}{s\cdot(k-h)}\left(\binom{m}{s\cdot h}+\binom{m}{s\cdot(k-2h)}\right)$$
$$= \sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(2m),m):\\ \frac{1}{3}<\frac{h}{k}<\frac{1}{2}}} \sum_{1\le s\le\left\lfloor\frac{m}{k-h}\right\rfloor} \binom{m}{s\cdot(k-h)}\left(\binom{m}{s\cdot h}+\binom{m}{s\cdot(k-2h)}\right)$$

$$= \sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(2m),m):\\ \frac{1}{2}<\frac{h}{k}<\frac{2}{3}}} \sum_{1\le s\le\left\lfloor\frac{m}{h}\right\rfloor} \binom{m}{s\cdot h}\left(\binom{m}{s\cdot(k-h)}+\binom{m}{s\cdot(2h-k)}\right)$$
$$= \sum_{\substack{\frac{h}{k}\in\mathcal{F}(\mathbb{B}(2m),m):\\ \frac{2}{3}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\le s\le\left\lfloor\frac{m}{h}\right\rfloor} \binom{m}{s\cdot h}\left(\binom{m}{s\cdot(k-h)}+\binom{m}{s\cdot(2h-k)}\right)$$
$$= 2^{2m-1} - 2^m - \tfrac{1}{2}\binom{2m}{m} - \sum_{1\le t\le\left\lfloor\frac{m}{2}\right\rfloor}\binom{m}{2t}\binom{m}{t} + 1\ .$$

The bijections between the Farey sequence $\mathcal{F}_m$ and the halfsequences of $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$, presented in Theorem 5, allow us to describe the properties of fractions from $\mathcal{F}_m$, analogous to those of fractions from $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$, presented in Proposition 7(ii,iii):

Corollary 8 *Let m be an integer, $m > 1$. The following combinatorial identities hold for fractions from the standard Farey sequence $\mathcal{F}_m$:*

(i)

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}_m:\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\leq s\leq\lfloor\frac{m}{k}\rfloor} \binom{m}{s\cdot h}\binom{m}{s\cdot k} = 2^{2m-1} - 2^m - \tfrac{1}{2}\binom{2m}{m} + 1\ .$$

(ii)

$$\begin{aligned}&\sum_{\substack{\frac{h}{k}\in\mathcal{F}_m:\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{2}}} \sum_{1\leq s\leq\lfloor\frac{m}{k}\rfloor} \binom{m}{s\cdot k}\left(\binom{m}{s\cdot h} + \binom{m}{s\cdot(k-h)}\right)\\ &= \sum_{\substack{\frac{h}{k}\in\mathcal{F}_m:\\ \frac{1}{2}<\frac{h}{k}<\frac{1}{1}}} \sum_{1\leq s\leq\lfloor\frac{m}{k}\rfloor} \binom{m}{s\cdot k}\left(\binom{m}{s\cdot h} + \binom{m}{s\cdot(k-h)}\right)\\ &= 2^{2m-1} - 2^m - \tfrac{1}{2}\binom{2m}{m} - \sum_{1\leq t\leq\lfloor\frac{m}{2}\rfloor}\binom{m}{2t}\binom{m}{t} + 1\ .\end{aligned}$$

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